Appl. Math. Inf. Sci. **14**, No. 5, 791-800 (2020)

Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/140506

Asymptotic Stability Of Illicit Drug Dynamics with Banditry Compartment

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Received: 12 Apr. 2020, Revised: 21 May 2020, Accepted: 25 May 2020
Published online: 1 Sep. 2020

Abstract: Substance abuse is one of the major global health and social problems. In view of this, a mathematical model was designed to analyze the dynamics of drug abuse and banditry among population. Necessary conditions for the existence and stability of drug abuse and banditry steady states are derived. The drug abuse and banditry reproduction number is evaluated. A locally asymptotically stable drug abuse and banditry-free equilibrium at the drug abuse and banditry basic reproduction number less than unity is proved via the analysis of characteristic equation. Whereas, the existence of a locally asymptotically stable drug abuse and banditry-present equilibrium is established at the basic reproduction number greater than unity based on the use of center manifold theory of bifurcation. Bifurcation analysis at the threshold, $\mathscr{R}_0 = 1$, is investigated. Moreover, the local asymptotic dynamics of the model related to the drug abuse and banditry is showed to exhibit backward bifurcation. In addition, a sensitivity analysis is performed to examine the contributory effects of the model parameters on the spread of the drug abuse and banditry menace with respect to the drug abuse and banditry basic reproduction number.

Keywords: Sensitivity analysis, Drug abuse and banditry model, Drug abuse and banditry reproduction number, Bifurcation analysis.

1 Introduction

Substance abuse remains one of the major global health and social problem [1]. Substance abuse (also known as drug abuse) was defined by D. Nutt et al. as wrong or bad use of a drug where the abusers consume a substance in amounts or methods that harm them or others. Production and abuse of such substance have increased dramatically in Africa since last decade. Thus, the demand for treatment services for drug addition has also increased [2]. Substance abuse has increased not only costs to the public health system, but also other epidemics, such as HIV and STDs among population. It remains a global problem with endless health consequences, high rates of suicide, crime and increased government expenditure [3, 4].

Every 6 months in South Africa, drug abuse data is collected by South African Community Epidemiology Network on Drug Use (SACENDU) as a regular treatment monitoring system [5]. SACENDU is a network of researchers, practitioners and policy makers from eight sites in South Africa. They meet twice a year and provide community-level public health surveillance information on substance abuse. It was reported that there has been a significant increase the in demand for drug abuse rehabilitation or treatment in South Africa [6].

Nyabadza and Coetzee [7] addressed a complex problem of drug abuse and drug-related crimes in communities in the Western Cape Province. A theoretical model to evaluate the epidemic of substance abuse and drug-related crimes within the Western Cape province of South Africa was constructed and explored. The dynamics of drug abuse and drug-related crimes within the Western Cape are simulated using STELLA sofware. The simulation results were consistent with the data from SACENDU and Crime Statistics of South Africa. The model has been a beneficial tool in designing and planning interventions strategies that combat substance abuse and the relevant problems.

Pang et al. [8] handled China, which is tobacco victimized country in the world and burdened by smoking-related illnesses. Their work concentrated on how to control tobacco smoking in China. A

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mathematical model, described by ordinary differential equations with a saturated incidence rate, was proposed to explore the effect of controlling smoking by setting up designed smoking areas and raising the price of cigarettes. The sufficient criteria for the local and global stabilities of the equilibria were obtained. Moreover, a sensitivity analysis was performed to identity which factors effectively control smoking in China. Furthermore, numerical simulations were carried out to illustrate and verify the analytic conclusions

Kalula and Nyabadza [9] formulated a six-state compartmental model including a core and non-core group, with fast and slow progression to addiction, to qualitatively investigate the dynamics of substance abuse and predict drug abuse trends. The analysis was presented in terms of the substance abuse epidemic threshold $\mathscr{R}_0$. A numerical simulations were performed to match between the model and the available data concerning methamphetamine abuse in the Western Cape and to define the role of some key parameters. It was also fitted to data on methamphetamine abusers who are rehabilitated using the least squares curve fitting method.

The model exhibits a backward bifurcation, where a stable drug-free equilibrium coexists with a stable drug-persistent equilibrium for a certain defined range of values of $\mathscr{R}_0$. The stabilities of the model equilibria were ascertained and persistence conditions were established. The results showed that it is insufficient to reduce $\mathscr{R}_0$ below unity to control the substance abuse epidemic. Also, threshold parameter was calculated, $\mathscr{R}_0c$. Moreover, substance abuse epidemic can be reduced by intervention programs targeted at light drug users and by increasing the treatment rate for the addicts. Methamphetamine drug is also associated with sexual risk behaviour, i.e. it increases the likelihood of exposure to sexually transmitted infections (STIs) and HIV [10]

The World Health Organisation (WHO) reported that 40% of the treatment of drug abuse is offered by the government and 60% by the private sector in a particular province in South Africa [11]. However, compared to other provinces in South Africa, the Western Cape Province has higher rates of the substance abuse [12]. This province causes trouble with the higher proportion of alcohol and drug positive arrestees [13] . According to WHO, Heroin drug is the most dangerous and addictive narcotic drug produced from the resin of the opium poppy [14]. It is mixed with other substances, such as chalk powder, zinc oxide and strychnine [15]. It leads to suppression of pain, heaviness of limbs and shallow breathing. It also damages liver and affects heart lining and valves. Pregnant women, who are addicted to heroin can bear addicted babies, which means that addiction to heroin can be vertically transfer [14, 16]. In addition, it destroys the chemical balance in the brain [11].

Another substance is Marijuana which is the dried leaves, flowers, stems and seeds from the hemp plant (Cannabis sativa) [17, 18]. Cannabis sativa is a plant which grows annually in all parts of the world, such as the Canadian-American border primarily by Asian gangs [17] and South Africa [19]. This plant contains mind-altering chemical delta 9 tetrahydrocannabino (THC) and other related compounds. In 1980, the number of natural compounds identified in Cannabis sativa was 423 and increased to 483 in 1995 [18, 20]. It is the most abused drug worldwide and 20% of the abusers are youth [21]. It is commonly self-administered by smoking through rolling marijuana leaves in tobacco paper to be smoked as a cigarette, which may produce a variety of pharmacological effects, such as sedation, euphoria, hallucinations and temporal distraction [22].

Alcohol is another type of substance abuse, which harms health. It causes a psychiatric problem, which requires recurring harmful abuse use of ethanol despite of its negative consequences [9, 23]. Moreover, it causes some complications for the fetus when used by pregnant woman; such as abnormal appearance, short height, low body weight, small head, poor coordination, low intelligence, as well as problems related to hearing, seeing , and behavior [16, 24] . Fetal alcohol syndrome is the pattern of physical abnormalities and the impairment of mental development which are frequent among children with alcoholic parents [16].

The National Institute on Drug Abuse (NIDA) defines Methamphetamine (MA) as an extremely addictive drug that is chemically similar to amphetamine (which stimulates the central nervous system) [22]. In the Western Cape province of South Africa, this drug is known as ”tik” and is the common drug. It is a much more potent version of its parent drug amphetamine, which was the first drug introduced into medical practice as a nasal decongestant, but it is limited to certain rare conditions [25].

According to [22, 24], the rates of methamphetamine abuse are on the increase in the world particularly in Africa, where it is commonly used by young people and the persons under the age of 20 [25]. Methamphetamine is an illegal drug similar to cocaine and other powerful drugs, which damage brain. It also increases the amount of neurotransmitter dopamine in the brain. Methamphetamine abusers experience nervousness, confusion, insomnia and mood disturbances. Moreover, long-term methamphetamine abusers suffer from skin sores caused by scratching. There is also high risk of contracting infectious diseases, such as HIV, which occurs because of sharing contaminated drug injection, having two or more partners, and practicing unsafe sex. Furthermore, it may worsen the progression of HIV/AIDS and its consequences [25].

In most Africa countries especially Cape Town, Nigeria and Ghana, SACENDU [6] reported that between January and June 2013, 76% of drug abusers were males, 71% were coloured people, 59% were unemployed, 67% were single, and 59% were between the ages of 15 and 29. In 2006, (summing up all drug types victims) received treatment as in and out-patients were 1428 [26]. The latest report from SACENDU shows that the number of

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people, who received treatment between January and June 2013, was 3137 and they were between the ages of 15 and 19. SACENDU also reported that 33% of drug abusers, who have been admitted for treatment in rehabilitation centers, abused Methamphetamine (tik), 22% abused alcohol, and 22% abused cannabis (marijuana). Tik is the most common drug in Cape Town because it is affordable and widely available, as well as its recipe is accessible in market.

This paper presents a mathematical model governed by system of nonlinear ordinary differential equations to explore the relationship between illicit drug and banditry among population. The rest of the study is organized, as follows: Section 2 is devoted to the formulation of the model and it's basic properties. In Section 3, asymptotic stability and sensitivity analysis of the illicit drug and bandit-free as well as illicit drug and bandit-present equilibria are examined. Conclusion is presented in Section 4.

2 Formulation of the Model

The total population at time t, denoted by $N(t)$, is sub-divided into six distinct classes of individuals that are susceptible $S(t)$ (those affected by abusing illicit drug and bandit practices), Prisoner $P(t)$ (those who are imprisoned for involved in illicit drug abuse and suspected bandit), individuals under detention $D(t)$ (those who are in police custody because of being involved in illicit drug abuse and suspected bandit), suspected bandit $B(t)$ (those who commit banditry or involve in bandit act), Illicit drug abuser $I(t)$ (those who abuse or take illicit substance) and rehabilitating population $R(t)$ (those who are under a form of treatment or rehabilitation due drug abuse or involvement in banditry). The population size is dynamic because the recruitment (births) is not balanced by removal (natural deaths) rates due to induced death rate δ. Figure 1 represents the schematic diagram for the illicit drug and banditry population dynamics, which is governed by the system of nonlinear ordinary differential equations (1).

The dynamics of the model is formulated under the following assumptions:

(i) That susceptible human progresses only to abuser not bandit.

(ii) That abuser, rehabilitated individual and prisoner move back to susceptible humans.

(iii) That abuser turns to bandit population.

(iv) That abuser and bandit will be first detained before being moved prison.

(v) That detainee can move either to prison or rehabilitation center.

$$
\begin{aligned}
\frac{dS}{dt} &= \pi + \alpha I(t) + \tau_1 R(t) + \tau_2 P(t) - \mu S(t) \\
&\quad -\lambda S(t) \\
\frac{dI}{dt} &= \lambda S(t) - (\alpha + \sigma_1 + \gamma_3 + \delta_1 + \mu) I(t) \\
\frac{dB}{dt} &= \gamma_2 I(t) - (\sigma_2 + \delta_2 + \mu) B(t) \\
\frac{dD}{dt} &= \sigma_1 I(t) + \sigma_2 B(t) - (\psi_1 + \psi_2 + \mu) D(t) \\
\frac{dP}{dt} &= \psi_1 D(t) - (\tau_2 + \gamma_1 + \delta_3 + \mu) P(t) \\
\frac{dR}{dt} &= \psi_2 D(t) + \gamma_1 p(t) - (\tau_1 + \mu) R(t),
\end{aligned} \tag{1}
$$

with initial conditions at $t = 0$.
Where

$$\lambda = \frac{\beta(I(t) + \eta B(t))}{N(t)}$$

$$
\begin{aligned}
&S(0) = S_0, \quad B(0) = B_0, \quad I(0) = I_0, \\
&D(0) = D_0, \quad P(0) = P_0, \quad R(0) = R_0.
\end{aligned} \tag{2}
$$

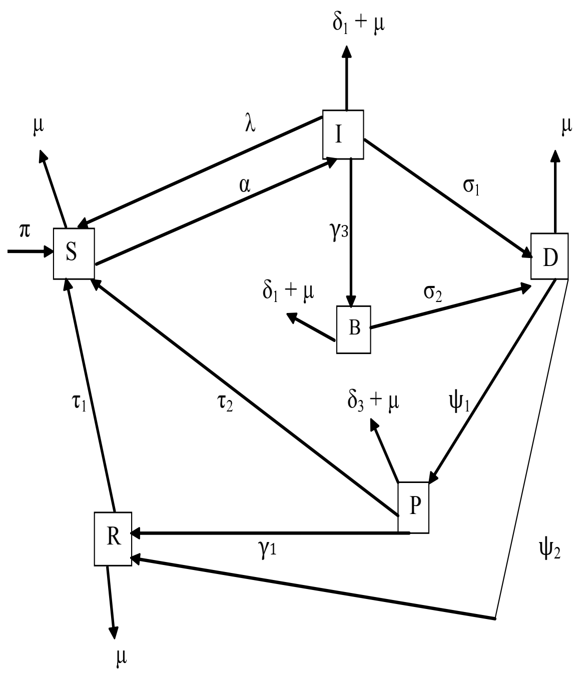

Fig. 1: Schematic diagram of the dynamic for drug abuse and banditry in a population

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For clarity, Table 1 gives the description of variables and parameters of drug abuse and banditry model.

Table 1. Description of variables and parameters

Variable/Parameter	Description
$S(t)$	population of susceptible individuals
$I(t)$	population of illicit drug user
$B(t)$	population of bandit
$D(t)$	population of individuals in detention
$P(t)$	population of individuals in prison
$R(t)$	population of individuals in rehabilitating center
π	recruitment rate into the susceptible population
α	rate at which illicit drug user become susceptible
τ_1	rate at which those in rehabilitating center are rehabilitated
τ_2	rate at which offenders complete sentence and become susceptible
μ	natural death rate
β	influence rate
σ_1	rate at which illicit drug user are apprehended and transferred to detention
σ_2	rate at which bandit are apprehended and transferred to detention
δ_1	induced death rate of illicit drug user
δ_2	induced death rate of bandit
δ_3	induced death rate of prisoner
ψ_1	rate at which detained individual are transferred to prison
ψ_2	rate at which detained individual are transferred to rehabilitating center
γ_1	rate at which those in prison move to rehabilitating center
γ_2	rate at which illicit drug user become bandit
η	modification parameter; showing that banditry is less effective when compare with illicit drug use, $\eta \in (0,1)$

It can be shown, using similar approach, that other state variables, $I(t)$; $B(t)$; $D(t)$; $P(t)$ and $R(t)$, are non-negative for all $t > 0$.

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2.1 Basic Qualitative Properties

2.1.1 Positivity and boundedness of solutions

Since model (1) monitors human population, all the parameters are non-negative. Thus, it is important to show that all the state variables are also non-negative for all time $t > 0$.

Theorem 1. *The state variables, $S(t)$; $I(t)$; $B(t)$; $D(t)$; $P(t)$ and $R(t)$, of the illicit drug and banditry model (1), with the non-negative initial data (2), remain non-negative for all $t > 0$.*

Proof. Recalling the equations in system (1)

One sees from the first equation of (1) that

$$\frac{dS}{dt} \geq -(\lambda+\mu)S(t),$$

so,

$$\frac{d}{dt}\left(S(t)\exp\left(\mu t+\int_0^t \lambda(\varpi)d\varpi\right)\right) \geq 0,$$

from which follows that

$$S(t) \geq S(0)\exp\left(-\left(\mu t+\int_0^t \lambda(\varpi)d\varpi\right)\right) > 0, \forall\, t > 0.$$

Next, consider the biologically feasible region, defined by $\Gamma \subset \mathbb{R}^6_+$
where:

$$\Gamma = \left\{(S,I,B,D,P,R) \in \mathbb{R}^6_+ : N \leq \frac{\pi}{\mu}\right\}$$

It is shown that Γ is positively invariant region.

Theorem 2. *The region Γ is positively invariant with respect to the model (1)*

Proof. The rate of change of the total population is given by

$$\frac{dN}{dt} = \pi - \mu N \tag{3}$$

it results in the solution;

$$N(t) = N(0)\exp(-\mu t) + \frac{\pi}{N}(1-\exp(-\mu t)).$$

it follows that $N(t) \longrightarrow \dfrac{\pi}{\mu}$ as t $\longrightarrow \infty$

in particular, $N(t) \leqslant \dfrac{\pi}{\mu}$ if $N(0) \leqslant \dfrac{\pi}{\mu}$ with respect to illicit drug and banditry model (1). Hence, it suffices to consider the dynamics of the model in Γ. In this region, illicit drug and banditry model is mathematically well-posed [27].

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3 Asymptotic Stability and Sensitivity Analysis

3.1 Existence of illicit drug and banditry-free Equilibrium (IDBFE ($\mathscr{D}_0$))

At illicit drug and banditry-free equilibrium $I_0 = B_0 = 0$ (No incidence of illicit drug and banditry in the population). Solving model (1) gives

$$S_0 = \frac{\pi}{\mu}$$

Therefore, the existence of illicit drug and banditry-free equilibrium is given by $\mathscr{D}_0$

$$\mathscr{D}_0 : (S_0, I_0, B_0, D_0, P_0, R_0) = \left(\frac{\pi}{\mu}, 0, 0, 0, 0, 0\right) \quad (4)$$

3.2 Illicit drug and banditry reproduction number

The illicit drug and banditry reproduction number, $\mathscr{R}_0$, given by (6), is a measure of the potential spread of illicit drug and banditry among naive or susceptible population. In other words, $\mathscr{R}_0$ represents the average number of secondary illicit drug and banditry cases influenced by a typical illicit drug and bandit in a completely susceptible population. The IDBFE is the steady state solution of the system (1) where no illicit drug and banditry existing among the population. Thus, the IDBFE of the system (1), denoted by $\mathscr{D}_0$, is given by

$$\mathscr{D}_0 = \left(\frac{\pi}{\mu}, 0, 0, 0, 0, 0\right) \quad (5)$$

Thus, using the next generation matrix approach [28], we observe that :

$$\frac{d}{dt}\begin{bmatrix} I \\ B \\ D \\ P \end{bmatrix} = \begin{bmatrix} \frac{\beta\mu(I+\eta B)}{\pi} \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} g_1 I \\ g_2 B - \gamma_2 I \\ g_3 D - \sigma_1 I - \sigma_2 B \\ g_4 P - \psi_1 D \end{bmatrix}$$

where :-

$$g_1 = (\alpha + \sigma_1 + \gamma_2 + \delta_1 + \mu)$$
$$g_2 = (\sigma_2 + \delta_2 + \mu)$$
$$g_3 = (\psi_1 + \psi_2 + \mu)$$
$$g_4 = (\tau_2 + \gamma_1 + \delta_3 + \mu)$$
$$g_5 = (\tau_1 + \mu)$$

Also,

$$F = \begin{bmatrix} \beta & \beta\eta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$V = \begin{bmatrix} g_1 & 0 & 0 & 0 \\ -\gamma_2 & g_2 & 0 & 0 \\ -\sigma_1 & -\sigma_2 & g_3 & 0 \\ 0 & 0 & -\psi_1 & g_4 \end{bmatrix}$$

Consequently, we obtain the spectral radius of the matrix FV^{-1}, known as the illicit drug and banditry reproduction number of the system (1), denoted by $\mathscr{R}_0$, and defined in the sense of [29–31] as

$$\mathscr{R}_0 = \frac{\beta g_6}{g_1 g_2}, \quad (6)$$

where:- $g_6 = \eta\gamma_2 + \sigma_2 + \delta_2 + \mu$

To examine the local stability of $\mathscr{D}_0$ given by (6), it is important to first obtain the illicit drug and banditry reproduction number, $\mathscr{R}_0$, given by (6), which is a measure of the potential spread of illicit drug and banditry among naive or susceptible population.

Theorem 3. *The illicit drug and banditry-free equilibrium, $\mathscr{D}_0$, of the system (1) is locally asymptotically stable if $\mathscr{R}_0 < 1$ and unstable $\mathscr{R}_0 > 1$.*

Proof. Now, to evaluate the local stability of illicit drug and banditry-free equilibrium, the Jacobian matrix evaluated at $\mathscr{D}_0$ corresponding to the following system is obtained as:

$$\mathscr{J}_0 = \begin{bmatrix} -\mu & \alpha-\beta & -\beta\eta & 0 & \tau_2 & \tau_1 \\ 0 & \beta-g_1 & \beta\eta & 0 & 0 & 0 \\ 0 & \gamma_3 & -g_2 & 0 & 0 & 0 \\ 0 & \sigma_1 & \sigma_2 & -g_3 & 0 & 0 \\ 0 & 0 & 0 & \psi_1 & -g_4 & 0 \\ 0 & 0 & 0 & \psi_2 & \gamma_1 & -g_5 \end{bmatrix}$$

A necessary and sufficient condition for local asymptotic stability is for the real part of the eigenvalue to be in the negative half plane [32]. Thus, we need to show that $J(\mathscr{D}_0)$ given the aforementioned the matrix has all its eigenvalues with negative real part. Thus, it is obvious from this matrix that $-\mu$, $-g_3$, $-g_4$ and , $-g_5$ are the four of the six eigenvalues of $J(\mathscr{D}_0)$ since the first, fourth, fifth and the sixth columns comprise only the diagonal terms. Hence, the other two eigenvalues can be obtained from the sub-matrix, $J^*(\mathscr{D}_0)$, given by

$$\mathscr{J}^*(\mathscr{D}_0) = \begin{bmatrix} \beta - g_1 & \beta\eta \\ \gamma_3 & -g_2 \end{bmatrix}$$

It follows from the characteristic equation of $J^*(\mathscr{D}_\mathscr{O})$ which is of the form $|J^*(\mathscr{D}_\mathscr{O}) - \lambda I| = 0$ given by :

$$\begin{bmatrix} (\beta - g_1) - \lambda & \beta\eta \\ \gamma_3 & -g_2 - \lambda \end{bmatrix} = 0$$

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Simplifying the above-mentioned matrix results in the polynomial below:

$$C_2\lambda^2 + C_1\lambda + C_0 = 0 \quad (7)$$

where:-

$C_2 = 1,$

$C_1 = g_2 + g_1 - \beta$

and

$C_0 = g_1g_2 - \beta g_2 - \beta\eta\gamma_3$

It is easy to see that C_0 can be written in terms of $\mathscr{R}_0$ as :

$$C_0 = g_1g_2[1 - \mathscr{R}_0] \quad (8)$$

If in (8) $\mathscr{R}_0 < 1$, then $C_0 > 0$. Since the coefficients $C_i, i = 1,2$ and the Hurwitz matrices of the polynomial (7) are positive, using Routh-Hurwitz criterion (see, [32, 33]), all the eigenvalues of (7) have negative real parts. Therefore, the illicit drug and banditry-free equilibrium, $\mathscr{D}_0$, is stable. Otherwise, when $\mathscr{R}_0 > 1$, $C_0 < 0$. By Descartes rule of signs [10], there exists one eigenvalue with positive real part. Hence, $\mathscr{D}_0$ is unstable for $\mathscr{R}_0 > 1$.

The implication of Theorem 3 is that reduction or elimination of illicit drug and banditry problem governed by model (1) can be eliminated from the population whenever an influx by illicit drug abuser and bandit individual is small such that $\mathscr{R}_0 < 1$.

3.3 Illicit drug and banditry-present equilibrium (IDBPE ($\mathscr{D}^*$))

The steady-state solution of model (1) when all the state variables are positive is referred to as the illicit drug and banditry-present equilibrium point denoted ($\mathscr{D}^*$) and given by

$$\mathscr{D}^* = (S^*, I^*, B^*, D^*, P^*, R^*). \quad (9)$$

It is burdensome to obtain the explicit form of the illicit drug and banditry-present equilibrium point of the model (1). However, the existence and local stability of $\mathscr{D}^*$ shall be explored using a center manifold theory of bifurcation analysis described in Castillo-Chavez and Song [34]. To this purpose, let the illicit drug and banditry model (1) be written in the vector form

$$\frac{dX}{dt} = F(X), \quad (10)$$

where:-

$\mathrm{X} = (x_1, x_2, x_3, x_4, x_5, x_6)^T$

and

$\mathrm{F} = (f_1, f_2, f_3, f_4, f_5, f_6)^T$

with

$S(t) = x_1, I(t) = x_2, B(t) = x_3, D(t) = x_4, P(t) = x_5,$
$R(t) = x_6.$

Then model (1) becomes:

$$\begin{aligned}
f_1 &= \frac{dx_1}{dt} = \pi + \alpha x_2 + \tau_1 x_6 + \tau_2 x_5 - \mu x_1 - \lambda x_1 \\
f_2 &= \frac{dx_2}{dt} = \lambda x_1 - g_1 x_2 \\
f_3 &= \frac{dx_3}{dt} = \gamma x_2 - g_2 x_3 \\
f_4 &= \frac{dx_4}{dt} = \sigma_1 x_2 + \sigma_2 x_3 - g_3 x_4 \\
f_5 &= \frac{dx_5}{dt} = \psi_1 x_4 - g_4 x_5 \\
f_6 &= \frac{dx_6}{dt} = \psi_2 x_4 + \gamma_1 x_5 - g_5 x_6
\end{aligned} \quad (11)$$

At $\mathscr{R}_0 = 1$ in (6), the bifurcation parameter β^* can be obtained as :

$$\beta^* = \frac{g_1g_2}{g_6} \quad (12)$$

The linearized matrix of the system (11) around the illicit drug and banditry-free equilibrium $\mathscr{D}_0$ and evaluated at β^* is given by

let $\quad b_2 = \beta^* - g_1 \qquad b_1 = \alpha - \beta^*$

$$\mathscr{J}_(D_0, \beta^*) = \begin{bmatrix} -\mu & b_1 & -\beta^*\eta & 0 & \tau_2 & \tau_1 \\ 0 & b_2 & \beta^*\eta & 0 & 0 & 0 \\ 0 & \gamma_3 & -g_2 & 0 & 0 & 0 \\ 0 & \sigma_1 & \sigma_2 & -g_3 & 0 & 0 \\ 0 & 0 & 0 & \psi_1 & -g_4 & 0 \\ 0 & 0 & 0 & \psi_2 & \gamma_1 & -g_5 \end{bmatrix}$$

The eigenvalues λ of $\mathscr{J}_(D_0, \beta^*)$ are the roots of the characteristic equation of the form :

$$(\lambda + \mu)(\lambda + g_3)(\lambda + g_4)(\lambda + g_5)P(\lambda) = 0 \quad (13)$$

where $P(\lambda)$ is a polynomial of degree two whose roots are all negative except one zero eigenvalue. The left eigenvector, $v = (v_1, v_2, \ldots, v_6)$, corresponding to the simple zero eigenvalue of (13) is obtained from $v\mathscr{J}_(\mathscr{D}_0, \beta^*) = 0$ as :

$$\begin{aligned}
v_1 = \quad v_4 = \quad v_5 = \quad v_6 &= 0; \\
v_2 &= \frac{g_2}{\beta^*\eta} V_3
\end{aligned} \quad (14)$$

Furthermore, the right eigenvector, $w = (w_1, w_2, \ldots, w_6)^T$, associated with this simple zero eigenvalue can be obtained from $v\mathscr{J}_(\mathscr{D}_0, \beta^*) = 0$. As a

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result, we have

$$
\begin{aligned}
w_1 &= \frac{g_2g_3g_4g_5(\alpha-\beta^*)-\beta^*\eta\gamma_3g_3g_4g_5+d_3}{g_3g_4g_5\gamma_3}w_3\\
w_2 &= \frac{g_2}{\gamma_3}w_3 \qquad w_4 = \frac{\sigma_1g_2+\sigma_2\gamma_3}{\gamma_3g_3}w_3\\
w_5 &= \frac{\psi_1(\sigma_1g_2+\sigma_2\gamma_3)}{\gamma_3g_3g_4}w_3\\
w_6 &= \frac{(g_4\psi_2-\gamma_1\psi_1)(\sigma_1g_2+\sigma_2\gamma_3)}{\gamma_3g_3g_4g_5}w_3
\end{aligned} \tag{15}
$$

where:-
$b_3 = (\sigma_1g_2+\sigma_2\gamma_3)(\tau_1(g_4\psi_2+\gamma_1\gamma_1\psi_1)+\tau_2g_5\psi_1$

It should be noted that the components of w and v are obtained, so $v.w = 1$ as required in [30, 34]. All the second-order partial derivatives of $f_i, i = i = 1,2...,6$, from the system (11) are zero at point $(\mathscr{D}_0,\beta^*)$ except the following :

$$\frac{\partial^2 f_1}{\partial x_1\partial x_2} = \frac{\partial^2 f_1}{\partial x_2\partial x_1} = \frac{-\beta^*\mu}{\pi},$$

$$\frac{\partial^2 f_1}{\partial x_1\partial x_3} = \frac{\partial^2 f_1}{\partial x_3\partial x_1} = \frac{-\beta^*\mu\eta}{\pi}$$

$$\frac{\partial^2 f_2}{\partial x_1\partial x_2} = \frac{\partial^2 f_2}{\partial x_2\partial x_1} = \frac{\beta^*\mu}{\pi},$$

$$\frac{\partial^2 f_2}{\partial x_1\partial x_3} = \frac{\partial^2 f_2}{\partial x_3\partial x_1} = \frac{\beta^*\mu\eta}{\pi}$$

with

$$\frac{\partial^2 f_1}{\partial x_2\partial\beta} = -1, \qquad \frac{\partial^2 f_1}{\partial x_3\partial\beta} = -\eta,$$

$$\frac{\partial^2 f_2}{\partial x_2\partial\beta} = 1, \qquad \frac{\partial^2 f_2}{\partial x_3\partial\beta} = \eta.$$

The direction of the bifurcation at $\mathscr{R}_0 = 1$ is defined by the signs of the bifurcation coefficients **a** and **b**, obtained from the above-mentioned partial derivatives, given, respectively, by

$$a = \sum_{k,i,j=1}^{6} v_kw_iw_j\frac{\partial^2 f_k}{\partial x_i\partial x_j}(\mathscr{D}_0,\beta^*)$$

$$= 2\left[\frac{g_2\eta b_4\mu}{d_2\gamma_3\pi(\eta\gamma_3+g_2)}\right]\left[\frac{g_2+\gamma_3\eta}{\gamma_3}\right]v_3w_3^2 \tag{16}$$

where:-
$d_1 = g_2g_3g_4g_5 \qquad d_2 = g_3g_4g_5$
$d_3 = \sigma_1g_2+\sigma_2\gamma_2 \qquad d_4 = g_4\psi_2+\gamma_1\psi_1$
$b_4 = (d_1\alpha+(\tau_2g_5\psi_1+\tau_1d_4)d_3)(\eta\gamma_3+g_2) - (d_1+\eta\gamma_3d_2)g_1g_2$

then,
$a > 0$ when $(d_1\alpha+(\tau_2g_5\psi_1+\tau_1d_4)d_3)(\eta\gamma_3+g_2) > (d_1+\eta\gamma_3d_2)g_1g_2$

and

$$b = \sum_{k,i=1}^{6} v_kw_i\frac{\partial^2 f_k}{\partial x_i\partial\beta}(\mathscr{D}_0,\beta^*)$$

$$= \left[\frac{(\gamma_3\eta+g_2)\eta}{g_1}\right]\left[\frac{g_2+\gamma_3\eta}{\gamma_3}\right]v_3w_3 \tag{17}$$

Since $a > 0$ and $b > 0$, it follows from Theorem 4.1(i) (see, [30, 34]) that the illicit drug and banditry model (1) exhibits a subcritical (backward) bifurcation and the illicit drug and banditry-free equilibrium $\mathscr{D}_0$ is locally stable but globally asymptotically unstable even when $\mathscr{R}_0 < 1$ except at some certain points when $\mathscr{R}_0 < \mathscr{R}_0c < 1$.

However, when $\mathscr{R}_0c < \mathscr{R}_0 < 1$, two illicit drug and banditry-free equilibria one is stable, and the other is unstable. The results of this model reveal how hard it may be to eliminate illicit drug and banditry if the following parameters (α, τ_1 and τ_2) are not watched closely.

Theorem 4.*The illicit drug and banditry model governed by (1) exhibits a backward bifurcation at the threshold* $\mathscr{R}_0 = 1$

3.4 Sensitivity Analysis

Following the idea in [29, 31, 35, 36], we perform a sensitivity analysis of the model (1) to define the contributory effects of the model parameters on the transmission and spread of the illicit drug and banditry menace.
Definition 3.1. The normalized forward-sensitivity index of variable (v) which depends differentiably on parameter (**p**) is defined as:

$$\Upsilon_p^v = \frac{\partial v}{\partial p}\times\frac{p}{v} \tag{18}$$

In particular, sensitivity indices of the illicit drug and banditry reproduction number, $\mathscr{R}_0$, with respect to the model parameters are computed. For example, using (18), the following are obtained:-

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$$
\begin{aligned}
\Upsilon_{\beta}^{\mathscr{R}_{\mathscr{O}}} &= \frac{\partial \mathscr{R}_{\mathscr{O}}}{\partial \beta} \times \frac{\beta}{\mathscr{R}_{\mathscr{O}}} = 1 \\
\Upsilon_{\eta}^{\mathscr{R}_{\mathscr{O}}} &= \frac{\partial \mathscr{R}_{\mathscr{O}}}{\partial \beta} \times \frac{\eta}{\mathscr{R}_{\mathscr{O}}} = \frac{\gamma_3 \eta}{\eta\gamma_3 + \mu + \delta_2 + \sigma_2} \\
\Upsilon(\gamma_3)^{\mathscr{R}_{\mathscr{O}}} &= \frac{\partial \mathscr{R}_{\mathscr{O}}}{\partial \gamma_3} \times \frac{\gamma_3}{\mathscr{R}_{\mathscr{O}}} \\
&= \frac{\gamma_3(\eta(\mu+\alpha+\delta_1+\sigma_1)-\mu-\delta_2-\sigma_2)}{(\eta\gamma_3+\mu+\delta_2+\sigma_2)(\alpha+\mu+\delta_1+\sigma_1+\gamma_3)} \\
\Upsilon(\sigma_2)^{\mathscr{R}_{\mathscr{O}}} &= \frac{\partial \mathscr{R}_{\mathscr{O}}}{\partial \sigma_2} \times \frac{\sigma_2}{\mathscr{R}_{\mathscr{O}}} \\
&= \frac{-\sigma_2\eta\gamma_3}{(\eta\gamma_3+\mu+\delta_2+\sigma_2)(\mu+\delta_2+\sigma_2)} \\
\Upsilon(\sigma_1)^{\mathscr{R}_{\mathscr{O}}} &= \frac{\partial \mathscr{R}_{\mathscr{O}}}{\partial \sigma_1} \times \frac{\sigma_1}{\mathscr{R}_{\mathscr{O}}} = \frac{-\sigma_1}{(\alpha+\sigma_1+\gamma_3+\mu+\delta_1)} \qquad (19) \\
\Upsilon(\alpha)^{\mathscr{R}_{\mathscr{O}}} &= \frac{\partial \mathscr{R}_{\mathscr{O}}}{\partial \alpha} \times \frac{\alpha}{\mathscr{R}_{\mathscr{O}}} = \frac{-\alpha}{(\alpha+\sigma_1+\gamma_3+\mu+\delta_1)} \\
\Upsilon(\delta_1)^{\mathscr{R}_{\mathscr{O}}} &= \frac{\partial \mathscr{R}_{\mathscr{O}}}{\partial \delta_1} \times \frac{\delta_1}{\mathscr{R}_{\mathscr{O}}} = \frac{-\delta_1}{(\alpha+\sigma_1+\gamma_3+\mu+\delta_1)} \\
\Upsilon(\delta_2)^{\mathscr{R}_{\mathscr{O}}} &= \frac{\partial \mathscr{R}_{\mathscr{O}}}{\partial \delta_2} \times \frac{\delta_2}{\mathscr{R}_{\mathscr{O}}} \\
&= \frac{-\delta_2\eta\gamma_3}{(\eta\gamma_3+\mu+\delta_2+\sigma_2)(\mu+\delta_2+\sigma_2)}
\end{aligned}
$$

Sensitivity index (S.I) of $\mathscr{R}_0$ with respect to other parameters of the model can be computed using method adopted in (19). We summarize the overall sensitivity analysis of the model by showing the signs of (S.I) in the following Table.

Table 2. Sensitivity indexes of $\mathscr{R}_0$ with respect to the model parameters.

Parameter	Sensitivity index
β	+ 1
α	-0.6506329114
η	+ 0.2586206897
σ_1	-0.2025316456
δ_1	-0.08860759494
σ_2	-0.01373207202
δ_2	-0.01089680723
γ_2	+0.005456132694

Table 3. Values of parameters of the model

Parameter	Range	Baseline value	Source
π	0.9 - 0.7	0.8	[7, 22]
α	0.4 - 0.1	0.257	[7, 22]
τ_1	0.1 - 0.04	0.057	[7]
τ_2	0.09 - 0.036	0.046	[7, 8]
μ	0.04 - 0.009	0.013	[23]
β	0.7 - 0.2	0.38	[8, 22]
σ_1	0.1 - 0.05	0.08	[8, 23]
σ_2	0.09 - 0.04	0.06	[7, 8]
δ_2	0.08 - 0.02	0.04	[7]
δ_3	0.3 - 0.08	0.14	[7]
ψ_1	0.09 - 0.02	0.05	Assumed
ψ_2	0.15 - 0.07	0.09	Assumed
γ_1	0.09 - 0.02	0.05	Assumed
η	0.5 - 0.1	0.3	[23]
γ_2	0.03 - 0.008	0.01	[22]
δ_1	0.06 - 0.028	0.035	[8]

The sign of the sensitivity index (S.I) plays a key role in defining how the parameters of the model relate to the illicit drug and banditry reproduction number, $\mathscr{R}_0$, of the model (1). In Table 2, the positive sign of (S.I) shows a direct relation between $\mathscr{R}_0$ and the parameters in this category while the negative sensitivity index exhibits an inverse relation between $\mathscr{R}_0$ and the parameters. With sensitivity analysis, one can get an insight on the appropriate intervention strategies that prevent and control the effect and spread of the illicit drug and banditry among population.

Accordingly, the sensitivity index of the illicit drug and banditry reproduction number, $\mathscr{R}_0$, of the model (1) related to the influence rate, β, is evaluated using [29–31] as $\Upsilon_(\alpha)^{\mathscr{R}_{\mathscr{O}}} = 1$. Similarly, the sensitivity indexes of $\mathscr{R}_0$ to each of the other eight parameters are computed and presented in Table 2. The parameters are arranged in a decreasing order of sensitivity indexes from the most sensitive to the least sensitive parameter. Table 2 indicates that the most sensitive parameter is β (influence rate). Since $\Upsilon_(\alpha)^{\mathscr{R}_{\mathscr{O}}} = 1$, increasing (or decreasing) β by 10% increases (or decreases) the illicit drug and banditry reproduction number, $\mathscr{R}_0$ by 10%. Similarly, for the other parameter with positive index, on the other,$\Upsilon_(\alpha)^{\mathscr{R}_{\mathscr{O}}}$ = -0.6506329114, increasing (or decreasing) α by 10% decreases (or increases) the illicit drug and banditry reproduction number, $\mathscr{R}_0$ by 6.5%.

This means that corrective intervention strategy should be provided to bring down the spate of illicit drug and banditry among population. Next to the most sensitive parameter is the modification parameter, η, which gives the positive index. According to Table 2, increasing (or decreasing) η by 10% increases (or decreases) $\mathscr{R}_0$ by 2.5%. This means that preventive intervention strategy should be provided to forestall the spread of illicit drug and banditry among population.

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4 Conclusion

A system of ordinary differential equations, which describes the dynamics of illicit drug and banditry among population, was presented and analysed. The results are defined, as follows:

(i)Illicit drug and banditry model was formulated to analyze the spread of menace among population with short and long time control, i.e detention and prison.
(ii)Existence of illicit drug and banditry-free equilibrium was carried out.
(iii)Illicit drug and banditry reproduction number was evaluated at illicit drug and banditry-free equilibrium and its local stability was shown.
(i)Bifurcation analysis of the mode was further carried out on the illicit drug and banditry present equilibrium and proved to be backward bifurcation.
(v)The most sensitive parameter of the model is the influence rate, followed by the modification parameter among other parameters. This shows that illicit drug and banditry-free population is dependent on the reduction of the influence rate on susceptible individuals and the modification parameter.

Accordingly, since the model exhibits backward bifurcation it implies that reducing $\mathscr{R}_{\mathscr{O}} < 1$ is insufficient to eradicate the problem of illicit drug and banditry. In addition, illicit drug and banditry free population is dependent on the reduction of the influence rate on susceptible individuals and the modification parameter. Finally, corrective strategy should be put in place to force down the two most sensitive parameters. However, strict enforcement of law against illicit drug and banditry will reduce illicit drug and banditry reproduction number which in-turn reduces illicit drug and banditry among population.

Acknowledgement

The author is grateful to the anonymous referee for the accurate checking of the details and the constructive comments that improved the present paper.

Conflict of Interest.
The authors declare that there is no conflict of interest regarding the publication of this article

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