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Generalized Fractional Hermite-Hadamard Type Inequalities for Convex Functions

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Abstract: In this article, we obtain some Hermite-Hadamard type inequalities for differentiable convex functions involving generalized fractional integrals. Some of our results are the extension of previously obtained results like (Dragomir and Agarwal in Appl. Math. Lett. 11(5): 91-95, 1998, Dragomir, Chob and Kimc in J. Math. Anal. Appl. 245(2):489-501, 2000, Yang, H. Wang and Tseng in Comput. Math. Appl. 47(2-3):207-216, 2004 and S. Qaisar et al. in J. Inequal. Appl. 2019(1):111, 2019). We also discuss some special cases.

Keywords: convex functions, generalized fractional integrals, Hermite-Hadamard inequalities.

1 Introduction

The Hermite–Hadamard inequality, discovered by C. Hermite and J. Hadamard, (see [1], [2, pp. 137]), is one of the most well established inequalities in the theory of convex functions with a geometrical interpretation and many applications. These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a,b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \quad (1)$$

Both inequalities in (1) hold in the reversed direction if f is concave. We note that Hermite–Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hermite–Hadamard inequality for convex functions has recently received renewed attention and a remarkable variety of refinements and generalizations has been investigated, see ([3–20]).
In [21], Dragomir and Agarwal obtained inequalities for differentiable convex mappings, which are connected with the right hand side of Hermite-Hadamard's (trapezoid) inequality and applied them to obtain some elementary inequalities for real numbers and in numerical integration, as follows.

Theorem 1.1. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I°, with $a < b$. If $|f'|^q$ is convex on $[a,b]$ for some $q \geq 1$, the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \quad (2)$$

In [22], Dragomir obtained inequalities for a Lipschitzian mapping, which are in connection with the right-hand side of Hermite-Hadamard's (trapezoid) inequality.

Theorem 1.2. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be $M-$Lipschitzian mapping on I where $x,y \in I$, with $x < y$, then we have the following inequality:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{M}{3}(b-a). \quad (3)$$

In [23], Qaisar et al. obtained inequalities for convex functions, which are related to the right hand side of the fractional Hermite-Hadamard inequalities.

Theorem 1.3. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I°, with $a < b$. If $|f'|^q$ is convex on $[a,b]$ for

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some $q \geq 1$, the following inequality holds:

$$\begin{aligned}
&\left| \left(\frac{(b-a)^\alpha - (b-x)^\alpha}{(b-a)^\alpha} + \frac{(x-a)^\alpha}{(b-a)^\alpha} \right) \frac{f(a)}{2} \right. \\
&+ \left(\frac{(b-a)^\alpha - (x-a)^\alpha}{(b-a)^\alpha} + \frac{(b-x)^\alpha}{(b-a)^\alpha} \right) \frac{f(b)}{2} \\
&\left. - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right| \\
\leq& \frac{(x-a)^{\alpha+2}}{(b-a)^\alpha} \left[\gamma_1^{1-\frac{1}{q}} \left(A \left| f'(x) \right|^q + B \left| f'(a) \right|^q \right)^{\frac{1}{q}} \right] \\
&+ \frac{(b-x)^{\alpha+2}}{(b-a)^\alpha} \left[\gamma_2^{1-\frac{1}{q}} \left(A \left| f'(x) \right|^q + B \left| f'(b) \right|^q \right)^{\frac{1}{q}} \right] \\
&+ \frac{(b-x)^{\alpha+2}}{(b-a)^\alpha} \left[\gamma_3^{1-\frac{1}{q}} \left(C \left| f'(x) \right|^q + D \left| f'(b) \right|^q \right)^{\frac{1}{q}} \right] \\
&+ \frac{(x-a)^{\alpha+2}}{(b-a)^\alpha} \left[\gamma_4^{1-\frac{1}{q}} \left(E \left| f'(x) \right|^q + F \left| f'(a) \right|^q \right)^{\frac{1}{q}} \right], \quad (4)
\end{aligned}$$

where

$$\begin{aligned}
\gamma_1 &= \frac{\alpha}{\alpha+1}, \ \gamma_2 = \frac{\alpha}{\alpha+1}, \\
\gamma_3 &= -\frac{1}{\alpha+1} \left(\frac{a-x}{x-b} \right)^{\alpha+1} - \left(\frac{a-x}{x-b} \right)^\alpha \\
&\quad + \frac{1}{\alpha+1} \left(\frac{a-b}{x-b} \right)^\alpha, \\
\gamma_4 &= \frac{1}{\alpha+1} \left(\frac{b-x}{x-a} \right)^{\alpha+1} - \left(\frac{b-x}{x-a} \right)^\alpha \\
&\quad - \frac{1}{\alpha+1} \left(\frac{b-a}{x-a} \right)^{\alpha+1},
\end{aligned}$$

and

$$\begin{aligned}
A &= \frac{\alpha}{2(\alpha+2)}, \ B = \frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)} \\
C &= -\frac{1}{(\alpha+1)(\alpha+2)} \left(\frac{a-x}{x-b} \right)^{\alpha+2} + \frac{1}{2} \left(\frac{a-x}{x-b} \right) \\
&\quad + \frac{1}{(\alpha+1)(\alpha+2)} \left(\frac{a-b}{x-b} \right)^{\alpha+2} \\
&\quad - \frac{1}{\alpha+1} \left(\frac{a-x}{x-b} \right)^{\alpha+1} - \left(\frac{a-x}{x-b} \right)^\alpha, \\
D &= \frac{1}{\alpha+1} \left(\frac{a-b}{x-b} \right)^{\alpha+1} + \frac{1}{(\alpha+1)(\alpha+2)} \left(\frac{a-x}{x-b} \right)^\alpha \\
&\quad - \frac{1}{2} \left(\frac{a-x}{x-b} \right) - \frac{1}{(\alpha+1)(\alpha+2)} \left(\frac{a-b}{x-b} \right)^{\alpha+2}, \\
E &= \left(\frac{b-x}{2(x-a)} \right) - \frac{1}{\alpha+1} \left(\frac{b-x}{x-a} \right)^{\alpha+1} - \frac{b-x}{x-a} \\
&\quad - \frac{1}{(\alpha+1)(\alpha+2)} \left(\frac{b-x}{x-a} \right)^{\alpha+2} \\
&\quad + \frac{1}{(\alpha+1)(\alpha+2)} \left(\frac{b-a}{x-a} \right)^{\alpha+2} \\
F &= \frac{1}{\alpha+1} \left(\frac{b-a}{x-a} \right)^{\alpha+2} - \frac{b-x}{2(x-a)} \\
&\quad + \frac{1}{(\alpha+1)(\alpha+2)} \left(\frac{b-x}{x-a} \right)^{\alpha+2} - E.
\end{aligned}$$

2 New Generalized Fractional Integral Operators

In this section, we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [14].
Let's define a function $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following conditions :

$$\int_0^1 \frac{\varphi(t)}{t} dt < \infty.$$

We define the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$$_{a^+}I_\varphi f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t) dt, \ \ x > a, \quad (5)$$

$$_{b^-}I_\varphi f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} f(t) dt, \ \ x < b. \quad (6)$$

Interested readers can check ([9, 24–28]) in which the authors obtained several inequalities related to Hermite-Hadamard type using generalized fractional integrals [14]. The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals, such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals,...etc. These important special cases of the integral operators (5) and (6) are mentioned below.
i) If we take $\varphi(t) = t$, the operators (5) and (6) reduce to the Riemann integral, as follows:

$$I_{a^+} f(x) = \int_a^x f(t) dt, \ \ x > a,$$

$$I_{b^-} f(x) = \int_x^b f(t) dt, \ \ x < b.$$

ii) If we take $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, the operators (5) and (6) reduce to the Riemann-Liouville fractional integral, as follows:

$$I_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \ \ x > a,$$

$$I^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \ \ x<b.$$

iii) If we take $\varphi(t) = \frac{1}{k\Gamma_k(\alpha)}t^{\frac{\alpha}{k}}$, the operators (5) and (6) reduce to the k-Riemann-Liouville fractional integral, as follows:

$$I^{\alpha}_{a^+,k}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \ \ x>a,$$

$$I^{\alpha}_{b^-,k}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \ \ x<b$$

where

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-\frac{t^k}{k}}dt, \ \ \mathscr{R}(\alpha)>0$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right), \ \ \mathscr{R}(\alpha)>0; k>0$$

are given by Mubeen and Habibullah in [29].
iv) If we take $\varphi(t) = t(x-t)^{\alpha-1}$, the operator (5) reduces to the conformable fractional operators, as follows:

$$I^{\alpha}_{a}f(x) = \int_a^x t^{\alpha-1} f(t)dt = \int_a^x f(t)d_{\alpha}t, \ \ x>a, \ \alpha\in(0,1)$$

is given by Khalil et al. in [30].
The present paper aims to obtain some inequalities connected with the right inequality of Hermite-Hadamard type for generalized fractional integrals using convexity for functions whose absolute values of the first derivative are convex.

3 Main Results

For brevity, in this section, we use following notations:

$$\Lambda_1(t) = \int_t^1 \frac{\varphi((x-a)u)}{u}du,$$

$$\Lambda_2(t) = \int_{\frac{x-a}{b-x}}^{\frac{b-a}{b-x}-t} \frac{\varphi((b-x)u)}{u}du,$$

$$\Lambda_3(t) = \int_t^1 \frac{\varphi((b-x)u)}{u}du$$

and

$$\Lambda_4(t) = \int_{\frac{b-x}{x-a}}^{\frac{b-a}{x-a}-t} \frac{\varphi((x-a)u)}{u}du.$$

Lemma 3.1. Let $f: I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be differentiable mapping on I°, with $a<b$. If $f'\in L[a,b]$, the following identity holds

$$\Lambda_1(0)f(a)+\Lambda_2(0)f(b)+\Lambda_3(0)f(b)+\Lambda_4(0)f(a)$$

$$-[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)] = \sum_{i=1}^{4}\Im_i, \quad (7)$$

where

$$\Im_1 = (a-x)\int_0^1 \Lambda_1(t)f'(tx+(1-t)a)dt,$$

$$\Im_2 = (b-x)\int_0^1 \Lambda_2(t)f'(tx+(1-t)b)dt,$$

$$\Im_3 = (b-x)\int_0^1 \Lambda_3(t)f'(tx+(1-t)b)dt,$$

$$\Im_4 = (a-x)\int_0^1 \Lambda_4(t)f'(tx+(1-t)a).$$

Proof. Applying integration by parts, we have

$$\Im_1 = (a-x)\int_0^1 \Lambda_1(t)f'(tx+(1-t)a)dt$$
$$= \Lambda_1(0)f(a) - \int_0^1 \frac{\varphi((x-a)t)}{t}f(tx+(1-t)a)dt$$

and changing the variable of integration, we have

$$\Im_1 = \Lambda_1(0)f(a) - \int_a^x \frac{\varphi(u-a)}{u-a}f(u)du. \quad (8)$$

Similarly, we have

$$\Im_2 = (b-x)\int_0^1 \Lambda_2(t)f'(tx+(1-t)b)dt$$
$$= \Lambda_2(0)f(b) - \int_0^1 \frac{\varphi((b-a)-(b-x)t)}{(b-a)-(b-x)t}f(tx+(1-t)b)dt$$
$$= \Lambda_2(0)f(b) - \int_x^b \frac{\varphi(u-a)}{u-a}du. \quad (9)$$

Adding (8) and (9), we have

$$\Im_1+\Im_2 = \Lambda_1(0)f(a)+\Lambda_2(0)f(b) - {}_{b-}I_{\varphi}f(a).$$

Consequently, we have

$$\Im_3+\Im_4 = \Lambda_3(0)f(b)+\Lambda_4(0)f(a) - {}_{a+}I_{\varphi}f(b).$$

The proof is completed.

Remark 3.1. Under the assumptions of Lemma 3.1, if we take $\varphi(t)=t$, we have following identity

$$\left(\frac{x-a}{b-a}\right)f(a)+\left(\frac{b-x}{b-a}\right)f(b)-\frac{1}{b-a}\int_a^b f(u)du$$
$$= \frac{(x-a)^2}{b-a}\int_0^1 (t-1)f'(tx+(1-t)b)dt$$
$$+\frac{(b-x)^2}{b-a}\int_0^1 (1-t)f(tx+(1-t)b)dt.$$

Remark 3.2. Under the assumptions of Lemma 3.1, if we use $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$, Lemma 3.1 reduces to ([23, Lemma 1]).

Corollary 3.1. Under the assumptions of Lemma 3.1, if we take $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we have the following new identity for $k-$frcational integrals:

$$\left[\left(\frac{(b-a)^{\frac{\alpha}{k}}-(b-x)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}+\frac{(x-a)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}\right)\frac{f(a)}{2}+\left(\frac{(b-a)^{\frac{\alpha}{k}}-(x-a)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}+\frac{(b-x)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}\right)\frac{f(b)}{2}-\frac{\Gamma_k(\alpha+1)}{2(b-a)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{a+,k}f(b)+J^{\alpha}_{b-,k}f(a)\right]=\sum_{i=1}^{4}A_i^*\right],$$

where

$$A_1^*=\frac{(x-a)^{\frac{\alpha}{k}+1}}{(b-a)^{\frac{\alpha}{k}}}\int_0^1(t^{\frac{\alpha}{k}}-1)f'(tx+(1-t)a)dt,$$

$$A_2^*=\frac{(b-x)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}\int_0^1(1-t^{\frac{\alpha}{k}})f'(tx+(1-t)b)dt,$$

$$A_3^*=\frac{(b-x)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}\int_0^1\left[\begin{array}{c}\left(\frac{a-b}{x-b}-t\right)^{\frac{\alpha}{k}}\\-\left(\frac{a-x}{x-b}\right)^{\frac{\alpha}{k}}\end{array}\right]f'(tx+(1-t)b)dt,$$

$$A_4^*=\frac{(x-a)^{\frac{\alpha}{k}+1}}{(b-a)^{\frac{\alpha}{k}}}\int_0^1\left[\begin{array}{c}\left(\frac{b-x}{b-a}\right)^{\frac{\alpha}{k}}\\-\left(\frac{b-a}{x-a}-t\right)^{\frac{\alpha}{k}}\end{array}\right]f'(tx+(1-t)a)dt.$$

Corollary 3.2. Under the assumptions of Lemma 3.1, if we take $\varphi(t)=t(x-t)^{\alpha-1}$, we have the following new identity for the conformable fractional integrals:

$$\Lambda_1^*(0)f(a)+\Lambda_2^*(0)f(b)+\Lambda_3^*(0)f(b)+\Lambda_4^*(0)f(a)$$

$$-2I_a^{\alpha}f(b)=\sum_{i=1}^{4}\Im_i^*$$

where

$$\Im_1^*=(a-x)\int_0^1\Lambda_1^*(t)f'(tx+(1-t)a)dt,$$

$$\Im_2^*=(b-x)\int_0^1\Lambda_2^*(t)f'(tx+(1-t)b)dt,$$

$$\Im_3^*=(b-x)\int_0^1\Lambda_3^*(t)f'(tx+(1-t)b)dt,$$

$$\Im_4^*=(a-x)\int_0^1\Lambda_4^*(t)f'(tx+(1-t)a),$$

and

$$\Lambda_1^*(t)=\frac{(-tx+at+1)^{\alpha}-(-x+a+1)^{\alpha}}{\alpha},$$

$$\Lambda_2^*(t)=\left[\left(\frac{\begin{array}{c}tx^2+((1-2b)t+b-a)x\\+(b^2-b)t-b^2+(a+1)b-a\end{array}}{b-x}\right)^{\alpha}-\left(\frac{x^2+(t-b-a)x-tb+(a+1)b-a}{b-x}\right)^{\alpha}\right],$$

$$\Lambda_3^*(t)=\frac{(tx-tb+1)^{\alpha}-(x-b+1)^{\alpha}}{\alpha},$$

$$\Lambda_4^*(t)=\left[\left(\frac{\begin{array}{c}tx^2+((-2a-1)t-b+a)x\\+(a^2+a)t-a^2+(a+1)b-a\end{array}}{x-a}\right)^{\alpha}-\left(\frac{x^2-(t+b+a)x+ta+(a+1)b-a}{x-a}\right)^{\alpha}\right].$$

Theorem 3.1. Let $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be differentiable mapping on I°, with $a<b$. If $|f'|$ is convex on $[a,b]$, the following inequality holds for the generalized fractional integrals:

$$\begin{aligned}&|\Lambda_1(0)f(a)+\Lambda_2(0)f(b)+\Lambda_3(0)f(b)+\Lambda_4(0)f(a)\\&-[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)]|\\&\le(x-a)\left[(A_1+A_7)\left|f'(x)\right|++(A_2+A_8)\left|f'(a)\right|\right]\\&+(b-x)\left[(A_3+A_5)\left|f'(x)\right|+(A_4+A_6)\left|f'(b)\right|\right],\end{aligned}\tag{10}$$

where

$$A_1=\int_0^1t\,|\Lambda_1(t)|\,dt,\ A_2=\int_0^1(1-t)\,|\Lambda_1(t)|\,dt,$$

$$A_3=\int_0^1t\,|\Lambda_2(t)|\,dt,\ A_4=\int_0^1(1-t)\,|\Lambda_2(t)|\,dt,$$

$$A_5=\int_0^1t\,|\Lambda_3(t)|\,dt,\ A_6=\int_0^1(1-t)\,|\Lambda_3(t)|\,dt,$$

$$A_7=\int_0^1t\,|\Lambda_4(t)|\,dt,\ A_8=\int_0^1(1-t)\,|\Lambda_4(t)|\,dt.$$

Proof. Using Lemma 3.1 and properties of modulus, we obtain

$$\begin{aligned}&|\Lambda_1(0)f(a)+\Lambda_2(0)f(b)+\Lambda_3(0)f(b)+\Lambda_4(0)f(a)\\&-[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)]|\\&\le|\Im_1|+|\Im_2|+|\Im_3|+|\Im_4|.\end{aligned}\tag{11}$$

Now, by convexity of $|f'|$, we obtain

$$\begin{aligned}|\Im_1|&=\left|(x-a)\int_0^1\Lambda_1(t)f'(tx+(1-t)a)dt\right|\\&\le(x-a)\int_0^1|\Lambda_1(t)|\left|f'(tx+(1-t)a)\right|\\&\le(x-a)\int_0^1|\Lambda_1(t)|\left(t\left|f'(x)\right|+(1-t)\left|f'(a)\right|\right)dt\\&=(x-a)\left[A_1\left|f'(x)\right|+A_2\left|f'(a)\right|\right].\end{aligned}\tag{12}$$

Similarly

$$\Im_2 \leq (b-x)\left[A_3\left|f'(x)\right|+A_4\left|f'(b)\right|\right], \tag{13}$$

$$\Im_3 \leq (b-x)[A_5\left|f'(x)\right|+A_6\left|f'(b)\right|], \tag{14}$$

$$\Im_4 = (x-a)[A_7\left|f'(x)\right|+A_8\left|f'(a)\right|]. \tag{15}$$

Using inequalities (12)-(15) in (11), we have required inequality (10).

Remark 3.3. Under the assumptions of Theorem 3.1, if we use $\varphi(t)=t$, we have the following inequality

$$\begin{aligned}&\left|\left(\frac{x-a}{b-a}\right)f(a)+\left(\frac{b-x}{b-a}\right)f(b)-\frac{1}{b-a}\int_a^b f(u)du\right|\\ &\leq \frac{(x-a)^2}{3}\left[\left|f'(x)\right|+2\left|f'(a)\right|\right]\\ &\quad+\frac{(b-x)^2}{3}\left[\left|f'(x)\right|+2\left|f'(b)\right|\right].\end{aligned}$$

Remark 3.4. If we set $x=\frac{a+b}{2}$ in Remark 3.3, Remark 3.3 reduces to the Theorem 1.1.

Remark 3.5. Under the assumptions of Theorem 3.1, if we take $\varphi(t)=\frac{t^{\alpha}}{\Gamma(a)}$, Theorem 3.1 reduces to ([23, Theorem 4]).

Corollary 3.3. Under the assumptions of Theorem 3.1, if we use $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we have the following inequality for the $k-$ fractional integrals:

$$\begin{aligned}&\left|\left(\frac{(b-a)^{\frac{\alpha}{k}}-(b-x)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}+\frac{(x-a)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}\right)\frac{f(a)}{2}\right.\\ &+\left(\frac{(b-a)^{\frac{\alpha}{k}}-(x-a)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}+\frac{(b-x)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}\right)\frac{f(b)}{2}\\ &\left.-\frac{\Gamma_k(\alpha+1)}{2(b-a)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{a+,k}f(b)+J^{\alpha}_{b-,k}f(a)\right]\right|\\ &\leq \frac{(x-a)^{\frac{\alpha}{k}+1}}{(b-a)^{\frac{\alpha}{k}+1}}\left[A^*\left|f'(x)\right|+B^*\left|f'(a)\right|\right]\\ &+\frac{(b-x)^{\frac{\alpha}{k}+1}}{(b-a)^{\frac{\alpha}{k}+1}}\left[A^*\left|f'(x)\right|+B^*\left|f'(b)\right|\right]\\ &+\frac{(b-x)^{\frac{\alpha}{k}+1}}{(b-a)^{\frac{\alpha}{k}+1}}\left[C^*\left|f'(x)\right|+D^*\left|f'(b)\right|\right]\\ &+\frac{(x-a)^{\frac{\alpha}{k}+1}}{(b-a)^{\frac{\alpha}{k}+1}}\left[E^*\left|f'(x)\right|+F^*\left|f'(a)\right|\right],\end{aligned}$$

where

$$\begin{aligned}A^* &= \frac{\alpha}{2(\alpha+2k)},\ B^*=\frac{3k\alpha}{2(\alpha+k)(\alpha+2k)}\\ C^* &= -\frac{k^2}{(\alpha+k)(\alpha+2k)}\left(\frac{a-x}{x-b}\right)^{\frac{\alpha}{k}+2}+\frac{1}{2}\left(\frac{a-x}{x-b}\right)\\ &\quad+\frac{k^2}{(\alpha+k)(\alpha+2k)}\left(\frac{a-b}{x-b}\right)^{\frac{\alpha}{k}+2}\\ &\quad-\frac{k}{\alpha+k}\left(\frac{a-x}{x-b}\right)^{\frac{\alpha}{k}+1}-\left(\frac{a-x}{x-b}\right)^{\frac{\alpha}{k}},\\ D^* &= \frac{k}{\alpha+k}\left(\frac{a-b}{x-b}\right)^{\frac{\alpha}{k}+1}+\frac{k^2}{(\alpha+k)(\alpha+2k)}\left(\frac{a-x}{x-b}\right)^{\frac{\alpha}{k}}\\ &\quad-\frac{1}{2}\left(\frac{a-x}{x-b}\right)-\frac{k^2}{(\alpha+k)(\alpha+2k)}\left(\frac{a-b}{x-b}\right)^{\frac{\alpha}{k}+2},\\ E^* &= \left(\frac{b-x}{2(x-a)}\right)-\frac{k}{\alpha+k}\left(\frac{b-x}{x-a}\right)^{\frac{\alpha}{k}+1}-\frac{b-x}{x-a}\\ &\quad-\frac{k^2}{(\alpha+k)(\alpha+2k)}\left(\frac{b-x}{x-a}\right)^{\frac{\alpha}{k}+2}\\ &\quad+\frac{k^2}{(\alpha+k)(\alpha+2k)}\left(\frac{b-a}{x-a}\right)^{\frac{\alpha}{k}+2},\\ F^* &= \frac{k}{\alpha+k}\left(\frac{b-a}{x-a}\right)^{\frac{\alpha}{k}+2}-\frac{b-x}{2(x-a)}\\ &\quad+\frac{k^2}{(\alpha+k)(\alpha+2k)}\left(\frac{b-x}{x-a}\right)^{\frac{\alpha}{k}+2}-E^*.\end{aligned}$$

Theorem 3.2. Let $f : I^{\circ} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable mapping on I°, with $a<b$. If $|f'|^q$, $q \geq 1$, is convex on $[a,b]$, the following inequality holds for the generalized fractional integrals:

$$\begin{aligned}&|\Lambda_1(0)f(a)+\Lambda_2(0)f(b)+\Lambda_3(0)f(b)+\Lambda_4(0)f(a)\\ &-[_{a+}I_{\varphi}f(b)+_{b-}I_{\varphi}f(a)]|\\ &\leq (x-a)\left[\lambda_1^{1-\frac{1}{q}}\left(A_1\left|f'(x)\right|^q+A_2\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right]\\ &+(b-x)\left[\lambda_2{}^{1-\frac{1}{q}}\left(A_3\left|f'(x)\right|^q+A_4\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]\\ &+(b-x)\left[\lambda_3{}^{1-\frac{1}{q}}\left(A_5\left|f'(x)\right|^q+A_6\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]\\ &+(x-a)\left[\lambda_4{}^{1-\frac{1}{q}}\left(A_7\left|f'(x)\right|^q+A_8\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right],\end{aligned} \tag{16}$$

where A_i's are defined in Theorem 3.1 and $\lambda_i=\int_0^1\Lambda_i(t)dt$.

Proof. Using Lemma 3.1 and properties of modulus, we obtain

$$\begin{aligned}&|\Lambda_1(0)f(a)+\Lambda_2(0)f(b)+\Lambda_3(0)f(b)+\Lambda_4(0)f(a)\\&-[{}_{a+}I_\varphi f(b)+{}_{b-}I_\varphi f(a)]|\\ \leq\ &|\Im_1|+|\Im_2|+|\Im_3|+|\Im_4|. \quad (17)\end{aligned}$$

Now, using well-known power mean inequality and convexity of $|f'|^q$, we have

$$\begin{aligned}|\Im_1| &= \left|(x-a)\int_0^1\Lambda_1(t)f'(tx+(1-t)a)dt\right|\\&\leq (x-a)\left(\int_0^1\Lambda_1(t)dt\right)^{1-\frac{1}{q}}\\&\quad\times\left(\int_0^1\Lambda_1(t)\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\&\leq (x-a)\left(\int_0^1\Lambda_1(t)dt\right)^{1-\frac{1}{q}}\\&\quad\times\left(\int_0^1\Lambda_1(t)[t\left|f'(x)\right|^q+(1-t)\left|f'(a)\right|^q]dt\right)^{\frac{1}{q}}\\&=(x-a)\left[\lambda_1{}^{1-\frac{1}{q}}\left(A_1\left|f'(x)\right|^q+A_2\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right]. \quad (18)\end{aligned}$$

Similarly, we have

$$|\Im_2|\leq (b-x)\left[(\lambda_2)^{1-\frac{1}{q}}\left(A_3\left|f'(x)\right|^q+A_4\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right] \quad (19)$$

$$|\Im_3|\leq (b-x)\left[(\lambda_3)^{1-\frac{1}{q}}\left(A_5\left|f'(x)\right|^q+A_6\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right] \quad (20)$$

$$|\Im_4|\leq (x-a)\left[(\lambda_4)^{1-\frac{1}{q}}\left(A_7\left|f'(x)\right|^q+A_8\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right] \quad (21)$$

Using inequalities (18)-(21) in inequality (17), we get our desired inequality (16).

Remark 3.6. Under the assumptions of Theorem 3.2, if we take $\varphi(t)=t$, we have the following inequality

$$\begin{aligned}&\left|\left(\frac{x-a}{b-a}\right)f(a)+\left(\frac{b-x}{b-a}\right)f(b)-\frac{1}{b-a}\int_a^b f(u)du\right|\\&\leq (x-a)\left(\frac{x-a}{2}\right)^{1-\frac{1}{q}}\\&\quad\times\left(\frac{x-a}{6}\left|f'(x)\right|^q+\frac{x-a}{3}\left|f'(a)\right|^q\right)^{\frac{1}{q}}\\&\quad+(b-x)\left(\frac{b-x}{2}\right)^{1-\frac{1}{q}}\\&\quad\times\left(\frac{b-x}{6}\left|f'(x)\right|^q+\frac{b-x}{3}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\\&\quad+(b-x)\left(\frac{b-x}{2}\right)^{1-\frac{1}{q}}\\&\quad\times\left(A_5\left|f'(x)\right|^q+A_6\left|f'(b)\right|^q\right)^{\frac{1}{q}}\\&\quad+(x-a)\left(\frac{x-a}{2}\right)^{1-\frac{1}{q}}\\&\quad\times\left(\frac{x-a}{6}\left|f'(x)\right|^q+\frac{x-a}{3}\left|f'(a)\right|^q\right)^{\frac{1}{q}}.\end{aligned}$$

Remark 3.7. If we set $x=\frac{a+b}{2}$ and $|f'(t)|\leq M$, $t\in[a,b]$ in Remark 3.6, Remark 3.6 reduces to Theorem 1.2.

Remark 3.8. Under the assumptions of Theorem 3.2, if we take $\varphi(t)=\frac{t^\alpha}{\Gamma(a)}$, Theorem 3.2 reduces to Theorem 1.3.

Corollary 3.4. Under the assumptions of Theorem 3.2, if we take $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha)}$, we have the following inequality for the $k-$fractional integrals:

$$\begin{aligned}&\left|\left(\frac{(b-a)^{\frac{\alpha}{k}}-(b-x)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}+\frac{(x-a)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}\right)\frac{f(a)}{2}\right.\\&+\left(\frac{(b-a)^{\frac{\alpha}{k}}-(x-a)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}+\frac{(b-x)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}\right)\frac{f(b)}{2}\\&\left.-\frac{\Gamma_k(\alpha+1)}{2(b-a)^{\frac{\alpha}{k}}}\left[J_{a+,k}^\alpha f(b)+J_{b-,k}^\alpha f(a)\right]\right|\\&\leq\frac{(x-a)^{\frac{\alpha}{k}+2}}{(b-a)^{\frac{\alpha}{k}}}\left[(\lambda_1^*)^{1-\frac{1}{q}}\left(A^*\left|f'(x)\right|^q+B^*\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right]\\&+\frac{(b-x)^{\frac{\alpha}{k}+2}}{(b-a)^{\frac{\alpha}{k}}}\left[(\lambda_2^*)^{1-\frac{1}{q}}\left(A^*\left|f'(x)\right|^q+B^*\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]\\&+\frac{(b-x)^{\frac{\alpha}{k}+2}}{(b-a)^{\frac{\alpha}{k}}}\left[(\lambda_3^*)^{1-\frac{1}{q}}\left(C^*\left|f'(x)\right|^q+D^*\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]\\&+\frac{(x-a)^{\frac{\alpha}{k}+2}}{(b-a)^{\frac{\alpha}{k}}}\left[(\lambda_3^*)^{1-\frac{1}{q}}\left(E^*\left|f'(x)\right|^q+F^*\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right],\end{aligned}$$

where

$$\lambda_1^* = \frac{\alpha k}{\alpha+k},\ \lambda_2^* = \frac{\alpha k}{\alpha+k},$$
$$\lambda_3^* = -\frac{k}{\alpha+k}\left(\frac{a-x}{x-b}\right)^{\frac{\alpha}{k}+1} - \left(\frac{a-x}{x-b}\right)^{\frac{\alpha}{k}} + \frac{k}{\alpha+k}\left(\frac{a-b}{x-b}\right)^{\frac{\alpha}{k}},$$
$$\lambda_4^* = \frac{k}{\alpha+k}\left(\frac{b-x}{x-a}\right)^{\frac{\alpha}{k}+1} - \left(\frac{b-x}{x-a}\right)^{\frac{\alpha}{k}} - \frac{k}{\alpha+k}\left(\frac{b-a}{x-a}\right)^{\frac{\alpha}{k}+1}.$$

Theorem 3.3. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I°, with $a<b$. If $|f'|^q$, $q>1$, is convex on $[a,b]$, the following inequality holds for the generalized fractional integrals:

$$|\Lambda_1(0)f(a)+\Lambda_2(0)f(b)+\Lambda_3(0)f(b)+\Lambda_4(0)f(a) - [{}_{a+}I_\varphi f(b) + {}_{b-}I_\varphi f(a)]|$$
$$\leq (x-a)\left[\lambda_1^{\frac{1}{p}}+\lambda_4^{\frac{1}{p}}\right]\left[\frac{|f'(x)|^q+|f'(a)|^q}{2}\right]^{\frac{1}{q}} + (b-x)\left[\lambda_2^{\frac{1}{p}}+\lambda_3^{\frac{1}{p}}\right]\left[\frac{|f'(x)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}, \tag{22}$$

where λ_i's are defined in Theorem 3.2 and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 3.1 and properties of modulus, we obtain

$$|\Lambda_1(0)f(a)+\Lambda_2(0)f(b)+\Lambda_3(0)f(b)+\Lambda_4(0)f(a) - [{}_{a+}I_\varphi f(b) + {}_{b-}I_\varphi f(a)]|$$
$$\leq |\mathfrak{I}_1|+|\mathfrak{I}_2|+|\mathfrak{I}_3|+|\mathfrak{I}_4|. \tag{23}$$

Now, applying well-known Hölder inequality and convexity of $|f'|^q$, we obtain

$$|\mathfrak{I}_1| = \left|(x-a)\int_0^1 \Lambda_1(t)f'(tx+(1-t)a)dt\right|$$
$$\leq (x-a)\left(\int_0^1 \Lambda_1(t)dt\right)^{\frac{1}{p}} \times \left(\int_0^1 |f'(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}}$$
$$\leq (x-a)\left(\int_0^1 \Lambda_1(t)dt\right)^{\frac{1}{p}} \times \left(\int_0^1 \left[t|f'(x)|^q+(1-t)|f'(a)|^q\right]dt\right)^{\frac{1}{q}}$$
$$= (x-a)\left[\lambda_1^{\frac{1}{p}}\left(\frac{|f'(x)|^q+|f'(a)|^q}{2}\right)^{\frac{1}{q}}\right]. \tag{24}$$

Similarly, we have

$$|\mathfrak{I}_2| \leq (b-x)\left[\lambda_2^{\frac{1}{p}}\left(\frac{|f'(x)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right], \tag{25}$$
$$|\mathfrak{I}_3| \leq (b-x)\left[\lambda_3^{\frac{1}{p}}\left(\frac{|f'(x)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right], \tag{26}$$
$$|\mathfrak{I}_4| \leq (x-a)\left[\lambda_4^{\frac{1}{p}}\left(\frac{|f'(x)|^q+|f'(a)|^q}{2}\right)^{\frac{1}{q}}\right]. \tag{27}$$

Using inequalities (24)-(27) in inequality (23), we obtain inequality (22).

Remark 3.9. Under the assumptions of Theorem 3.3, if we take $\varphi(t)=t$, we have following inequality

$$\left|\left(\frac{x-a}{b-a}\right)f(a)+\left(\frac{b-x}{b-a}\right)f(b)-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq (x-a)\left[2\left(\frac{x-a}{2}\right)^{\frac{1}{p}}\right]\left[\frac{|f'(x)|^q+|f'(a)|^q}{2}\right]^{\frac{1}{q}} + (b-x)\left[2\left(\frac{b-x}{2}\right)^{\frac{1}{p}}\right]\left[\frac{|f'(x)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}.$$

Remark 3.10. Under the Assumptions of Theorem 3.3, if we use $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$, we have the following new inequality

$$\left|\left(\frac{(b-a)^\alpha-(b-x)^\alpha}{(b-a)^\alpha}+\frac{(x-a)^\alpha}{(b-a)^\alpha}\right)\frac{f(a)}{2} + \left(\frac{(b-a)^\alpha-(x-a)^\alpha}{(b-a)^\alpha}+\frac{(b-x)^\alpha}{(b-a)^\alpha}\right)\frac{f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|$$
$$\leq (x-a)\left[\gamma_1^{\frac{1}{p}}+\gamma_4^{\frac{1}{p}}\right]\left[\frac{|f'(x)|^q+|f'(a)|^q}{2}\right]^{\frac{1}{q}} + (b-x)\left[\gamma_2^{\frac{1}{p}}+\gamma_3^{\frac{1}{p}}\right]\left[\frac{|f'(x)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}.$$

Corollary 3.5. Under the assumptions of Theorem 3.3, if we set $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we have the following new inequality

$$\left|\left(\frac{(b-a)^{\frac{\alpha}{k}}-(b-x)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}+\frac{(x-a)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}\right)\frac{f(a)}{2}\right.$$
$$+\left(\frac{(b-a)^{\frac{\alpha}{k}}-(x-a)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}+\frac{(b-x)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}\right)\frac{f(b)}{2}$$
$$\left.-\frac{\Gamma_k(\alpha+1)}{2(b-a)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{a+,k}f(b)+J^{\alpha}_{b-,k}f(a)\right]\right|$$
$$\leq (x-a)\left[(\lambda_1^*)^{\frac{1}{p}}+(\lambda_4^*)^{\frac{1}{p}}\right]\left[\frac{|f'(x)|^q+|f'(a)|^q}{2}\right]^{\frac{1}{q}}$$
$$+(b-x)\left[(\lambda_2^*)^{\frac{1}{p}}+(\lambda_3^*)^{\frac{1}{p}}\right]\left[\frac{|f'(x)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}.$$

Theorem 3.4. Let $f: I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be differentiable mapping on I°, with $a<b$. If $|f'|^q$, $q>1$, is concave on $[a,b]$, the following inequality holds for the generalized fractional integrals:

$$|\Lambda_1(0)f(a)+\Lambda_2(0)f(b)+\Lambda_3(0)f(b)+\Lambda_4(0)f(a)$$
$$-[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)]|$$
$$\leq (x-a)\left[\lambda_1\left|f'\left(\frac{A_1x+A_2a}{\lambda_1}\right)\right|+\lambda_4\left|f'\left(\frac{A_7x+A_8a}{\lambda_4}\right)\right|\right]$$
$$+(b-x)\left[\lambda_2\left|f'\left(\frac{A_3x+A_4b}{\lambda_2}\right)\right|+\lambda_3\left|f'\left(\frac{A_5x+A_6b}{\lambda_3}\right)\right|\right],$$

where λ_i's are defined in Theorem 3.2 and A_i's are defined in Theorem 3.1.

Proof. First, we note that

$$\left|f'(tx+(1-t)y)\right|^q \geq t\left|f'(x)\right|^q+(1-t)\left|f'(y)\right|^q$$
$$\geq \left(t\left|f'(x)\right|+(1-t)\left|f'(y)\right|\right)^q.$$

Hence,

$$\left|f'(tx+(1-t)y)\right|\geq t\left|f'(x)\right|+(1-t)\left|f'(y)\right|,$$

which shows that $|f'|$ is also concave. Now, using well-known Jensen's integral inequality, we have

$$|\mathfrak{I}_1|\leq (x-a)\left(\int_0^1|\Lambda_1(t)dt|\right)$$
$$\times\left|f'\left(\frac{\int_0^1|\Lambda_1(t)|(tx+(1-t)a)dt}{\int_0^1|\Lambda_1(t)|dt}\right)\right|$$
$$=(x-a)\lambda_1\left|f'\left(\frac{A_1x+A_2a}{\lambda_1}\right)\right|. \quad (28)$$

Similarly, we have

$$|\mathfrak{I}_2|\leq (b-x)\lambda_2\left|f'\left(\frac{A_3x+A_4b}{\lambda_2}\right)\right|, \quad (29)$$

$$|\mathfrak{I}_3|\leq (b-x)\lambda_3\left|f'\left(\frac{A_5x+A_6b}{\lambda_3}\right)\right|, \quad (30)$$

$$|\mathfrak{I}_4|\leq (x-a)\lambda_4\left|f'\left(\frac{A_7x+A_8a}{\lambda_4}\right)\right|. \quad (31)$$

The proof of theorem is completed.

Remark 3.11. Under the assumptions of Theorem 3.4, if we take $\varphi(t)=t$, we have the following inequality

$$\left|\left(\frac{x-a}{b-a}\right)f(a)+\left(\frac{b-x}{b-a}\right)f(b)-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq (x-a)^2\left|f'\left(\frac{x+2a}{3}\right)\right|+(b-x)^2\left|f'\left(\frac{x+2b}{3}\right)\right|.$$

Remark 3.12. If we set $x=\frac{a+b}{2}$ in Remark 3.11, Remark 3.11 reduces to the [23, Corollary 1].

Remark 3.13. Under the assumptions of Theorem 3.4, if we use $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, Theorem 3.4 reduces to ([23, Theorem 6]).

Corollary 3.6. Under the assumption of Theorem 3.4, we take $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have the following new inequality for the $k-$fractional integrals:

$$\left|\left(\frac{(b-a)^{\frac{\alpha}{k}}-(b-x)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}+\frac{(x-a)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}\right)\frac{f(a)}{2}\right.$$
$$+\left(\frac{(b-a)^{\frac{\alpha}{k}}-(x-a)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}+\frac{(b-x)^{\frac{\alpha}{k}}}{(b-a)^{\frac{\alpha}{k}}}\right)\frac{f(b)}{2}$$
$$\left.-\frac{\Gamma_k(\alpha+1)}{2(b-a)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{a+,k}f(b)+J^{\alpha}_{b-,k}f(a)\right]\right|$$
$$\leq \lambda_1^*\left|f'\left((\alpha+k)\left[\frac{A^*x+B^*a}{\alpha}\right]\right)\right|$$
$$+\lambda_2^*\left|f'\left((\alpha+k)\left[\frac{A^*x+B^*b}{\alpha}\right]\right)\right|$$
$$+\lambda_3^*\left|f'\left((\alpha+k)\left[\frac{C^*x+D^*b}{\alpha}\right]\right)\right|$$
$$+\lambda_4^*\left|f'\left((\alpha+k)\left[\frac{E^*x+F^*a}{\alpha}\right]\right)\right|.$$

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article

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