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Numerical Studies of the Fractional Optimal Control Problem of Awareness and Trial Advertising Model

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Abstract: In this article, a new nonstandard finite difference (NSFD) scheme was formulated to find an approximation numerical solution of the optimal control of the system of a generalized fractional model of the advertising problem. This system describes the flow of the customers from the unaware set to the aware or bought group. Properties of the studied system were introduced numerically and analytically. The utilized NSFD scheme keeps the attributes of the analytic solutions of the proposed system as positivity and boundedness. To test the behavior of the proposed method, numerical examples are run out with some comprising with standard finite difference method and forward-backward sweep method. The results show the simplicity, accuracy, and applicability of this approximation method.

Keywords: Financial models for awareness and trial, fractional optimal control problem, nonstandard finite difference method, Caputo fractional derivative.

1 Introduction

The simple definition of the optimal control problem is the procedure of determining both the state variables and the control trajectories for the studied dynamical model during interval of time to minimize or maximize an attitude index. The state variable (or function) is the set of variables (functions) that utilized for describing the mathematical status of the model. Historically, optimal control is an amplification of the calculus of variations. To solve the basic optimal control problems, some set of the necessary conditions should be satisfied. In the 1950's, Pontryagin, a Russian mathematician and his co-authors introduced these conditions. They proposed the adjoint function to place the studied differential equation to a function which called an objective functional. As the Lagrange multipliers in multivariable calculus, these functions do an identical target.

It is widely known that the goal of the advertisements is to persued the people to buy specific goods, which consists on the focusing on the necessity of the goods in generic and by appearing the difference between a particular kind over the other goods for motivating the audiences to purchases the product. There are many strategies to modify the thought of audience on goods or services. The advertising messages are one of these methods. These missions may be through magazines, newspapers, television and radio, which known as the body media. The missions, also, maybe through soft media like letters, text websites, and speech [1].

It is very important to study the strategies of advertisement to raise the rate of sales and to obtain superior earn for the companies.Therefore, building and studying suitable dynamic advertising models for describing the sales, which depends on the time and depends on the public population [2]. There are many proposed systems for describing the issue of the advertisement that study these issues from viewpoint of economic, operations management and marketing ([1] and [3]), such that analyzing of the advertisement policies are done over time using dynamical systems ([4] and [5]). Most of these dynamical systems are considered to be differential models, such that the market subpopulations, sales, share and the all significant status variables are supposed to be continuously changeable with respect to the time. The

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advertisement models will be constructed depending on the purposes of the advertising. Kinds of these models are aimed at the comparison between more than two products. Other may be aimed to inject a new product to the market.

Usually, the effect of the advertising is delayed in time, thus we need to incorporate the memory to systems of advertisement. So, to describe the strategies of the advertisement, the models in which the present status follows all of its prior statuses are more suitable than those in which the present status follows only its first previous one. It is widely known: the derivatives of fractional order are defined using integrals during the whole studied interval [6], and so on the models which described depending on derivative of fractional order are most proper for the advertising phenomena. In a lot of epidemic models, the systems which introduced such that the derivatives are of non-integer (fractional) order were seen to produce more superior real results than the systems which builder using on derivative of integer-order [7].

These days, calculus of fractional order has obtained major fame and gravity depending on its attractive applications as a new mathematical modelling which labor in engineering and scientific fields, like physics ([8], [9], [10]) thermoelasticity [11], viscoelasticity ([12], [13], [14]), hydrology [15], fractional dynamics ([8], [16]) and system control ([6], [17] and [18]). Differential equations which have fractional order is consider to be the best method to depict the fractional systems. Unfortunately, rarely and in some easy cases we can take out the proper solutions of fractional differential equations. Therefore, it is very important to use numerical techniques to find approximation solutions of these models, these methods require great effort.

In the recent years, fractional optimal control problems (FOCPs) refers to the maximization (minimization) of a specific objective functional under some constraints, most probably dynamic conditions, for the control and state independent variables. These conditions have some fractional order derivative in many sense. Many approximation techniques to approximate the solutions of different types of FOCPs were listed and used in ([19]-[21]) and the references cited therein.

On the other hand, Mickens ([22]-[24]) introduced the nonstandard finite difference methods (NSFDM) for getting better particular discretizations of derivative terms and some nonlocal terms of the studied differential equations. In this manners: depending on the particular discretization and using specific denominator functions, these methods can have larger stable region and will be more accurate than standard methods [25], being in mind these methods are easy to formulate. NSFM is useful to applied in fields of chemistry, physics, engineering ([26], [27], [28] and [29]). The most glamorous implementations of NSFDM are especially in mathematical ecology and biology ([30], [31]). Also, for solving fractional-order system the NSFDM give well performed for the dynamic preserving properties. Some fractional systems like, the fractional-order Rössler model [32], the fractional-order neuron model [33].

The main contribution of present work was: numerical study of the approximation solutions of the following fractional optimal control problem of advertising model:

$$J=\int_0^{t_{final}} e^{-rt}[cz(t)-\frac{B_1}{2}(1-u(t))^2-\frac{B_2}{2}(1-v(t))^2]dt, \tag{1}$$

subject to equations:

$$\begin{aligned} {}_0^cD_t^\alpha x(t) &= -u(t)x(t)-\frac{k^\alpha}{N(t)}x(t)(N(t)-x(t))+\mu_b^\alpha N(t)-\mu_d^\alpha x(t),\\ {}_0^cD_t^\alpha z(t) &= (a^\alpha+v(t))(N(t)-x(t)-z(t))-\delta^\alpha z(t)-\mu_d^\alpha z(t). \end{aligned} \tag{2}$$

The problem here is to pick out the control variables $u(t)$ and $v(t)$ such as to maximize the objective functional J. This study will depend on developing a straightforward and efficient method based on the NSFD technique. System (2) is generalization of the system which was proposed, in case $\alpha=1$, by Muller [4]. That system is abbreviation of the following system [4]

$$\begin{aligned} x'(t) &= -ux(t)-\frac{k}{N(t)}x(t)(N(t)-x(t)),\\ y'(t) &= ux(t)+\frac{k}{N(t)}x(t)(N(t)-x(t))-(a+v)y(t)+\delta z(t),\\ z'(t) &= (a+v)y(t)-\delta z(t). \end{aligned} \tag{3}$$

It describes how the consumers flows from group into different one. Where the whole number of persons $ux(t)$ flow to the potential set $y(t)$ depending on advertisement. Furthermore, $(N(t)-x(t))$ consumers whom know about the output, inform and contact the whole of $k(N(t)-x(t))$, out of them only the quotient $x(t)/N(t)$ are latterly acquainted. In our previous work [34] we find approximation solution of the fractional system of this phenomena without control. Also, Benito *et al.* [2] solved a fractional case of this model without control using a numerical technique.

Definitions of the parameters and variables in the above equations are given in table (1). It is necessary to mention here that all the parameters in the new system follow α, the fractional derivative order. We will omit the symbol α from the above of the parameters to make it easier, in the notation in the succession of the current article.

Table 1: Notations and parameters of the studied system (1)-(2) with their identification

Symbol	Definition
$N(t)$	$N(t) = x(t) + y(t) + z(t)$, summation of the unknowns, the whole number of the population.
${}_0^cD_t^\alpha$	Fractional derivative operator in Caputo sense.
α	The fractional derivative order.
t	$t \geq 0$, time.
u	Awareness, which switches the audience from $x(t)$, the set of who do not aware, into the prospective one $y(t)$ by letting them know the products.
v	Trial advertisement, which switches the people from $y(t)$, the prospective group, into the bought set $z(t)$ by encouraging them to purchased the products.
$x(t)$	The number of the group of people who did not realize anything about the goods.
$y(t)$	The number of the group of people who realize the products but they did not purchase it till now.
$z(t)$	The number of the set of people who purchased the products.
a	First purchase, (Trial rate).
k	Rate of contact.
r	Rate of Discount.
δ	Rate of switching.
c	$c = p(r + \delta + g)$.
g	Repeat purchase.
p	Net price.
μ_b	Birth rate.
μ_d	Death rate.

We concern in the case of derivative of fractional-order because the action of advertisements is not immediate, then combining the memory is so necessary to understand and explain the advertisement with both elements: trial and awareness advertisement. We refer to ([2], [4] and [5]), to understand deeply the elaboration of this model.

This article is structured as follows: In the following section, we introduce the preliminaries of NSFDM and recollect pertinent definitions on fractional calculus. in section 3 we discuss a few characteristics of the solutions of the proposed model. Necessary optimality conditions for the proposed model was introduced in Section 4. In Section 5, the nonstandard finite difference scheme (NSFDS) for system (2) was proposed then we demonstrated that this schema preserves the boundedness and the positivity of the actual solutions of the proposed model, we also, studied the stability of the proposed discretization scheme. In Section 6, we simulate the numerical solution of the introduced system and we stated the outcomes to show the efficiency and the applicability of NSFDM. Finally, in the last section, the conclusion of the paper is presented.

2 Preliminaries and notations

Here, we will mention a few necessary preparations for posterior debates. Firstly, we define the NSFDM. Secondly, we mention few helpful definitions and mathematical preliminaries of calculus of fractional order.

2.1 The nonstandard finite difference technique

Mickens ([23], [24]) was the first mathematician who modified the finite difference method to the NSFDM approach. The adaptive method depends on construction a modulated discretize numerical schema for ordinary differential equations (ODEs) or partial differential equations (PDEs). This technique can preserve the attributes of the exact solution of any differential model depending on some specific steps:

1. The terms which are not linear should be approximated using a nonlocal manner.
2. The denominator of the approximated derivatives should be a function of the step size.
3. The NSFD schema must not have a solution that does not consort to the solutions of the original differential equation.
4. Any specific properties that the solutions of the differential equations have should be also properties for the schema of the nonstandard finite difference method.
5. The actual derivatives which appear in the studied differential equations and the corresponding approximation derivatives should have the same order.

Briefly, using Euler's method to approximate $\frac{dy}{dt}$ instead of using $\frac{y(t+h)-y(t)}{h}$, we utilize $\frac{y(t+h)-y(t)}{\phi(h)}$, such that $\phi(h)$ is a function of the step size h which must be continuous function, and when $h \longrightarrow 0$ then

$$\phi(h) = h + O(h^2), \;\; 0 < \phi(h) < 1.$$

In addition to this replacement, the nonlinear terms in the differential equation, if it exists, we may change it to a linear term by approximating it as non-local discretization, as follows:

$$yx \longrightarrow \begin{cases} y_n x_{n+1}, \\ y_n x_{n-1}, \\ y_{n+1} x_n. \end{cases}$$

2.2 *Definitions of fractional calculus*

The definitions of the derivatives of fractional order were introduced in many literatures (see e.g., [6], [35], [36]). Usually, Riemann-Liouville definition, Grünwald-Letnikov definition, and Caputo definition are utilized to define the time derivatives of fractional order. Recently, the Caputo's operator of the derivative of the fractional-order is one of the most widespread fractional derivatives in many branches of applied science and engineering because this operator deals in a appropriate technique with differential equations associated by some specific initial conditions.

Definition 1.*The Caputo's definition of derivative of fractional order α, $\alpha \in \mathbb{R}^+$, is introduced by (Caputo, 1967)*

$$({}_0^cD_t^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(x)}{(t-x)^{1-n+\alpha}} dx, \quad t > 0, \tag{4}$$

such that $n = [\alpha] + 1$, $f(x) \in C^n[0,\infty[$.

Depending on the above operator definition anyone notices that the derivative of any constant depending on Caputo's definition will be zero, and

$${}_0^cD_t^\alpha t^\beta = \begin{cases} 0, & \text{for } \beta < \alpha, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, & \text{for } \beta \geq \alpha. \end{cases} \tag{5}$$

Furthermore, as to the standard derivative of integer-order, the Caputo's fractional operator satisfy the linearity property, i.e.

$${}_0^cD_t^\alpha (\lambda f(t) + \gamma g(t) = \lambda \, {}_0^cD_t^\alpha f(t) + \gamma \, {}_0^cD_t^\alpha g(t).$$

Notice, if $\alpha \in \mathbb{N}$ then Caputo's derivative will matches with the standard derivative of an integer order.

3 Properties of the solutions of the introduced system

3.1 *Positivity of the solutions*

It is well known that the populations must always be positive then we can show that the solutions of the introduced model are non-negative. For proving the theorem of the positivity we will use Lemma 1 in the following (which called generalized mean value theorem):

Lemma 1.*[37] If the function* $g(t) \in C[a,b]$ *and* ${}_a^cD_t^\alpha g(t) \in C]a,b]$, *when* $0 < \alpha \leq 1$. *Therefore:*

$$g(t) = g(a) + {}_0^cD_t^\alpha g(\xi) \frac{(t-a)^\alpha}{\Gamma(\alpha)},$$

with $0 \leq \xi \leq t$.

So, If $g(t) \in C[0,b]$ and ${}_0^cD_t^\alpha g(t) \in C]0,b]$ and if ${}_0^cD_t^\alpha g(t) \geq 0$ then g is nondecreasing function.

Theorem 1.*System (2) has unique solution and this solution is positive.*

*Proof.*Depending on the results which were introduced in [38], there is a unique solution of (2). Using the above lemma we can write the following:

$$
\begin{aligned}
&{}_0^c D_t^{\alpha} x(t)|_{x=0} \geq 0, \\
&{}_0^c D_t^{\alpha} z(t)|_{z=0} \geq 0.
\end{aligned} \tag{6}
$$

So, both of $x(t)$ and $z(t)$, for any t, are greater than zero.

3.2 Stability analysis of model

Theorem 2.*[39] The equilibrium points of system (2), are carried out by finding t which satisfy: $g(t) = 0$, such that $g(t)$ is the function on the right-hand side of the model (2). The obtained equilibrium points will be asymptotically local stable when λ_i the eigenvalues of the matrix $J = \frac{\partial g}{\partial t}$ that calculated at these points are satisfy the condition: $|arg(\lambda_i)| > \frac{\alpha\pi}{2}$. The associated matrix $J = \frac{\partial g}{\partial t}$ called Jacobian matrix.*

The equilibrium points of system (2) are $x = 0,\ z = \frac{N(a+v)}{a+v+\delta}$.

The Jacobian matrix (J) for the proposed system that calculated at the obtained equilibrium point can be as the following:

$$
\boldsymbol{J} = \begin{pmatrix} -u-k & 0 \\ -a-v & -a-v-\delta \end{pmatrix},
$$

where the eigenvalues are found to be as the following

$$
\sigma_1 = -k-u, \quad \sigma_2 = -a-\delta-v.
$$

So, all equilibrium points for the model are locally asymptotically stable, for all t.

4 Necessary optimality conditions for the proposed model

Depending on state system (2) and the objective function (1), the Hamiltonian function of the studied fractional optimal control problem is introduced, using a Lagrange multiplier technique, in the following form:

$$
\begin{aligned}
H &= e^{-rt}[cz(t) - \frac{B_1}{2}(1-u(t))^2 - \frac{B_2}{2}(1-v(t))^2] \\
&+ \lambda_1[-u(t)x(t) - \frac{k^{\alpha}}{N(t)}x(t)(N(t)-x(t)) + \mu_b^{\alpha}N(t) - \mu_d^{\alpha}x(t)] \\
&+ \lambda_2[(a^{\alpha}+v(t))(N(t)-x(t)-z(t)) - \delta^{\alpha}z(t) - \mu_d^{\alpha}z(t)].
\end{aligned} \tag{7}
$$

Where λ_1 and λ_2 are the co-state variables (Lagrange multipliers). Agrawal [40] derived the necessary optimality conditions for the fraction optimal control problem when the derivatives are defined using Caputo fractional operator. therefore, the necessary optimality conditions of the proposed model are can be derived in the following form:

Let the optimal control variables u^*, v^* and the solutions of the associated state system (2) x and z then there are co-state variables λ_1 and λ_2 such that:

– State equations

$$
\begin{aligned}
{}_0^c D_t^{\alpha} x(t) &= -u(t)x(t) - \frac{k}{N(t)}x(t)(N(t)-x(t)) + \mu_b N(t) - \mu_d x(t), \quad x(0) = N, \\
{}_0^c D_t^{\alpha} z(t) &= (a+v(t))(N(t)-x(t)-z(t)) - \delta z(t) - \mu_d z(t), \quad z(0) = 0.
\end{aligned} \tag{8}
$$

– Co-state equations

$$
\begin{aligned}
{}_t^c D_{t_{final}}^{\alpha} \lambda_1 &= \frac{\partial H}{\partial x} = -u(t)\lambda_1(t) - \frac{k}{N(t)}\lambda_1(N(t)-x(t)) + \frac{k}{N(t)}\lambda_1 x(t) - \lambda_2(a+v(t)) - \mu_d\lambda_1, \\
{}_t^c D_{t_{final}}^{\alpha} \lambda_2 &= \frac{\partial H}{\partial x} = ce^{-rt} - \lambda_2(a+v(t)) - \delta\lambda_2 - \mu_d\lambda_2.
\end{aligned} \tag{9}
$$

– Stationarity conditions

$$\frac{\partial H}{\partial u} = 0, \quad \frac{\partial H}{\partial v} = 0,$$

i.e.,

$$u^* = min\left\{max\left\{1 - \frac{\lambda_1 x}{B_1 e^{-rt}}, 0\right\}, 1\right\}, \quad v^* = min\left\{max\left\{1 + \frac{\lambda_2 (N - x - z)}{B_2 e^{-rt}}, 0\right\}, 1\right\}. \tag{10}$$

– Transversality conditions

$$\lambda_1(t_{final}) = 0, \quad \lambda_2(t_{final}) = 0. \tag{11}$$

The state system (8) coupled with the co-state equations (9) and the transversality conditions (11) together with the relationship (10) construct the optimality system.

5 Numerical algorithm

In this section, explicit discretization schema of the optimality system was constructed using NSFDM. To build the novel schema, an approximations of the derivatives is used with function in the denominator and an appropriate nonlocal approximation is applied to discretize the nonlinear terms.

Let the mesh points have the following coordinate:

$$t_n = n\triangle t, \quad n = 0,\ 1,\ 2,\ ...,\ N_n,$$

where N_n is a specific positive integer number and

$$h := \triangle t = \frac{t_{final}}{N_n}.$$

The approximation values of $x,\ z,\ u,$ and v at the mesh points (t_n) are signified by $x_n,\ z_n,\ u_n$ and v_n respectively. The approximation of Caputo derivative operator using nonstandard technique is built depending on the Grünwald-Letnikov method:

$${}_0^cD_t^\alpha x(t)\Big|_{t=t_n} = \frac{1}{(\phi(\triangle t))^\alpha}(x_{n+1} - \sum_{i=1}^{n+1} w_i x_{n+1-i} - q_{n+1} x_0), \tag{12}$$

where

$$w_i = (-1)^{i-1}\binom{\alpha}{i}, \quad w_1 = \alpha,$$

$$q_i = \frac{i^{-\alpha}}{\Gamma(1-\alpha)}, \quad i = 1,\ 2,\ \ldots,\ n+1.$$

Theorem 3.*[41] Let $\alpha \in (0,1)$, then the following two relations:*

$$0 < q_{i+1} < q_i < ... < q_1 = \frac{1}{\Gamma(1-\alpha)}, \tag{13}$$

$$0 < w_{i+1} < w_i < ... < w_1 = \alpha < 1. \tag{14}$$

are satisfied for the coefficients q_i and w_i where $i \geq 1$.

*Proof.*see [41].

Depending on nonstandard method and relation (12) we have, for system (8), the following nonstandard schema, which is built to be explicit schema:

$$x_{n+1} - \sum_{i=1}^{n+1} w_i x_{n+1-i} - q_{n+1} x_0 = (\phi(h))^\alpha(-u_n x_{n+1} - \frac{k}{N} x_{n+1}(N - x_n) + \mu_b N - \mu_d x_{n+1}),$$

$$z_{n+1} - \sum_{i=1}^{n+1} w_i z_{n+1-i} - q_{n+1} z_0 = (\phi(h))^\alpha((a + v_n)(N - x_n - z_{n+1}) - \delta^\alpha z_{n+1} - \mu_d z_{n+1}).$$

Since each of these equations is linear in x_{n+1} and z_{n+1} so, after some calculations we get the following discrtize equations, which are explicit:

$$x_{n+1} = \frac{1}{1+(\phi(h))^{\alpha}(u_n+\frac{k}{N}(N-x_n)+\mu_d)}[\sum_{i=1}^{n+1} w_i x_{n+1-i} + q_{n+1}x_0 + (\phi(h))^{\alpha}\mu_b N],$$
$$z_{n+1} = \frac{1}{1+(\phi(h))^{\alpha}(a+v_n+\delta+\mu_d)}[\sum_{i=1}^{n+1} w_i z_{n+1-i} + q_{n+1}z_0 + (\phi(h))^{\alpha}(a+v_n)(N-x_n)]. \tag{15}$$

5.1 Boundedness and positivity of the NSFD solutions

In the current subsection, we study some properties of the introduced scheme (15).

Theorem 4.*(Boundedness). Let $x_0 = N,\ z_0 = 0$ are the initial conditions such that $x_0 + z_0 = N$, then for all $n = 1,\ 2,\ \ldots$ we have x_n and z_n to be bounded*

*Proof.*By multiplying each single equation of the system (15) by its denominator we get:

$$\begin{aligned} x_{n+1}(1 + (\phi(h))^{\alpha}(u_n+\frac{k}{N}(N-x_n)+\mu_d)) + z_{n+1}(1+(\phi(h))^{\alpha}(a+v_n+\delta+\mu_d)) \\ = \sum_{i=1}^{n+1} w_i(x_{n+1-i}+z_{n+1-i}) + q_{n+1}N + (\phi(h))^{\alpha}(\mu_b N + (a+v_n)(N-x_n)), \\ \leq \sum_{i=1}^{n+1} w_i(x_{n+1-i}+z_{n+1-i}) + q_{n+1}N + N(\phi(h))^{\alpha}(\mu_b + a + v_n), \end{aligned} \tag{16}$$

using induction, let $n = 0$, then:

$$\begin{aligned} x_1(1 + (\phi(h))^{\alpha}(u_0+\frac{k}{N}(N-x_0)+\mu_d)) + z_1(1+(\phi(h))^{\alpha}(a+v_0+\delta+\mu_d)) \\ = x_1(1+(\phi(h))^{\alpha}(u_0+\mu_d)) + z_1(1+(\phi(h))^{\alpha}(a+v_0+\delta+\mu_d)), \\ \leq w_1(x_0+z_0) + q_1 N + N(\phi(h))^{\alpha}(\mu_b+a+v_0), \\ = N(\alpha + \frac{1}{\Gamma(1-\alpha)} + (\phi(h))^{\alpha}(\mu_b+a+v_0)), \\ = N(\alpha + \frac{1}{\Gamma(1-\alpha)} + (\phi(h))^{\alpha}m_0), \\ = NM_0, \end{aligned} \tag{17}$$

such that $m_0 = \mu_b + a + v_0$ and $M_0 = \alpha + \frac{1}{\Gamma(1-\alpha)} + (\phi(h))^{\alpha}m_0$.

So, we have

$$x_1 \leq \frac{NM_0}{(1+(\phi(h))^{\alpha}(u_0+\mu_d))}, \quad z_1 \leq \frac{NM_0}{(1+(\phi(h))^{\alpha}(a+v_0+\delta+\mu_d))},$$

i.e.,

$$x_1 \leq NM_0, \quad z_1 \leq NM_0.$$

Let $n = 1$, we have:

$$\begin{aligned} x_2(1 + (\phi(h))^{\alpha}(u_1+\frac{k}{N}(N-x_1)+\mu_d)) + z_2(1+(\phi(h))^{\alpha}(a+v_1+\delta+\mu_d)) \\ \leq w_1(x_1+z_1) + w_2(x_0+z_0) + q_2 N + N(\phi(h))^{\alpha}(\mu_b+a+v_1), \\ \leq w_1 2NM_0 + w_1 N + q_1 N + N(\phi(h))^{\alpha}(\mu_b+a+v_1), \\ = N(2\alpha M_0 + \alpha + \frac{1}{\Gamma(1-\alpha)} + (\phi(h))^{\alpha}m_1), \\ = NM_1, \end{aligned} \tag{18}$$

such that $m_1 = \mu_b + a + v_1$ and $M_1 = 2\alpha M_0 + \alpha + \frac{1}{\Gamma(1-\alpha)} + (\phi(h))^{\alpha} m_1$.

So,

$$x_2 \le NM_1, \qquad z_2 \le NM_1.$$

Let $n = 2$, we have:

$$\begin{aligned}
x_3(1 + (\phi(h))^{\alpha}(u_2 + \frac{k}{N}(N - x_2) + \mu_d)) + z_3(1 + (\phi(h))^{\alpha}(a + v_2 + \delta + \mu_d))\\
&\le w_1(x_2 + z_2) + w_2(x_1 + z_1) + w_3(x_0 + z_0) + q_3 N + N(\phi(h))^{\alpha}(\mu_b + a + v_2),\\
&\le w_1 2NM_1 + w_1 2NM_0 + w_1 N + q_1 N + N(\phi(h))^{\alpha}(\mu_b + a + v_1),\\
&= N(2\alpha M_1 + 2\alpha M_0 + \alpha + \frac{1}{\Gamma(1-\alpha)} + (\phi(h))^{\alpha} m_2),\\
&= NM_2,
\end{aligned} \tag{19}$$

such that $m_2 = \mu_b + a + v_2$ and $M_2 = 2\alpha M_0 + 2\alpha M_1 + \alpha + \frac{1}{\Gamma(1-\alpha)} + (\phi(h))^{\alpha} m_2$.

So,

$$x_3 \le NM_2, \qquad z_3 \le NM_2.$$

Let $n = 3$, we have:

$$\begin{aligned}
x_4(1 + (\phi(h))^{\alpha}(u_3 + \frac{k}{N}(N - x_3) + \mu_d)) + z_4(1 + (\phi(h))^{\alpha}(a + v_3 + \delta + \mu_d))\\
&\le w_1(x_3 + z_3) + w_2(x_2 + z_2) + w_3(x_1 + z_1) + w_4(x_0 + z_0) + q_4 N + N(\phi(h))^{\alpha}(\mu_b + a + v_3),\\
&\le w_1 2NM_2 + w_1 2NM_1 + w_1 2NM_0 + w_1 N + q_1 N + N(\phi(h))^{\alpha}(\mu_b + a + v_3),\\
&= N(2\alpha M_2 + 2\alpha M_1 + 2\alpha M_0 + \alpha + \frac{1}{\Gamma(1-\alpha)} + (\phi(h))^{\alpha} m_3),\\
&= NM_3,
\end{aligned} \tag{20}$$

such that $m_3 = \mu_b + a + v_3$ and $M_3 = 2\alpha M_0 + 2\alpha M_1 + 2\alpha M_2 + \alpha + \frac{1}{\Gamma(1-\alpha)} + (\phi(h))^{\alpha} m_3$.

So,

$$x_4 \le NM_3, \qquad z_4 \le NM_3.$$

Let $n = 4$, we have:

$$\begin{aligned}
x_5(1 + (\phi(h))^{\alpha}(u_4 + \frac{k}{N}(N - x_4) + \mu_d)) + z_5(1 + (\phi(h))^{\alpha}(a + v_4 + \delta + \mu_d))\\
&\le w_1(x_4 + z_4) + w_2(x_3 + z_3) + w_3(x_2 + z_2) + w_4(x_1 + z_1) + w_5(x_0 + z_0) + q_5 N\\
&\qquad + q_5 N + N(\phi(h))^{\alpha}(\mu_b + a + v_4),\\
&\le w_1 2NM_3 + w_1 2NM_2 + w_1 2NM_1 + w_1 2NM_0 + w_1 N + q_1 N + N(\phi(h))^{\alpha}(\mu_b + a + v_4),\\
&= N(2\alpha M_3 + 2\alpha M_2 + 2\alpha M_1 + 2\alpha M_0 + \alpha + \frac{1}{\Gamma(1-\alpha)} + (\phi(h))^{\alpha} m_4),\\
&= NM_4,
\end{aligned} \tag{21}$$

such that $m_4 = \mu_b + a + v_4$ and $M_4 = 2\alpha M_0 + 2\alpha M_1 + 2\alpha M_2 + 2\alpha M_3 + \alpha + \frac{1}{\Gamma(1-\alpha)} + (\phi(h))^{\alpha} m_4$.

So,

$$x_5 \le NM_4, \qquad z_5 \le NM_4.$$

In this stage let us suppose the following

$$\begin{aligned}
x_n(1 + (\phi(h))^{\alpha}(u_{n-1} + \frac{k}{N}(N - x_{n-1}) + \mu_d)) + z_n(1 + (\phi(h))^{\alpha}(a + v_{n-1} + \delta + \mu_d))\\
&\le N(2\alpha M_{n-2} + \ldots + 2\alpha M_3 + 2\alpha M_2 + 2\alpha M_1 + 2\alpha M_0 + \alpha + \frac{1}{\Gamma(1-\alpha)} + (\phi(h))^{\alpha} m_{n-1}),\\
&= NM_{n-1}.
\end{aligned} \tag{22}$$

i.e.,

$$x_n \le NM_{n-1}, \qquad z_n \le NM_{n-1},$$

where,

$$m_{n-1} = \mu_b + a + v_{n-1},$$

and

$$M_{n-1} = 2\alpha M_0 + 2\alpha M_1 + 2\alpha M_2 + 2\alpha M_3 + ... + 2\alpha M_{n-2} + \alpha + \frac{1}{\Gamma(1-\alpha)} + (\phi(h))^{\alpha} m_{n-1}.$$

Now, we will proof

$$x_{n+1} \leq NM_n, \qquad z_{n+1} \leq NM_n,$$

where

$$M_n = (2\alpha M_{n-1} + ... + 2\alpha M_3 + 2\alpha M_2 + 2\alpha M_1 + 2\alpha M_0 + \alpha + \frac{1}{\Gamma(1-\alpha)} + (\phi(h))^{\alpha} m_n.$$

and

$$m_n = \mu_b + a + v_n.$$

Using Eq. (16) we get

$$\begin{aligned} x_{n+1}(1 + (\phi(h))^{\alpha}(u_n + \frac{k}{N}(N-x_n) + \mu_d)) + z_{n+1}(1 + (\phi(h))^{\alpha}(a + v_n + \delta + \mu_d)) \\ \leq w_1(x_n + z_n) + w_2(x_{n-1} + z_{n-1}) + ... + w_{n-2}(x_2 + z_2) + w_{n-1}(x_1 + z_1) + w_n(x_0 + z_0) \\ + q_n N + N(\phi(h))^{\alpha}(\mu_b + a + v_n), \\ \leq w_1 2NM_{n-1} + w_1 2NM_{n-2} + ... + w_1 2NM_2 + w_1 2NM_1 + w_1 2NM_0 + w_1 N, \\ + q_1 N + N(\phi(h))^{\alpha}(\mu_b + a + v_n), \\ = N(2\alpha M_{n-1} + ... + 2\alpha M_3 + 2\alpha M_2 + 2\alpha M_1 + 2\alpha M_0 + \alpha + \frac{1}{\Gamma(1-\alpha)} + (\phi(h))^{\alpha} m_n), \\ = NM_n, \end{aligned} \tag{23}$$

so,

$$x_{n+1} \leq NM_n, \qquad z_{n+1} \leq NM_n.$$

Theorem 5.*(Positivity). Let $x_0 \geq 0,\ z_0 \geq 0$, then $x_n > 0,\ z_n > 0$ for all $n = 1,\ 2,\ ...$ is satisfied and all the parameter of this system are positive.*

*Proof.*Using induction. Let $n = 0$, then from system (15) we have:

$$\begin{aligned} x_1 &= \frac{1}{1 + (\phi(h))^{\alpha}(u_0 + \frac{k}{N}(N-x_0) + \mu_d)}[w_1 x_0 + q_1 x_0 + (\phi(h))^{\alpha}\mu_b N] \geq 0, \\ z_1 &= \frac{1}{1 + (\phi(h))^{\alpha}(a + v_0 + \delta + \mu_d)}[w_1 z_0 + q_1 z_0 + (\phi(h))^{\alpha}(a + v_0)(N - x_0)] \geq 0. \end{aligned} \tag{24}$$

Supposing, for all $n < n+1$, that $x_n \geq 0,\ y_n \geq 0$ and $z_n \geq 0$. Then we have for the index $n+1$

$$\begin{aligned} x_{n+1} &= \frac{1}{1 + (\phi(h))^{\alpha}(u_n + \frac{k}{N}(N-x_n) + \mu_d)}[\sum_{i=1}^{n+1} w_i x_{n+1-i} + q_{n+1} x_0 + (\phi(h))^{\alpha}\mu_b N] \geq 0, \\ z_{n+1} &= \frac{1}{1 + (\phi(h))^{\alpha}(a + v_n + \delta + \mu_d)}[\sum_{i=1}^{n+1} w_i z_{n+1-i} + q_{n+1} z_0 + (\phi(h))^{\alpha}(a + v_n)(N - x_n)] \geq 0. \end{aligned} \tag{25}$$

5.2 Stability

Definition 2.*The proposed schema (15) is called asymptotically stable, if L_1 and L_2 are exist as constants when $\alpha \longrightarrow 1$, such that*

$$x_{n+1} \leq L_1 \text{and } z_{n+1} \leq L_2,$$

satisfy for $0 < x_0 + z_0 = N$, the arbitrary initial values.

Benefiting from the boundedness theorem we deduce that the introduced NSFDS (15) is asymptotically stable.

6 Numerical results

NSFDM can minimize computational's time because it can utilize larger discretize steps. In the current section, we use NSFDM to avert a long calculation's time. For the following numerical dealing we utilized $\phi(h) = 1 - e^{-h}$. In all computational run, we depend on a computer machine with intel (R) Core i3-3110M @ 2.40GHz and 4GB RAM to run our code which written using MATLAB 2015a .

Let the $N = 1000$ total population, switching rate $\delta = 0.2$, first purchase rate $a = 0.02$, advertising trial rate $r = 0.1$, $g = 0.1$, $p = 0.2$, and $k = 0.01$, where $\mu_b = \mu_d = 0$ and $B1 = B2 = 500$. The conditions at the beginning time are $x(0) = N$ and $z(0) = 0$.

Figure (1) compares between the solutions utilizing NSFDM and forward-backward sweep method (FBSM) when $\alpha = 1$ at different final time for x, z.

Figure (2) illustrates that the solutions (obtained using NSFDM) of control variables u, v change when $\alpha = 1$ at $t_{final} = 10$ and $t_{final} = 15$.

Figure (3) illustrates that the solutions (obtained using NSFDM) of system (2) (x, z, u, v) change such that α, the order of the fractional derivative, takes several values at $t_{final} = 10$.

Figure (4) illustrates that the solutions (obtained using NSFDM) of system (2) (x, z, u, v) change such that α, the order of the fractional derivative, takes several at $t_{final} = 15$.

Figure (5) illustrates that the solutions (obtained using NSFDM) of system (2) (x, z) change when we use control variable and without using control variable at $t_{final} = 10$ when α takes different values.

Figure (6) compares between the solutions obtained by NSFDM and standard finite difference method (SFDM) when $\alpha = 0.7$, $t_{final} = 20$. We can see from this figure that NSFDM still stable when the final time is big whereas SFDM will be unstable.

Table (2) shows the value of the objective functional and the final values of x and z for different α when final time is 10 which obtained using NSFDM and SFDM.

Comparing between changing of the populations in the advertisement model with derivative of a natural order and the inhabitants in the advertisement system with non integer order derivative we notice depending on the shown diagrams that the variations are less fast in the model with the fractional derivatives. That is consequent to the action of memory. Furthermore, from Fig.(5) we see that the number of consumers who are not informed about the existence of the product is decreased rapidly when the control variables are used and the number of individuals who have purchased the product is go up more faster when we use control variables than without using these control variables.

Table 2: Values of objective functional and the values of the states variables utilizing NSFDM and SFDM when $t_{final} = 10$ with different values of the differentiation order α.

α	NSFDM			SFDM		
	J	x	z	J	x	z
1	286.14	0	834	285.35	0	834
0.9	267.09	17	804	266.67	18	804
0.8	246.61	42	762	244.22	43	760
0.7	225.53	77	709	223.34	79	708

7 Conclusion

In this paper, a numerical method for approximating the solutions of a nonlinear fractional optimal control problem of trial advertising and awareness model is presented. The proposed fractional advertisement system, like the most models of fractional-order derivative, has produced solutions which appropriate actual informations which more convenient than the models with the derivatives of integer-order, that due to the advertising proceedings have a souvenir impact on the individuals. The used method to study numerically the optimality system of the proposed problem was NSFDM. The proposed method is preserve the properties of the analytic solutions like positivity and boundedness. Numerical consequences are given to demonstrate the applicability and validity of the introduced scheme. By utilizing the NSFD scheme, numerical instabilities and the false solutions can be taken away, and accurate numerical solutions are accomplished for each time-step size.

From these numerical outcomes, it is clear that the introduced scheme exhibits perfect efficiency and accuracy.

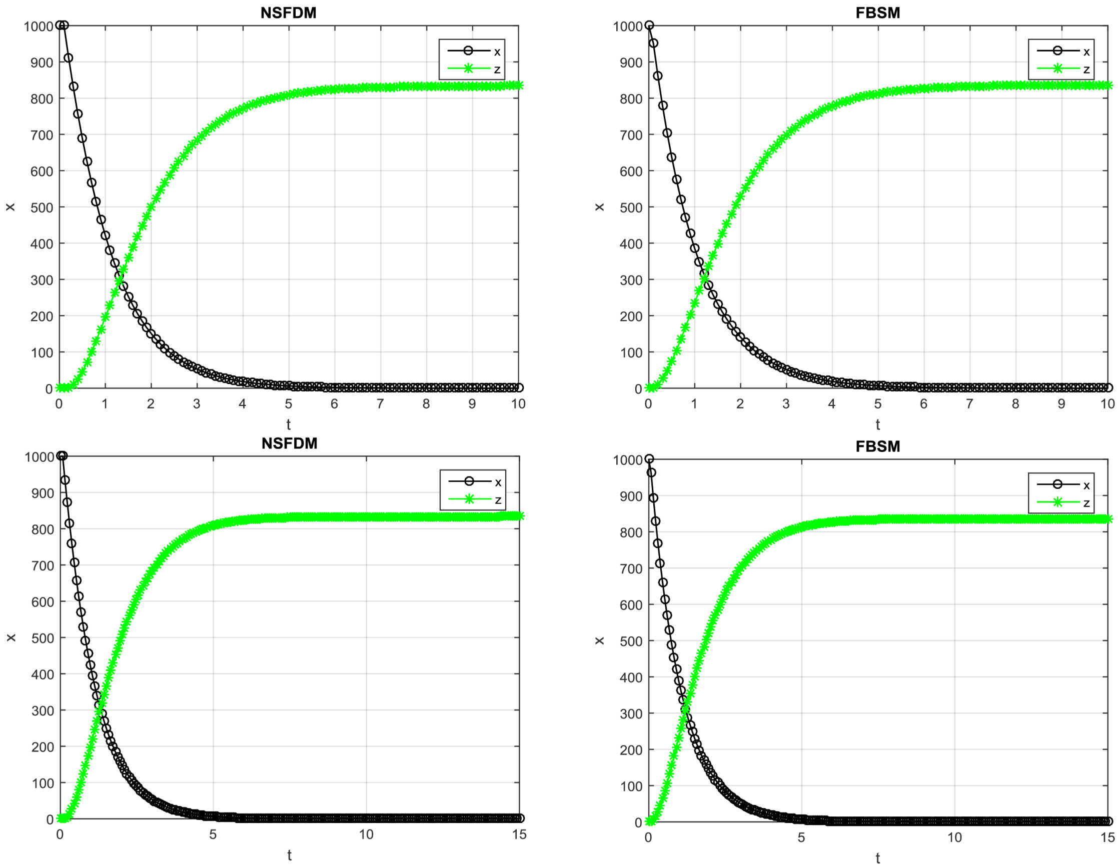

Fig. 1: Comparing between the results of $x,\ z$ using NSFDM and FBSM when $\alpha = 1$ at different final time.

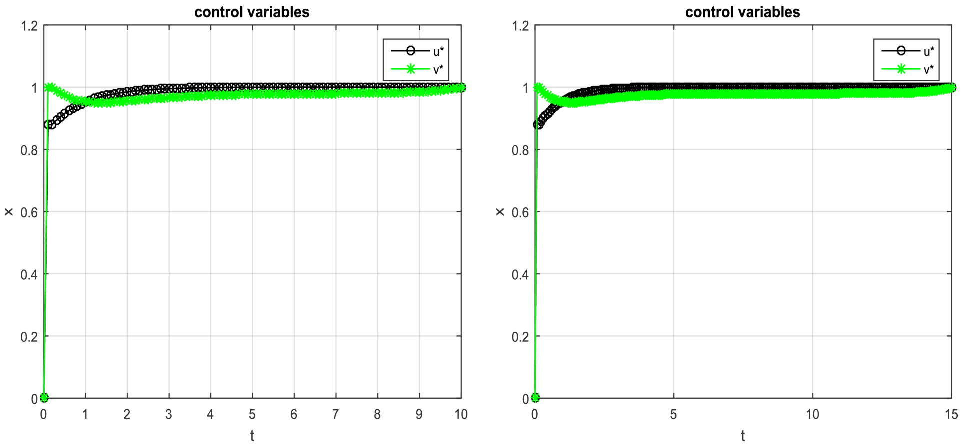

Fig. 2: Solutions of control variables in different final time.

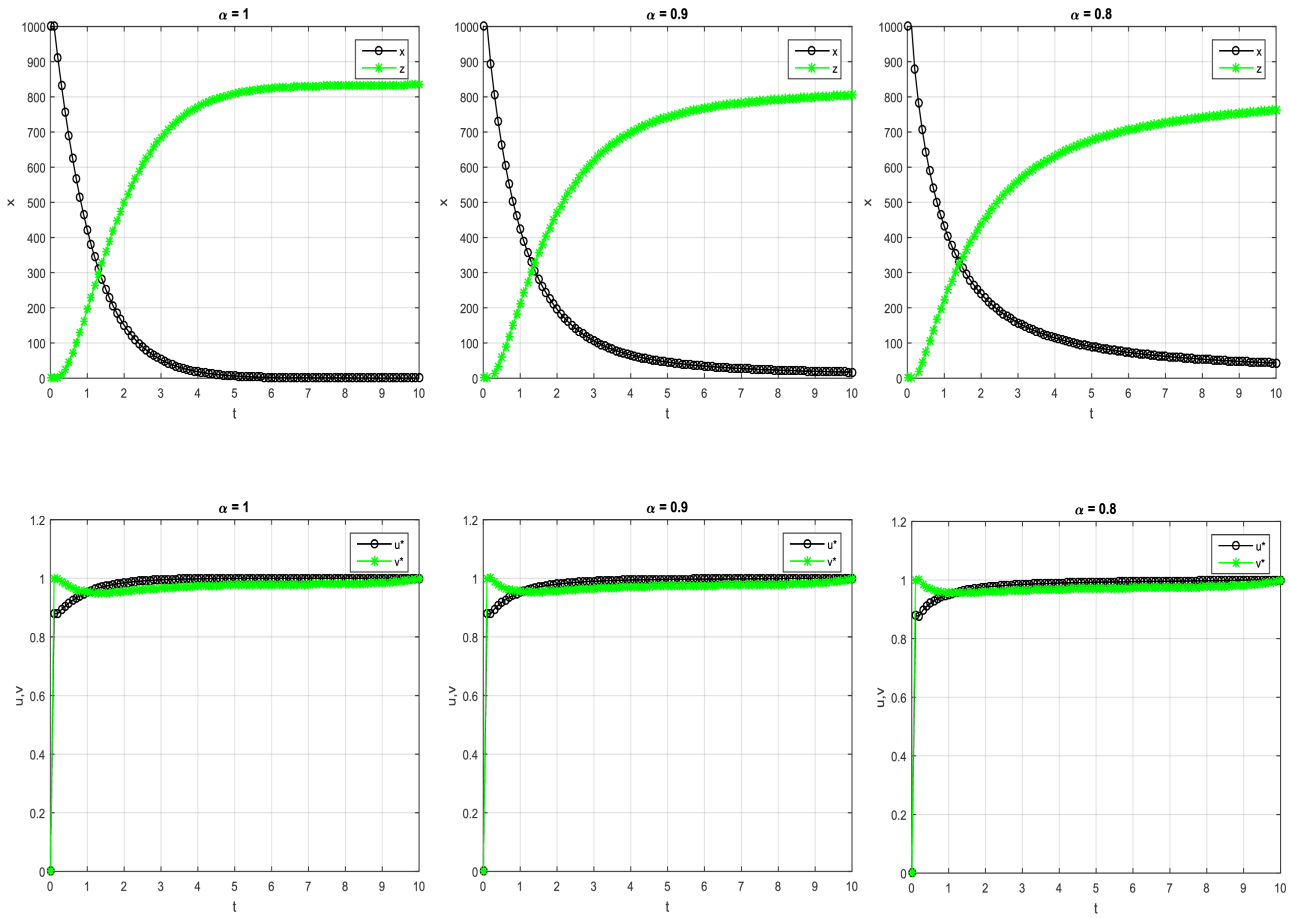

Fig. 3: Comparison between the solutions at different values of α, for x, z, u, v.

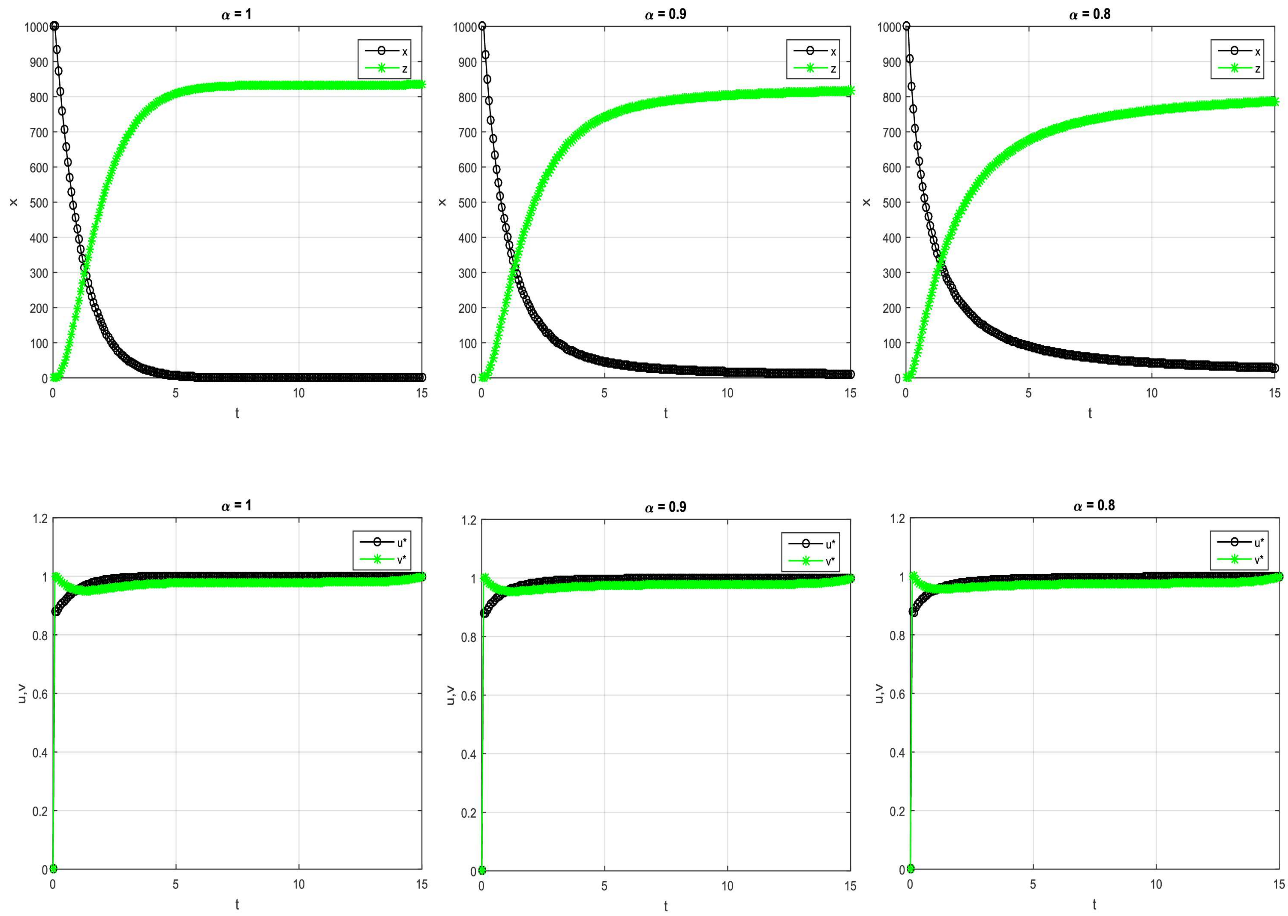

Fig. 4: Comparison between the solutions of $x,\ z,\ u,\ v$ at different values of α.

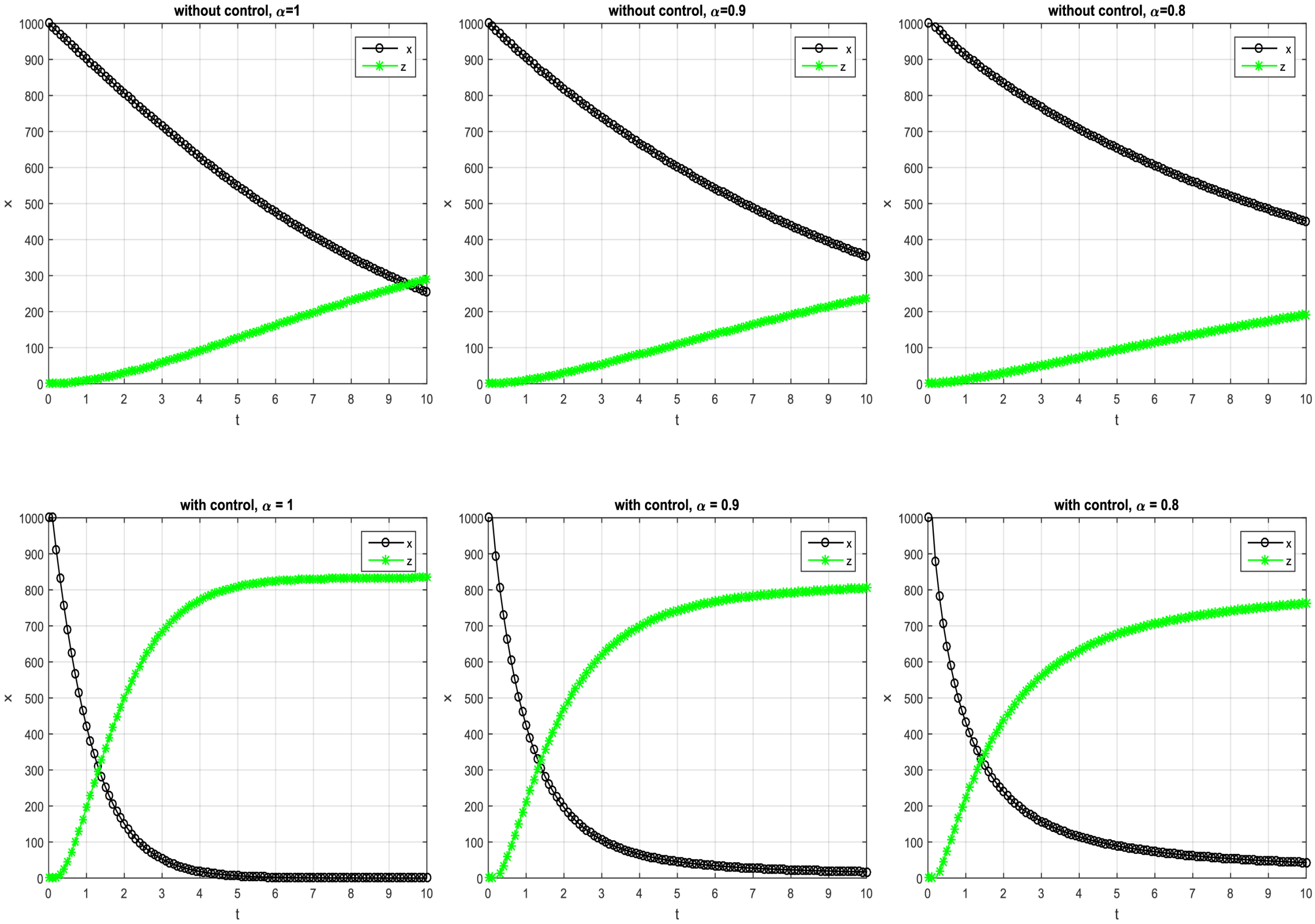

Fig. 5: Comparing between the results of $x,\ z$ without control and with control when α takes various values.

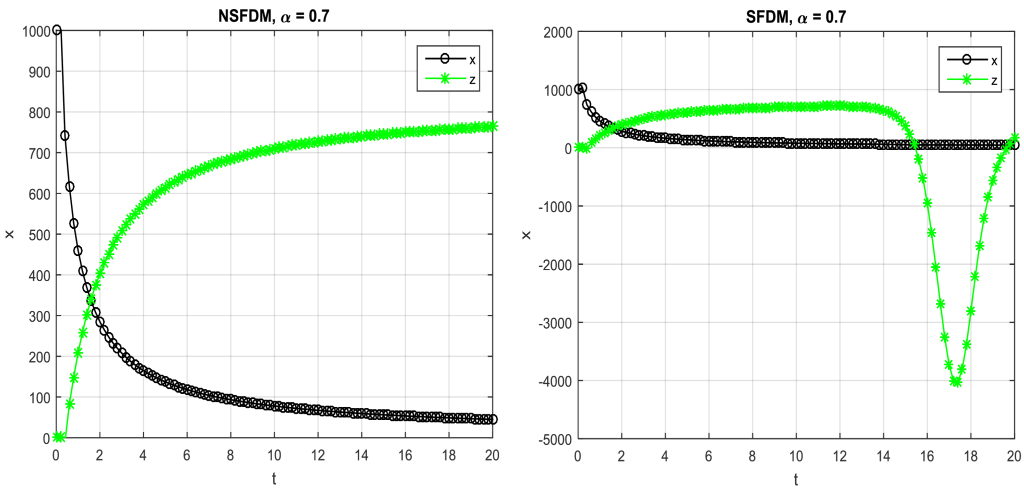

Fig. 6: Stable solutions using NSFDM and unstable solutions using SFDM.

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