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Multiplication Operators with Closed Range in Operator Algebras

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Abstract: Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators from a Hilbert space $\mathcal{H}$ into itself. Let $T \in \mathcal{B}(\mathcal{H})$. Define $L_T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by $L_T(S) = TS$ and define $R_T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by $R_T(S) = ST$. Consider the following three statements: (i) T has closed range in $\mathcal{H}$, (ii) L_T has closed range in $\mathcal{B}(\mathcal{H})$, (iii) R_T has closed range in $\mathcal{B}(\mathcal{H})$. It is proved that all these three statements are equivalent. Some possibilities of extending this result to Banach spaces have been discussed.

Keywords: Closed range operator ; Hahn-Banach extension property.

1. Introduction

Many of the concrete applications of mathematics in science and engineering, eventually result in a problem involving operator equations. This problem can be usually represented as an operator equation

$$Tx = y \tag{1}$$

where $T : X \to Y$ is a linear or nonlinear operator (between certain function spaces or Euclidean spaces) such as differential operator or integral operator or a matrix. The spaces X and Y are linear spaces endowed with certain norms on them. Solving linear equations with infinitely many variables is a problem of functional analysis, while solving equations with finitely many variables is one of the main themes of linear algebra.

The problem of solving the equation (1) is well-posed if it asserts existence and uniqueness of a solution of (1) and the continuous dependence of the solution on the data y. It is well-known that the problem of solving the operator equation (1) is essentially well-posed if the range of T is closed. The study of operators with closed range plays a vital role in perturbation theory.

The composition of closed range continuous linear operators between Fréchet spaces does not have closed range, in general. We have given necessary and sufficient conditions for the composition operator TS to have closed range in [3, 8]. Bouldin [1, 2] has given a geometric characterization in terms of the angle between two linear subspaces for operators on Hilbert spaces and for the operators on Banach spaces. The closedness of range of a linear operator T is helpful in establishing the Hyers-Ulam stability of T between Fréchet spaces [7]. The range and null spaces of a linear operator T are denoted by $R(T)$ and $N(T)$ respectively. In this paper, we first discuss the closedness of multiplication operators on Hilbert spaces.

Kulkarni and Nair [5] proved that for a bounded linear operator T from a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{K}$, T has closed range in $\mathcal{K}$ if and only if $\sigma(T^*T) \subseteq \{0\} \cup [\gamma, \|T\|^2]$, for some $\gamma > 0$. In this paper it is proved that $\sigma(x^*x) \subseteq \{0\} \cup [\gamma, \|x\|^2]$, for some $\gamma > 0$ if and only if $L_x : A \to A$ defined by $L_x(y) = xy$ has closed range in A where x is any element of a commutative C^*-algebra A with identity. Moreover, the multiplication operators with closed range are found on some classical Banach algebras. Throughout the paper, the following characterization for closed range operators on Banach spaces is used.

Theorem 1. *[3] Let $T : X \to Y$ be a bounded linear operator from a Banach space X into a Banach space Y. Then T has closed range in Y if and only if there is a constant $c > 0$ such that for given $x \in X$, there is an element $y \in X$ such that $Tx = Ty$ and $\|y\| \le c\|Tx\|$.*

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We shall apply this theorem to multiplication operators on Banach algebras. For a given Banach algebra A and a given element x in A, L_x and R_x will denote the left multiplication operator and the right multiplication operator, respectively; that is; $L_x(y) = xy$ and $R_x(y) = yx$, for every $y \in A$.

(1) If A is a Banach algebra with a multiplicative identity element e, then L_0 and L_e have closed ranges.

(2) If x is invertible, then L_x has closed range ; this result follows from the next proposition.

(3) If A is a commutative Banach algebra with an identity element e and with a non trivial idempotent element e', then $L_{e'}$ has closed range (because $e'x_n \to x$ implies that $e'x_n = e'e'x_n \to e'x = x$).

Proposition 2 *Let A be a Banach algebra with multiplicative identity e. Then $x \in A$ is right invertible in A if and only if $L_x(A) = A$.*

Proof. Suppose x is right invertible in A with right inverse x^{-1}. Then for $y \in A$, $L_x(x^{-1}y) = xx^{-1}y = ey = y$. Thus $L_x(A) = A$. Conversely, assume that $L_x(A) = A$. Then for $e \in A$, there is a $y \in A$ such that $e = L_x(y) = xy$ so that x is right invertible with the right inverse y.

We now find multiplication operators with closed ranges in some classical Banach algebras.

Proposition 3 *For $1 \leq p < \infty$, the collection of all $x \in \ell_p$ such that L_x has closed range is the space*

$$c_{00} = \{x = (x_n)_{n=1}^{\infty} : x_n = 0 \text{ for all except for finitely many } n\}.$$

*Proof.*If $x \in c_{00}$, then the range of L_x is finite dimensional and hence it is closed in ℓ_p. If $x = (x_n)_{n=1}^{\infty} \notin c_{00}$, then there is a subsequence $(x_{n_m})_{m=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $|x_{n_m}| < \frac{1}{m}$ and $x_{n_m} \neq 0$ for every m. For each n_m, define $x^{(m)} \in \ell_p$ by

$$x_i^{(m)} = \begin{cases} m & \text{if } i = n_m \\ 0 & \text{elsewhere.} \end{cases}$$

If, for given m, there is a $z^{(m)} \in \ell_p$ such that $L_x z^{(m)} = L_x x^{(m)}$, then $z_{n_m}^{(m)} = m$ so that $\|z^{(m)}\|_p \geq m$ and $\|L_x x^{(m)}\|_p \leq 1$. In this case, there is no positive constant c such that $\|z^{(m)}\|_p \leq c\|L_x x^{(m)}\|_p$, for every m. Therefore L_x does not have closed range in ℓ_p; by theorem 1.

Proposition 4 *For a given element $x \in \ell_\infty$, L_x has closed range in ℓ_∞ if and only if $x = (0, 0, 0, \ldots)$, or, $\inf_n \{|x_n| : x_n \neq 0\} > 0$.*

This proposition follows from the next proposition. The argument used in the previous proof shall be extended to the following proposition.

Proposition 5 *Let (X, M, μ) be a (non-negative) positive measure space. For a given element $f \in L_\infty(X)$, L_f has closed range in $L_\infty(X)$ if and only if $f = 0$ almost everywhere on X or ess. inf. $\{|f(x)| : x \in X, f(x) \neq 0\} > 0$ on $\{x \in X : f(x) \neq 0\}$.*

*Proof.*If $f = 0$, then $L_f(L_\infty(X)) = \{0\}$ so that it has closed range in $L_\infty(X)$. Suppose $f \neq 0$ almost everywhere on X and assume that

$$0 < \text{ess. inf.} \{|f(x)| : x \in X, f(x) \neq 0\} = c, \text{say},$$

on $\{x \in X : \ f(x) \neq 0\}$. Then for each $g \in L_\infty(X)$, define h such that

$$h(x) = \begin{cases} g(x) & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 \end{cases}$$

then $c\|h\|_\infty \leq \|fg\|_\infty = \|L_f(g)\|_\infty$ and $L_f(g) = L_f(h)$. Thus, by theorem 1, L_f has closed range in $L_\infty(X)$. Suppose $f \neq 0$ almost everywhere on X and

$$\text{ess. inf.} \{|f(x)| : x \in X, f(x) \neq 0\} = 0$$

on $\{x \in X : f(x) \neq 0\}$. Then we can find a sequence $(E_n)_{n=1}^{\infty}$ of disjoint measurable subsets of X with positive measures such that $|f(x)| \leq \frac{1}{n}$, for all $x \in E_n$, and $f(x) \neq 0$ almost everywhere on E_n. For each integer n, define $g_n \in L_\infty(X)$ by

$$g_n(x) = \begin{cases} n & \text{if } x \in E_n \\ 0 & \text{elsewhere.} \end{cases}$$

If, for each n, there is a $h_n \in L_\infty(X)$ such that $L_f(h_n) = L_f(g_n)$, then $h_n(x) \geq n$ on E_n almost everywhere, $\|h_n\|_\infty \geq n$ and $\|L_f(g_n)\| \leq 1$. In this case, there is no positive constant $c > 0$ such that $c\|h_n\|_\infty \leq \|L_f(g_n)\|_\infty$, for every n. Therefore L_f does not have closed range in $L_\infty(X)$, by theorem 1.

It is possible to find multiplication operators with closed range to each individual Banach algebra. The following theorem gives a characterization of closed range multiplication operators on commutative C^*-algebras with multiplicative identity elements.

Theorem 6. *Let A be a commutative C^*-algebra over $\mathbb{C}$ with a multiplicative identity element. Let $x \in A$. Then the following three statements are equivalent:*
(1) L_x has closed range in A.
(2) 0 is not a limit point of $\sigma(x)$.
(3) $\sigma(x^*x) \subseteq \{0\} \cup [\gamma, \|x\|^2]$, for some $\gamma > 0$.

Proof.
By the Gelfand-Naimark theorem [10], let us assume that $A = C(\Delta)$, the algebra of all complex valued continuous functions on the maximal ideal space Δ of A, where Δ is a compact Hausdorff space under the Gelfand-topology. Fix $x \in A$.

Note that L_x has closed range in A if and only if there is a positive constant $c > 0$ such that for each $y \in A$ there is a $z \in A$ such that $\|z\| \leq c\|xy\|$ and $xy = xz$; that is,

$$\sup_{m\in\Delta} |z(m)| \leq c \sup_{m\in\Delta} |x(m)y(m)| \text{ and } xy = xz. \tag{2}$$

If $\{m \in \Delta : x(m) = 0\}$ is open in Δ, then we can take $c = \sup_{m\in\Delta} \frac{1}{|x(m)|}$ and

$$z(m) = \begin{cases} y(m) & \text{if } x(m) \neq 0 \\ 0 & \text{if } x(m) = 0 \end{cases}$$

so that the previous condition (2) is satisfied.

Suppose $\{m \in \Delta : x(m) = 0\}$ is not open in Δ. Then, for each $n > 0$, find $y_n \in A$ such that $y_n(m) \in [0, 1]$, for all $m \in \Delta$, $y_n(m) = 0$ for $|x(m)| \geq \frac{1}{n}$ and $y_n(m) = 1$ for $|x(m)| \leq \frac{1}{2n}$. This is possible by the Urysohn's lemma.

If there is an $z_n \in A$ such that $xy_n = xz_n$, then $z_n(m) = y_n(m)$ for some m with $0 < |x(m)| \leq \frac{1}{2n}$ and hence $\sup_{m\in\Delta} |z_n(m)| \geq 1 = \sup_{m\in\Delta} |y_n(m)|$ but $\sup_{m\in\Delta} |x(m)y_n(m)| \leq \frac{1}{n}$ and hence there is no positive constant c such that $\sup_{m\in\Delta} |z_n(m)| \leq c \ \sup_{m\in\Delta} |x(m)y_n(m)|$, for each $n = 1, 2, \ldots$. Thus the condition (2) fails to be true in this case. Therefore, we conclude that L_x has closed range in A if and only if $\{m \in \Delta : x(m) = 0\}$ is open in Δ. That is, L_x has closed range in A if and only if 0 is not a limit point of $\sigma(x)$. Thus (6) and (6) are equivalent. It is clear that (6) and (6) are equivalent.

We now discuss the closedness of range of multiplication operators on Hilbert spaces. Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators from a Hilbert space $\mathcal{H}$ into itself. For given $T \in \mathcal{B}(H)$, let $L_T : \mathcal{B}(H) \to \mathcal{B}(H)$ and $R_T : \mathcal{B}(H) \to \mathcal{B}(H)$ be the left and right multiplication operators defined by $L_T(S) = TS$ and $R_T(S) = ST$, respectively. The following theorem gives a relation between the closedness of range of T and the range of left/right multiplication operators.

Theorem 7. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent :*
(1) T has closed range in $\mathcal{H}$.
(2) L_T has closed range in $\mathcal{B}(\mathcal{H})$.
(3) R_T has closed range in $\mathcal{B}(\mathcal{H})$.

Proof. Suppose that the range $R(T)$ is closed in $\mathcal{H}$. Suppose $R_T(T_n) = T_nT$ converges to S in $\mathcal{B}(\mathcal{H})$, as $n \to \infty$, where $T_n \in \mathcal{B}(\mathcal{H})$. If $x \in N(T)$, then $S(x) = 0 = \lim_{n\to\infty} T_n(T(x))$. For each $x \in \mathcal{H}$, define $\Lambda(T(x)) = Sx$. If $Tx = Ty$, then $x - y \in N(T)$ and hence $S(x - y) = 0$ so that $Sx = Sy$. Thus Λ is well defined on $R(T)$. The linearity of T and S imply the linearity of Λ on $R(T)$. Since $R(T)$ is closed in $\mathcal{H}$, there is a positive constant $c > 0$ such that for given $x \in \mathcal{H}$, there is a $y \in \mathcal{H}$ satisfying $\|y\| \leq c\|Tx\|$ and $Tx = Ty$. For such elements x and y we have

$$\|\Lambda(Tx)\| = \|\Lambda(Ty)\| = \|Sy\| \leq \|S\| \ \|y\| \leq \|S\| \ c \ \|Tx\|.$$

This proves that Λ is continuous on $R(T)$.

Extend Λ to $\Lambda' : \mathcal{H} \to \mathcal{H}$ by $\Lambda'(y_1 + y_2) = \Lambda(y_1)$ for $y_1 \in R(T)$ and $y_2 \in R(T)^\perp$. Then $\Lambda' \in \mathcal{B}(\mathcal{H})$. Also $S = \Lambda T = \Lambda' T = R_T(\Lambda')$. Thus $S \in R_T(\mathcal{B}(\mathcal{H}))$, the range of R_T. This proves that the range of R_T is closed in $\mathcal{B}(\mathcal{H})$. Therefore (7) implies (7).

Now, assume that $R(T)$ is not closed in $\mathcal{H}$. Then there is a sequence (x_n) in $\mathcal{H}$ and element $y_0 \in \mathcal{H}$ such that $Tx_n \to y_0$ as $n \to \infty$ and $y_0 \notin R(T)$. Fix a non-zero continuous linear finctional f on $\mathcal{H}$. To each n, define $T_n : \mathcal{H} \to \mathcal{H}$

by $T_n(x) = f(x)x_n$. Then each $T_n \in \mathcal{B}(\mathcal{H})$ and $TT_n \in \mathcal{B}(\mathcal{H})$. Also TT_n converges to S defined by $S(x) = f(x)y_0$, in $\mathcal{B}(\mathcal{H})$. Therefore S is in the closure of $L_T(\mathcal{B}(\mathcal{H}))$. If there is some $T' \in \mathcal{B}(\mathcal{H})$ such that $S = TT'$, then $y_0 \in R(S) \subseteq R(T)$, which is impossible. Therefore $S \notin L_T(\mathcal{B}(\mathcal{H}))$. Thus $L_T(\mathcal{B}(\mathcal{H}))$ is not closed in $\mathcal{B}(\mathcal{H})$. Therefore, if L_T has closed range in $\mathcal{B}(\mathcal{H})$, then $R(T)$ is closed in $\mathcal{H}$. Thus, (7) is proved.

Assume that (7) is true. Therefore $\{ST : S \in \mathcal{B}(\mathcal{H})\}$ is closed in $\mathcal{B}(\mathcal{H})$. Then $\{(ST)^* : S \in \mathcal{B}(\mathcal{H})\}$ is closed in $\mathcal{B}(\mathcal{H})$. By Banach's closed range theorem [10], $\{T^*S^* : S^* \in \mathcal{B}(\mathcal{H})\}$ is closed in $\mathcal{B}(\mathcal{H})$. That is, $L_{T^*}(\mathcal{B}(\mathcal{H}))$ is closed in $\mathcal{B}(\mathcal{H})$. Therefore, the previous part of the proof implies that T^* has closed range in $\mathcal{H}$. The first part of the proof implies that $\{S^*T^* : S^* \in \mathcal{B}(\mathcal{H})\}$ is closed in $\mathcal{B}(\mathcal{H})$. That is, $\{(TS)^* : S \in \mathcal{B}(\mathcal{H})\}$ is closed in $\mathcal{B}(\mathcal{H})$. Therefore, $\{TS : S \in \mathcal{B}(\mathcal{H})\}$ is closed in $\mathcal{B}(\mathcal{H})$. Thus (7) is proved.

The author is not able to extend this theorem to Banach spaces. However some partial extensions are obtained. It is proved in the next section that any R_T has closed range in $\mathcal{B}(X)$, if X has Hahn-Banach extension property and T has closed range in X. It is also observed that T has closed range in any Banach space X, if L_T or R_T has closed range in $\mathcal{B}(X)$.

2. Partial Extensions

Definition 1. *Let Z be a normed space. Z is said to have Hahn-Banach extension property if for any linear subspace Y of a normed space and for any continuous linear operator $T : Y \to Z$, there is a continuous linear extension $\widetilde{T} : X \to Z$ to T such that $\|T\| = \|\widetilde{T}\|$.*

There are many characterizations for normed spaces with Hahn-Banach property in [4] and [9].

Theorem 8. *Let X be a Banach space with Hahn-Banach extension property. Let $T \in \mathcal{B}(X)$ has closed range in X. Then $L_T : \mathcal{B}(X) \to \mathcal{B}(X)$ has closed range.*

Proof. Suppose T_nT converges to some S in $\mathcal{B}(X)$, for a sequence (T_n) in $\mathcal{B}(X)$. If $x \in N(T)$, then $Sx = 0 = \lim_{n\to\infty} T_nTx$. For each $x \in X$, define $\Lambda(Tx) = Sx$. If $Tx = Ty$, then $x - y \in N(T)$, $S(x - y) = 0$ and $Sx = Sy$. Thus Λ is well defined on $R(T)$.

The linearity of T and S imply the linearity of Λ on $R(T)$. Since $R(T)$ is closed in X, by theorem 1, there is a positive constant $c > 0$ such that for given $x \in X$, there is an element $y \in X$ satisfying $\|y\| \leq c\|Tx\|$ and $Tx = Ty$. For such elements x and y we have

$$\|\Lambda(Tx)\| = \|\Lambda(Ty)\| = \|Sy\| \leq \|S\|\ \|y\| \leq c\|S\|\ \|Tx\|.$$

This proves that Λ is continuous on $R(T)$.

Extend Λ to a continuous linear operator $\Lambda' : X \to X$, which is possible because X has Hahn-Banach extension property. Then $S = \Lambda T = \Lambda' T = R_T(\Lambda')$ and hence $S \in R_T(\mathcal{B}(X))$, the range of R_T. This proves that the range of R_T is closed in $\mathcal{B}(X)$.

Theorem 9. *Let X be a Banach space. Let $T \in \mathcal{B}(X)$. Suppose $L_T : \mathcal{B}(X) \to \mathcal{B}(X)$ has closed range. Then T has closed range in X.*

Proof. On the contrary, assume that $R(T)$ is not closed in X. Then there is a sequence $(x_n)_{n=1}^{\infty}$ in X and an element $y_0 \in X$ such that $Tx_n \to y_0$ as $n \to \infty$, and $y_0 \in R(T)$. Fix a non zero continuous linear functional f on X. To each n, define $T_n : X \to X$ by $T_n(x) = f(x)x_n$. Then each $T_n \in \mathcal{B}(X)$ and $TT_n \in \mathcal{B}(X)$. Also TT_n converges to S defined by $S(x) = f(x)y_0$ in $\mathcal{B}(X)$. Therefore S is in the closure of $L_T(\mathcal{B}(X))$. If there is some $T' \in \mathcal{B}(X)$ such that $S = TT'$ then $y_0 \in R(S) \subseteq R(T)$, which is impossible. Therefore $S \in L_T(\mathcal{B}(X))$. Thus $L_T(\mathcal{B}(X))$ is not closed in $\mathcal{B}(X)$, which is a contradiction. This proves the result.

Theorem 10. *Let X be a Banach space. Let $T \in \mathcal{B}(X)$. Suppose $R_T : \mathcal{B}(X) \to \mathcal{B}(X)$ has closed range. Then T has closed range in X.*

Proof. Since $\|S\| = \|S^*\|$ and $(ST)^* = T^*S^*$, for all $S \in \mathcal{B}(X)$, and since $\{ST : S \in \mathcal{B}(X)\}$ is closed in $\mathcal{B}(X)$, $\{ T^*S^* : S^* \in \mathcal{B}(X^*) \}$ is also closed in $\mathcal{B}(X^*)$, where X^* is the dual of X and $T^* : X^* \to X^*$ is the adjoint of T. That is, $L_{T^*} : \mathcal{B}(X^*) \to \mathcal{B}(X^*)$ has closed range. So, by theorem 9, T^* has closed range in X^*. This implies that T has closed range in X, because it is known in [10] that $S \in \mathcal{B}(X)$ has closed range in X if and only if $S^* \in \mathcal{B}(X^*)$ has closed range in X^*.

Further Remarks. If $X = \mathcal{H}$ is a Hilbert space and T^* is the Hilbert adjoint of $T \in \mathcal{B}(\mathcal{H})$, then $\{ST : S \in \mathcal{B}(\mathcal{H})\}$ is closed in $\mathcal{B}(\mathcal{H})$ if and only if $\{T^*S^* : S^* \in \mathcal{B}(\mathcal{H})\}$ is closed in $\mathcal{B}(\mathcal{H})$. So, $L_T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ has closed range if and only if $R_T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ has closed range. In the proof of the theorem 7, Λ can be extended to Λ' by making $\Lambda'(R(T)^{\perp}) = \{0\}$, if X is a Hilbert space. This type of extension is possible only in Hilbert spaces because every closed subspace is complemented in a Banach space if and only if the Banach space is a Hilbert space [6]. Some new arguments are expected to extend the theorem 7 to Banach spaces.

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