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Iterative Methods for Solving Seventh-Order Nonlinear Time Fractional Equations

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Abstract: The present paper aims to investigate the numerical solutions of the seventh order Caputo fractional time Kaup-Kupershmidt, Sawada-Kotera and Lax's Korteweg-de Vries equations using two reliable techniques, namely, the fractional reduced differential transform method and q-homotopy analysis transform method. These equations are the mathematical formulation of physical phenomena that arise in chemistry, engineering and physics. For instance, in the motions of long waves in shallow water under gravity, nonlinear optics, quantum mechanics, plasma physics, fluid mechanics and so on. With these two methods, we construct series solution to these problems in the recurrence relation form. We present error estimates to further investigate the accuracy and reliability of the proposed techniques. The outcome of the study reveals that the two techniques used are computationally accurate, reliable and easy to implement when solving fractional nonlinear complex phenomena that arise in physics, biology, chemistry and mathematics.

Keywords: q-Homotopy analysis transform method, fractional reduced differential transform method, Kaup-Kupershmidt seventh-order equation, Lax's seventh-order Korteweg-de Vries equation, Sawada-Kotera seventh-order equation.

1 Introduction and preliminaries

Recently, a lot of attention has been dedicated to fractional calculus by many authors by virtue of its immense applications in the fields of natural sciences and engineering which are already established in signal and image processing, biotechnology, electrodynamics, viscoelasticity, random walk, financial models, nanotechnology, anomalous diffusion, anomalous transport and many other fields [1–11]. Some essential characteristics of fractional calculus have been outlined by many authors in [12–15].

The nonlinear fractional partial differential equations are more complicated to be solved than the classical type, for its operator has been defined by integral. The nonlinear problems considered in the present work are the seventh order Kaup-Kupershmidt (7TFK-K), Sawada-Kotera (7TFS-K) and Lax's Korteweg-de Vries (7TFLK-dV) equations of the Caputo fractional time derivative. These nonlinear equations represent the mathematical models of physical phenomena that arise in chemistry, engineering and physics, for instance, in the motions of long waves in shallow water under gravity, nonlinear optics,quantum mechanics, plasma physics, fluid mechanics, and so on [16]. Several analytical schemes have developed and been implemented by different authors in seeking solutions to these nonlinear problems. Some of the techniques are the Adomian decomposition method [17, 18], pseudospectral method [19], variational iteration method [20, 21], q-homotopy analysis method [16], lie symmetry analysis method [22] and generalized Cole-Hopf transformation method [23]. See [24–28] for other methods used in obtaining solutions to nonlinear fractional partial differential equations.

In 1992, Liao proposed homotopy analysis method (HAM) [13, 29–31] which was later modified by El-Tawil et al.

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in [32]. The modification was named q-homotopy analysis method, (q-HAM). The q-HAM has been employed by many researchers to obtain solutions to linear and non-linear problems [16, 33–41]. It introduced additional parameter n to the already existing auxiliary parameter $\hbar$ in HAM which provides much flexibility compared to the HAM in controlling and adjusting the convergence of the series solution. In the current study, we consider the seventh order nonlinear fractional partial differential equation,(NFPDE) in the form

$$\mathcal{D}_t^{\alpha} u = \mathcal{F}(t, x, u, u^2, u^3, u_x, u_{xx}, u_{xxx}, u_{xxxx}, u_{xxxxxx}, u_{xxxxxxx}), \quad 0 < \alpha \leq 1, \tag{1}$$

subject to the initial condition

$$u(x, 0) = f(x), \tag{2}$$

where α is the fractional order. The proposed methods used in the present investigation are the fractional reduced differential transform method, (FRDTM) and q-homotopy analysis transform method, (q-HATM). The FRDTM was proposed by Keskin and Oturanc [42] and the q-HATM which is a mixture of q-HAM and Laplace transform method was proposed by Singh et al. [43]. Both methods overcome a very huge computations that may arise in other methods used to obtain approximate and exact solutions to nonlinear problem with high accuracy and minimal computations. The use of Laplace transform gives q-HATM an advantage over the HAM and q-HAM. In [44–46], the convergence analysis of q-HATM solution is presented. We construct both FRDTM and q-HATM solution in the form of recurrence relations. With these two methods, we reveal series form solution to these problems.

2 Preliminaries

Here, we present some essential theory of calculus related to fractional order and the classical Laplace transform that will be used in the present investigation.

Definition 1. *Let $u(x,t) \in C_\mu$ $(\mu \geq -1)$ be a function. Then the Riemann-Liouville fractional integral $\big($denoted by $J^{\alpha}u(x,t)\big)$ is defined as [15, 47, 48],*

$$J^{\alpha}u(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-w)^{\alpha-1}\, u(x,w)\, dw, \quad \alpha,\ t > 0, \tag{3}$$

where $J^0 u(x,t) = u(x,t)$ and Γ is the classical gamma function.

Definition 2. *The Caputo fractional derivative, (CFD) of $u(x,t) \in C_l^{\varphi}$, $\big($denoted by $\mathcal{D}^{\alpha}u(x,t)\big)$, where $\varphi - 1 < \alpha < \varphi$, $\varphi \in \mathbb{N}$ is presented by [15, 47]:*

$$\mathcal{D}^{\alpha}u(x,t) = \begin{cases} u^{(\varphi)}(x,t), & \alpha = \varphi, \\ J^{\varphi-\alpha}u^{(\varphi)}(x,t), & \varphi - 1 < \alpha < \varphi, \end{cases} \tag{4}$$

where

$$J^{\varphi-\alpha}u^{(\varphi)}(x,t) = \frac{1}{\Gamma(\varphi-\alpha)} \int_0^t (t-w)^{\varphi-\alpha-1} u^{(\varphi)}(x,w)\, dw, \quad \alpha,\ t > 0. \tag{5}$$

Definition 3. *The Laplace transform, (LT) of CFD is given by*

$$\mathscr{L}\Big[\mathcal{D}_t^{\alpha}u(x,t)\Big] = s^{\alpha}\mathscr{L}[u(x,t)] - \sum_{r=0}^{\varphi-1} s^{\alpha-r-1}u^{(r)}(x,0^+), \quad \varphi - 1 < \alpha \leq \varphi. \tag{6}$$

3 Analysis of the proposed methods

To present the concept of proposed methods, we examine the NFPDE

$$\mathcal{D}_t^{\alpha}u(x,t) + Lu(x,t) + \mathcal{N}u(x,t) = f(x,t), \qquad \varphi - 1 < \alpha \leq \varphi, \tag{7}$$

where $\mathcal{D}^{\alpha}$ denotes the CFD, L is the linear differential operator, $\mathcal{N}$ signifies the nonlinear differential operator, $u(x,t)$ is the unknown function and $f(x,t)$ specifies the source term.

3.1 Analysis of FRDTM

Consider a function $u(x,t)$ which is analytic and differentiated continuously in some specified domain and suppose $u(x,t)$ can be expressed as $u(x,t) = g(x)h(t)$. In regard to the properties of differential transform, function $u(x,t)$ can be written as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^{k\alpha}, \tag{8}$$

where

$$U_k(x) = \frac{1}{\Gamma(\alpha k+1)} \left[\frac{\partial^{k\alpha} u(x,t)}{\partial t^{k\alpha}} \right]_{t=0}. \tag{9}$$

Here, α is the fractional order and the t-dimensional spectrum function $U_k(x)$ is the transformed function of $u(x,t)$. Consider Eq. (7) as

$$\mathcal{D}_t^{\alpha} u(x,t) + Lu(x,t) + \mathcal{N}u(x,t) = f(x,t), \tag{10}$$

subject to the initial condition

$$u(x,0) = f(x). \tag{11}$$

According to Table 1, the iteration formula for Eq. (10) is

$$\frac{\Gamma(\alpha k+\alpha+1)}{\Gamma(\alpha k+1)} U_{(k+1)}(x) = F_k(x) - LU_k(x) - \mathcal{N}U_k(x), \tag{12}$$

where $F_k(x)$ designates the transformation of function $f(x,t)$. From initial condition Eq. (11), we write

$$U_0 = f(x). \tag{13}$$

Substituting Eq. (13) into Eq. (12), we obtain the $U_k(x)$ values. The inverse transformation of the set $\{U_k(x)\}_{k=0}^{M}$ gives

$$\mathcal{U}^{(M)}(x,t) = \sum_{k=0}^{M} U_k(x) t^{k\alpha}. \tag{14}$$

and

$$u(x,t) = \lim_{M\to\infty} \mathcal{U}^{(M)}(x,t) = \sum_{k=0}^{\infty} U_k(x) t^{k\alpha}, \tag{15}$$

gives the exact solution of Eq. (7). In Table 1, we presents some essential properties of FRDTM, where $\mathrm{w} = \mathrm{w}(x,t),\ u = u(x,t),\ \mathrm{v} = \mathrm{v}(x,t),\ W_k = W_k(x),\ U_k = U_k(x)$.

Table 1: The essential operations of FRDTM

Functional Form	Transformed function
u	$U_k = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha} u}{\partial t^{k\alpha}} \right]_{t=0}$
$\mathrm{w} = \mathcal{S}_1 u + \mathcal{S}_2 \mathrm{v}$	$W_k = \mathcal{S}_1 U_k + \mathcal{S}_2 V_k$, $\mathcal{S}_1$ and $\mathcal{S}_2$ are constants
$\mathrm{w} = u\mathrm{v}$	$W_k = \sum_{i=0}^{k} U_i V_{(k-i)} = \sum_{i=0}^{k} V_i U_{(k-i)}$
$\mathrm{w} = \mathcal{D}_t^{r\alpha} u$	$W_k = \frac{\Gamma(k\alpha+r\alpha+1)}{\Gamma(k\alpha+1)} U_{(k+r)}$
$\mathrm{w} = \frac{\partial^{\omega} u}{\partial x^{\omega}}$	$W_k = \frac{\partial^{\omega} U_k}{\partial x^{\omega}}$
$\mathrm{w} = x^r t^m$	$W_k = x^r \delta(k-m)$, $\delta(k-m) = \begin{cases} 1, & k=m \\ 0, & k \neq m \end{cases}$

3.2 Fundamentals of the q-HATM

We begin by first considering Eq. (7) given as

$$\mathcal{D}_t^{\alpha}u(x,t)+Lu(x,t)+\mathcal{N}u(x,t)-f(x,t)=0. \tag{16}$$

Apply LT on Eq. (16) and simplify to obtain

$$\mathscr{L}[u(x,t)]-\frac{1}{s^{\alpha}}\sum_{m=0}^{\varphi-1}s^{\alpha-m-1}u^{(m)}(x,0)+\frac{1}{s^{\alpha}}\Big(\mathscr{L}\Big[Lu(x,t)+\mathcal{N}u(x,t)-f(x,t)\Big]\Big)=0. \tag{17}$$

To epitomize the concept of homotopy method [29], we construct zeroth-order deformation equation for $0\le q\le \frac{1}{n},\ n\ge 1$, as

$$(1-nq)\mathscr{L}\Big(\Phi(x,t;q)-u_0(x,t)\Big)=\hbar q\mathcal{H}(x,t)N[\Phi(x,t;q)], \tag{18}$$

and define $N\big[\Phi(x,t;q)\big]$ as

$$\begin{aligned}N\big[\Phi(x,t;q)\big]=&\mathscr{L}[\Phi(x,t;q)]-\frac{1}{s^{\alpha}}\sum_{m=0}^{\varphi-1}s^{\alpha-m-1}\Phi^{(m)}(x,t;q)(0^+)\\&+\frac{1}{s^{\alpha}}\Big(\mathscr{L}\big[L\Phi(x,t;q)\big]+\mathscr{L}\big[\mathcal{N}\Phi(x,t;q)\big]-\mathscr{L}\big[f(x,t)\big]\Big).\end{aligned} \tag{19}$$

Here, q is the embedded parameter, the non-zero $\hbar$ is the auxiliary parameter, $\mathscr{L}$ is the LT and $\mathcal{H}(x,t)\neq 0$ represents the auxiliary function. Considering Eq. (18) with $q=0,\ \frac{1}{n}$, we get

$$\Phi\Big(x,t;0\Big)=u_0(x,t),\qquad \Phi\Big(x,t;\frac{1}{n}\Big)=u(x,t). \tag{20}$$

When q rises from 0 to $\frac{1}{n}$, the solutions $\Phi(x,t;q)$ range from the initial guess, u_0 to the solution, u. In case that u_0, $\mathcal{H}$, and $\hbar$ are all chosen accordingly, then the solutions $\Phi(x,t;q)$ in Eq. (18) hold as much as $0\le q\le\frac{1}{n}$. Accordingly, the Taylor series expansion of $\Phi(x,t;q)$ is given as

$$\Phi(x,t;q)=u_0(x,t)+\sum_{k=1}^{\infty}u_k(x,t)q^k, \tag{21}$$

where

$$u_k(x,t)=\frac{1}{k!}\frac{\partial^k\Phi(x,t;q)}{\partial q^k}\bigg|_{q=0}. \tag{22}$$

If we choose u_0, $\hbar$, and $\mathcal{H}$ adequately, Eq. (21) converges at $q=\frac{1}{n}$. From Eq. (20), we obtain

$$u(x,t)=u_0(x,t)+\sum_{k=1}^{\infty}u_k(x,t)\left(\frac{1}{n}\right)^k. \tag{23}$$

Differentiating Eq. (18) k-times w.r.t to "q", setting $q=0$ and lastly multiplying by $\frac{1}{k!}$ gives

$$\mathscr{L}\big[u_k(x,t)-\Upsilon_k^{*}u_{k-1}(x,t)\big]=\hbar\mathcal{H}(x,t)\mathcal{R}_k\big(\mathbf{u}_{k-1}(x,t)\big). \tag{24}$$

Here, the vector $\mathbf{u}_k$, is defined as

$$\mathbf{u}_k(x,t)=\{u_0(x,t),u_1(x,t),\cdots,u_k(x,t)\}. \tag{25}$$

Application of the inverse LT on Eq. (24) gives

$$u_k(x,t)=\Upsilon_k^{*}u_{k-1}(x,t)+\hbar\mathscr{L}^{-1}\big[\mathcal{H}(x,t)\mathcal{R}_k\big(\mathbf{u}_{k-1}(x,t)\big)\big]. \tag{26}$$

where

$$\mathcal{R}_k\big(\mathbf{u}_{k-1}(x,t)\big) = \mathscr{L}\big[u_{k-1}(x,t)\big] - \left(1-\frac{\Upsilon_k^*}{n}\right)\left(\sum_{m=0}^{\varphi-1} s^{\alpha-m-1}u^{(m)}(x,0) + \frac{1}{s^{\alpha}}\mathscr{L}\big[f(x,t)\big]\right) + \frac{1}{s^{\alpha}}\mathscr{L}\big[Lu(x,t)+\mathrm{H}_{k-1}\big], \tag{27}$$

and

$$\Upsilon_k^* = \begin{cases} 0 & k \leqslant 1, \\ n & otherwise. \end{cases} \tag{28}$$

Here, H is the homotopy polynomial and defined as

$$\mathrm{H}_k = \frac{1}{k!}\frac{\partial^k \Phi(x,t;q)}{\partial q^k}\bigg|_{q=0}, \quad \Phi(x,t;q) = \Phi_0 + q\Phi_1 + q^2\Phi_2 + q^3\Phi_3 + \cdots. \tag{29}$$

4 Applications

Here, we employ the two proposed methods to obtain solutions to the 7TFLK-dV, 7TFS-K and 7TFK-K equations. Given the generalised form of 7TFLK-dV, 7TFS-K and 7TFK-K equations

$$\mathcal{D}_t^{\alpha}u + \mathcal{S}_1u^3u_x + \mathcal{S}_2u_x^3 + \mathcal{S}_3uu_xu_{xx} + \mathcal{S}_4u^2u_{xxx} + \mathcal{S}_5u_{xx}u_{xxx} + \mathcal{S}_6u_xu_{xxxx} + \mathcal{S}_7uu_{xxxxx} + u_{xxxxxxx} = 0, \tag{30}$$

with the initial condition

$$u(x,0) = f(x). \tag{31}$$

Here, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}_4$, $\mathcal{S}_5$, $\mathcal{S}_6$, and $\mathcal{S}_7$ are real constants.

4.1 FRDTM solution

In view of Table 1, the differential transform of Eq. (30) reads

$$\begin{aligned} U_{k+1}(x) = -\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\Bigg[&\mathcal{S}_1\sum_{i=0}^{k}\sum_{l=0}^{i}\sum_{j=0}^{l}U_jU_{(l-j)}U_{(i-l)}\frac{\partial U_{(k-i)}}{\partial x} \\ &+ \mathcal{S}_2\sum_{i=0}^{k}\sum_{l=0}^{i}\frac{\partial U_l}{\partial x}\frac{\partial U_{(i-l)}}{\partial x}\frac{\partial U_{(k-i)}}{\partial x} + \mathcal{S}_3\sum_{i=0}^{k}\sum_{l=0}^{i}U_l\frac{\partial U_{(i-l)}}{\partial x}\frac{\partial^2 u_{(k-i)}}{\partial x^2} \\ &+ \mathcal{S}_4\sum_{i=0}^{k}\sum_{l=0}^{i}U_lU_{(i-l)}\frac{\partial^3 U_{(k-i)}}{\partial x^3} + \mathcal{S}_5\sum_{i=0}^{k}\frac{\partial^2 U_i}{\partial x^2}\frac{\partial^3 u_{(k-i)}}{\partial x^3} \\ &+ \mathcal{S}_6\sum_{i=0}^{k}\frac{\partial U_i}{\partial x}\frac{\partial^4 U_{(k-i)}}{\partial x^4} + \mathcal{S}_7\sum_{i=0}^{k}U_i\frac{\partial^5 U_{(k-i)}}{\partial x^5} + \frac{\partial^7 U_k}{\partial x^7}\Bigg], \end{aligned} \tag{32}$$

where $U_k = U_k(x)$ and $k = 0, 1, 2, \cdots$. The series solution by FRDTM is presented by

$$\mathcal{U}^{(M)}(x,t) = \sum_{k=0}^{M} U_k(x)t^{k\alpha}. \tag{33}$$

4.2 q-HATM solution

Performing LT on Eq. (30) with Eq. (31), we obtain

$$\mathscr{L}[u(x,t)] - \frac{1}{s}\{f(x)\} + \frac{1}{s^\alpha}\mathscr{L}\left[\mathcal{S}_1 u^3\frac{\partial u}{\partial x} + \mathcal{S}_2\left(\frac{\partial u}{\partial x}\right)^3 + \mathcal{S}_3 u\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \mathcal{S}_4 u^2\frac{\partial^3 u}{\partial x^3} + \mathcal{S}_5\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^3} + \mathcal{S}_6\frac{\partial u}{\partial x}\frac{\partial^4 u}{\partial x^4} + \mathcal{S}_7 u\frac{\partial^5 u}{\partial x^5} + \frac{\partial^7 u}{\partial x^7}\right] = 0. \tag{34}$$

We set out the nonlinear operator $\mathcal{Z}(\Phi(x,t;q))$ as

$$\begin{aligned}\mathcal{Z}\big(\Phi(x,t;q)\big) =& \mathscr{L}[\Phi(x,t;q)] - \frac{1}{s}\{f(x)\} + \frac{1}{s^\alpha}\mathscr{L}\Bigg[\mathcal{S}_1\Phi^3(x,t;q)\frac{\partial\Phi(x,t;q)}{\partial x} + \mathcal{S}_2\left(\frac{\partial\Phi(x,t;q)}{\partial x}\right)^3\\ &+ \mathcal{S}_3\Phi(x,t;q)\frac{\partial\Phi(x,t;q)}{\partial x}\frac{\partial^2\Phi(x,t;q)}{\partial x^2} + \mathcal{S}_4\Phi^2(x,t;q)\frac{\partial^3\Phi(x,t;q)}{\partial x^3}\\ &+ \mathcal{S}_5\frac{\partial^2\Phi(x,t;q)}{\partial x^2}\frac{\partial^3\Phi(x,t;q)}{\partial x^3} + \mathcal{S}_6\frac{\partial\Phi(x,t;q)}{\partial x}\frac{\partial^4\Phi(x,t;q)}{\partial x^4}\\ &+ \mathcal{S}_7\Phi(x,t;q)\frac{\partial^5\Phi(x,t;q)}{\partial x^5} + \frac{\partial^7\Phi(x,t;q)}{\partial x^7}\Bigg].\end{aligned} \tag{35}$$

Setting $\mathcal{H}(x,t) = 1$ in Eq. (24), the kth order deformation equation is given as

$$\mathscr{L}\big[u_k(x,t) - \Upsilon_k^* u_{k-1}(x,t)\big] = \hbar\mathcal{R}_k\big(\mathbf{u}_{k-1}(x,t)\big), \tag{36}$$

where

$$\begin{aligned}\mathcal{R}_k\big(\mathbf{u}_{k-1}\big) =& \mathscr{L}\big[u_{k-1}(x,t)\big] - \left(1 - \frac{\Upsilon_k^*}{n}\right)\frac{1}{s}\{f(x)\} + \frac{1}{s^\alpha}\mathscr{L}\Bigg[\mathcal{S}_1\sum_{i=0}^{k-1}\sum_{l=0}^{i}\sum_{j=0}^{l} u_j u_{(l-j)} u_{(i-l)}\frac{\partial u_{(k-1-i)}}{\partial x}\\ &+ \mathcal{S}_2\sum_{i=0}^{k-1}\sum_{l=0}^{i}\frac{\partial u_l}{\partial x}\frac{\partial u_{(i-l)}}{\partial x}\frac{\partial u_{(k-1-i)}}{\partial x} + \mathcal{S}_3\sum_{i=0}^{k-1}\sum_{l=0}^{i} u_l\frac{\partial u_{(i-l)}}{\partial x}\frac{\partial^2 u_{(k-1-i)}}{\partial x^2}\\ &+ \mathcal{S}_4\sum_{i=0}^{k-1}\sum_{l=0}^{i} u_l u_{(i-l)}\frac{\partial^3 u_{(k-1-i)}}{\partial x^3} + \mathcal{S}_5\sum_{i=0}^{k-1}\frac{\partial^2 u_i}{\partial x^2}\frac{\partial^3 u_{(k-1-i)}}{\partial x^3}\\ &+ \mathcal{S}_6\sum_{i=0}^{k-1}\frac{\partial u_i}{\partial x}\frac{\partial^4 u_{(k-1-i)}}{\partial x^4} + \mathcal{S}_7\sum_{i=0}^{k-1} u_i\frac{\partial^5 u_{(k-1-i)}}{\partial x^5} + \frac{\partial^7 u_{(k-1)}}{\partial x^7}\Bigg].\end{aligned} \tag{37}$$

Using inverse LT on Eq. (36), for $k \geq 1$, we obtain

$$u_k(x,t) = \Upsilon_k^* u_{k-1}(x,t) + \hbar\mathscr{L}^{-1}\left[\mathcal{R}_k\left(\mathbf{u}_{k-1}(x,t)\right)\right]. \tag{38}$$

The series solution by q-HATM is presented as

$$\mathcal{U}^{(M)}(x,t,n,\hbar) = u_0(x,t) + \sum_{k=1}^{M} u_k(x,t)\left(\frac{1}{n}\right)^k. \tag{39}$$

*Example 1.*Consider the 7TFLK-dV equation [18, 21, 49]

$$\mathcal{D}_t^\alpha u + 140u^3 u_x + 70u_x^3 + 280uu_x u_{xx} + 70u^2 u_{xxx} + 70u_{xx}u_{xxx} + 42u_x u_{xxxx} + 14uu_{xxxxx} + u_{xxxxxxx} = 0, \tag{40}$$

with the initial condition

$$u(x,0) = f(x) = 2\lambda^2\operatorname{sech}^2(\lambda x). \tag{41}$$

The exact solution of Eq. (40) for a special case when $\alpha = 1$ is

$$u(x,t) = 2\lambda^2\operatorname{sech}^2\big(\lambda(x - 64\lambda^6 t)\big), \tag{42}$$

where λ is an arbitrary constant.

FRDTM solution:

From Eq. (32) with $\mathcal{S}_1 = 140$, $\mathcal{S}_2 = 70$, $\mathcal{S}_3 = 280$, $\mathcal{S}_4 = 70$, $\mathcal{S}_5 = 70$, $\mathcal{S}_6 = 42$, and $\mathcal{S}_7 = 14$, we obtain

$$\begin{aligned} U_{k+1}(x) = &- \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)} \Bigg[140 \sum_{i=0}^{k}\sum_{l=0}^{i}\sum_{j=0}^{l} U_j U_{(l-j)} U_{(i-l)} \frac{\partial U_{(k-i)}}{\partial x} \\ &+ 70 \sum_{i=0}^{k}\sum_{l=0}^{i} \frac{\partial U_l}{\partial x}\frac{\partial U_{(i-l)}}{\partial x}\frac{\partial U_{(k-i)}}{\partial x} + 280 \sum_{i=0}^{k}\sum_{l=0}^{i} U_l \frac{\partial U_{(i-l)}}{\partial x}\frac{\partial^2 u_{(k-i)}}{\partial x^2} \\ &+ 70 \sum_{i=0}^{k}\sum_{l=0}^{i} U_l U_{(i-l)} \frac{\partial^3 U_{(k-i)}}{\partial x^3} + 70 \sum_{i=0}^{k} \frac{\partial^2 U_i}{\partial x^2}\frac{\partial^3 u_{(k-i)}}{\partial x^3} \\ &+ 42 \sum_{i=0}^{k} \frac{\partial U_i}{\partial x}\frac{\partial^4 U_{(k-i)}}{\partial x^4} + 14 \sum_{i=0}^{k} U_i \frac{\partial^5 U_{(k-i)}}{\partial x^5} + \frac{\partial^7 U_k}{\partial x^7} \Bigg]. \end{aligned} \tag{43}$$

From initial condition Eq. (41), we write

$$U_0 = 2\lambda^2 \operatorname{sech}^2(\lambda x). \tag{44}$$

Using Eq. (43) with initial condition Eq. (44) for $k = 0, 1, 2, \cdots$, we obtain the following successive solutions:

$$U_0 = 2\lambda^2 \operatorname{sech}^2(\lambda x),$$

$$U_1 = \frac{256\lambda^9 \tanh(\lambda x) \operatorname{sech}^2(\lambda x)}{\Gamma(\alpha+1)},$$

$$U_2 = \frac{16384\lambda^{16} \big(\cosh(2\lambda x) - 2\big) \operatorname{sech}^4(\lambda x)}{\Gamma(2\alpha+1)},$$

$$\begin{aligned} U_3 = &\frac{32768\lambda^{23}\big(16756\cosh(4\lambda x) - 125508\cosh(2\lambda x)\big)\tanh(\lambda x)\operatorname{sech}^{10}(\lambda x)}{\Gamma(3\alpha+1)} \\ &+ \frac{32768\lambda^{23}\big(\cosh(8\lambda x) - 508\cosh(6\lambda x) + 129387\big)\tanh(\lambda x)\operatorname{sech}^{10}(\lambda x)}{\Gamma(3\alpha+1)} \\ &- \frac{917504\lambda^{23}\Gamma(2\alpha+1)\big(300\cosh(4\lambda x) - 2239\cosh(2\lambda x)\big)\tanh(\lambda x)\operatorname{sech}^{10}(\lambda x)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} \\ &+ \frac{917504\lambda^{23}\Gamma(2\alpha+1)\big(9\cosh(6\lambda x) - 2312\big)\tanh(\lambda x)\operatorname{sech}^{10}(\lambda x)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}. \end{aligned}$$

In this way, the remaining terms can be obtained.

q-HATM solution:

From Eq. (37) with $\mathcal{S}_1 = 140, \mathcal{S}_2 = 70, \mathcal{S}_3 = 280, \mathcal{S}_4 = 70, \mathcal{S}_5 = 70, \mathcal{S}_6 = 42$, and $\mathcal{S}_7 = 14$, we have

$$\begin{aligned}\mathcal{R}_k\left(\mathbf{u}_{k-1}\right) =& \mathscr{L}\left[u_{k-1}(x,t)\right] - \left(1-\frac{\Upsilon_k^*}{n}\right)\frac{1}{s}\left(2\lambda^2\operatorname{sech}^2(\lambda x)\right)\\ &+ \frac{1}{s^\alpha}\mathscr{L}\Bigg[140\sum_{i=0}^{k-1}\sum_{l=0}^{i}\sum_{j=0}^{l}u_j u_{(l-j)}u_{(i-l)}\frac{\partial u_{(k-1-i)}}{\partial x}\\ &+ 70\sum_{i=0}^{k-1}\sum_{l=0}^{i}\frac{\partial u_l}{\partial x}\frac{\partial u_{(i-l)}}{\partial x}\frac{\partial u_{(k-1-i)}}{\partial x} + 280\sum_{i=0}^{k-1}\sum_{l=0}^{i}u_l\frac{\partial u_{(i-l)}}{\partial x}\frac{\partial^2 u_{(k-1-i)}}{\partial x^2}\\ &+ 70\sum_{i=0}^{k-1}\sum_{l=0}^{i}u_l u_{(i-l)}\frac{\partial^3 u_{(k-1-i)}}{\partial x^3} + 70\sum_{i=0}^{k-1}\frac{\partial^2 u_i}{\partial x^2}\frac{\partial^3 u_{(k-1-i)}}{\partial x^3}\\ &+ 42\sum_{i=0}^{k-1}\frac{\partial u_i}{\partial x}\frac{\partial^4 u_{(k-1-i)}}{\partial x^4} + 14\sum_{i=0}^{k-1}u_i\frac{\partial^5 u_{(k-1-i)}}{\partial x^5} + \frac{\partial^7 u_{(k-1)}}{\partial x^7}\Bigg].\end{aligned} \tag{45}$$

Considering Eq. (38) using Eq. (45), we obtain the sequential solutions as follows:

$$u_0 = 2\lambda^2\operatorname{sech}^2(\lambda x),$$

$$u_1 = -\frac{256\lambda^9\hbar\tanh(\lambda x)\operatorname{sech}^2(\lambda x)}{\Gamma(\alpha+1)}t^\alpha,$$

$$u_2 = (n+\hbar)\,u_1 + \frac{16384\lambda^{16}\hbar^2\left(\cosh(2\lambda x)-2\right)\operatorname{sech}^4(\lambda x)}{\Gamma(2\alpha+1)}t^{2\alpha},$$

$$\begin{aligned}u_3 =& (n+\hbar)\,u_2 + \frac{16384\lambda^{16}\hbar^2(n+\hbar)\left(\cosh(2\lambda x)-2\right)\operatorname{sech}^4(\lambda x)}{\Gamma(2\alpha+1)}t^{2\alpha}\\ &- \frac{32768\lambda^{23}\hbar^3\left(16756\cosh(4\lambda x)-125508\cosh(2\lambda x)\right)\tanh(\lambda x)\operatorname{sech}^{10}(\lambda x)}{\Gamma(3\alpha+1)}t^{3\alpha}\\ &- \frac{32768\lambda^{23}\hbar^3\left(\cosh(8\lambda x)-508\cosh(6\lambda x)+129387\right)\tanh(\lambda x)\operatorname{sech}^{10}(\lambda x)}{\Gamma(3\alpha+1)}t^{3\alpha}\\ &+ \frac{917504\lambda^{23}\hbar^3\Gamma(2\alpha+1)\left(300\cosh(4\lambda x)-2239\cosh(2\lambda x)\right)\tanh(\lambda x)\operatorname{sech}^{10}(\lambda x)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}t^{3\alpha}\\ &- \frac{917504\lambda^{23}\hbar^3\Gamma(2\alpha+1)\left(9\cosh(6\lambda x)-2312\right)\tanh(\lambda x)\operatorname{sech}^{10}(\lambda x)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}t^{3\alpha}.\end{aligned}$$

Following the same approach, we can obtain the remaining terms. The special case when $\alpha = 1$, the FRDTM and q-HATM with $n = 1,\ \hbar = -1$ series solution reduces to

$$\begin{aligned}u(x,t) =& 2\lambda^2\operatorname{sech}^2(\lambda x) + 256\lambda^9\tanh(\lambda x)\operatorname{sech}^2(\lambda x)t\\ &+ 8192\lambda^{16}\left(\cosh(2\lambda x)-2\right)\operatorname{sech}^4(\lambda x)t^2\\ &+ \frac{1048567}{3}\lambda^{23}\left(\cosh(2\lambda x)-5\right)\tanh(\lambda x)\operatorname{sech}^4(\lambda x)t^3 + \cdots,\end{aligned} \tag{46}$$

which represent the first four terms series expansion of the exact solution Eq. (42).

*Example 2.*Consider the 7TFS-K equation [18, 49]

$$\mathcal{D}_t^\alpha u + 252u^3u_x + 63u_x^3 + 378uu_xu_{xx} + 126u^2u_{xxx} + 63u_{xx}u_{xxx} + 42u_xu_{xxxx} + 21uu_{xxxxx} + u_{xxxxxxx} = 0, \tag{47}$$

with the initial condition

$$u(x,0)=f(x)=\frac{4\lambda^2}{3}\Big(2-3\tanh^2(\lambda x)\Big). \tag{48}$$

The exact solution of Eq. (47) for a special case when $\alpha=1$ is

$$u(x,t)=\frac{4\lambda^2}{3}\left(2-3\tanh^2\left(\lambda\left[x+\frac{256\lambda^6}{3}t\right]\right)\right), \tag{49}$$

where λ is an arbitrary constant.

FRDTM solution:

From Eq. (32) with $\mathcal{S}_1=252$, $\mathcal{S}_2=63$, $\mathcal{S}_3=378$, $\mathcal{S}_4=126$, $\mathcal{S}_5=63$, $\mathcal{S}_6=42$, and $\mathcal{S}_7=21$, we obtain

$$\begin{aligned}U_{k+1}(x)=&-\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left[252\sum_{i=0}^{k}\sum_{l=0}^{i}\sum_{j=0}^{l}U_jU_{(l-j)}U_{(i-l)}\frac{\partial U_{(k-i)}}{\partial x}\right.\\&+63\sum_{i=0}^{k}\sum_{l=0}^{i}\frac{\partial U_l}{\partial x}\frac{\partial U_{(i-l)}}{\partial x}\frac{\partial U_{(k-i)}}{\partial x}+378\sum_{i=0}^{k}\sum_{l=0}^{i}U_l\frac{\partial U_{(i-l)}}{\partial x}\frac{\partial^2 u_{(k-i)}}{\partial x^2}\\&+126\sum_{i=0}^{k}\sum_{l=0}^{i}U_lU_{(i-l)}\frac{\partial^3 U_{(k-i)}}{\partial x^3}+68\sum_{i=0}^{k}\frac{\partial^2 U_i}{\partial x^2}\frac{\partial^3 u_{(k-i)}}{\partial x^3}\\&\left.+42\sum_{i=0}^{k}\frac{\partial U_i}{\partial x}\frac{\partial^4 U_{(k-i)}}{\partial x^4}+21\sum_{i=0}^{k}U_i\frac{\partial^5 U_{(k-i)}}{\partial x^5}+\frac{\partial^7 U_k}{\partial x^7}\right].\end{aligned} \tag{50}$$

From initial condition Eq. (48), we write

$$U_0=\frac{4\lambda^2}{3}\Big(2-3\tanh^2(\lambda x)\Big). \tag{51}$$

Using Eq. (50) with initial condition Eq. (51) for $k=0,1,2,\cdots$, we obtain the following successive solutions:

$$U_0=\frac{4\lambda^2}{3}\Big(2-3\tanh^2(\lambda x)\Big),$$

$$U_1=-\frac{2048\lambda^9\tanh(\lambda x)\operatorname{sech}^2(\lambda x)}{3\Gamma(\alpha+1)},$$

$$U_2=\frac{524288\lambda^{16}\big(\cosh(2\lambda x)-2\big)\operatorname{sech}^4(\lambda x)}{9\Gamma(2\alpha+1)},$$

$$U_3=-\frac{268435456\lambda^{23}\tanh(\lambda x)\operatorname{sech}^4(\lambda x)\big(\cosh(2\lambda x)-5\big)}{27\Gamma(3\alpha+1)}.$$

In this way, the remaining terms can be obtained.

q-HATM solution:

From Eq. (37) with $\mathcal{S}_1 = 252$, $\mathcal{S}_2 = 63$, $\mathcal{S}_3 = 378$, $\mathcal{S}_4 = 126$, $\mathcal{S}_5 = 63$, $\mathcal{S}_6 = 42$, and $\mathcal{S}_7 = 21$, we obtain

$$\begin{aligned}\mathcal{R}_k\left(\mathbf{u}_{k-1}\right) =& \mathscr{L}\left[u_{k-1}(x,t)\right] - \left(1-\frac{\Upsilon_k^*}{n}\right)\frac{1}{s}\left\{\frac{4\lambda^2}{3}\left(2-3\tanh^2(\lambda x)\right)\right\}\\ &+ \frac{1}{s^\alpha}\mathscr{L}\left[252\sum_{i=0}^{k-1}\sum_{l=0}^{i}\sum_{j=0}^{l}u_j u_{(l-j)}u_{(i-l)}\frac{\partial u_{(k-1-i)}}{\partial x}\right.\\ &+ 63\sum_{i=0}^{k-1}\sum_{l=0}^{i}\frac{\partial u_l}{\partial x}\frac{\partial u_{(i-l)}}{\partial x}\frac{\partial u_{(k-1-i)}}{\partial x} + 378\sum_{i=0}^{k-1}\sum_{l=0}^{i}u_l\frac{\partial u_{(i-l)}}{\partial x}\frac{\partial^2 u_{(k-1-i)}}{\partial x^2}\\ &+ 126\sum_{i=0}^{k-1}\sum_{l=0}^{i}u_l u_{(i-l)}\frac{\partial^3 u_{(k-1-i)}}{\partial x^3} + 63\sum_{i=0}^{k-1}\frac{\partial^2 u_i}{\partial x^2}\frac{\partial^3 u_{(k-1-i)}}{\partial x^3}\\ &\left.+ 42\sum_{i=0}^{k-1}\frac{\partial u_i}{\partial x}\frac{\partial^4 u_{(k-1-i)}}{\partial x^4} + 21\sum_{i=0}^{k-1}u_i\frac{\partial^5 u_{(k-1-i)}}{\partial x^5} + \frac{\partial^7 u_{(k-1)}}{\partial x^7}\right].\end{aligned} \tag{52}$$

Considering Eq. (38) using Eq. (52), we obtain the sequential solutions:

$$u_0 = \frac{4\lambda^2}{3}\left(2-3\tanh^2(\lambda x)\right),$$

$$u_1 = \frac{2048\lambda^9\hbar\tanh(\lambda x)\operatorname{sech}^2(\lambda x)}{3\Gamma(\alpha+1)}t^\alpha,$$

$$u_2 = (n+\hbar)\,u_1 + \frac{524288\lambda^{16}\hbar^2\left(\cosh(2\lambda x)-2\right)\operatorname{sech}^4(\lambda x)}{9\Gamma(2\alpha+1)}t^{2\alpha},$$

$$\begin{aligned}u_3 =& (n+\hbar)\,u_2 + \frac{524288\lambda^{16}\hbar^2(n+\hbar)\operatorname{sech}^4(\lambda x)\left(\cosh(2\lambda x)-2\right)}{9\Gamma(2\alpha+1)}t^{2\alpha}\\ &+ \frac{268435456\lambda^{23}\hbar^3\tanh(\lambda x)\operatorname{sech}^4(\lambda x)\left(\cosh(2\lambda x)-5\right)}{27\Gamma(3\alpha+1)}t^{3\alpha}.\end{aligned}$$

Following the same approach, we can obtain the remaining terms. The special case when $\alpha = 1$, the FRDTM and q-HATM with $n = 1,\ \hbar = -1$ series solution reduces to

$$\begin{aligned}u(x,t) =& \frac{4}{3}\lambda^2\left(2-3\tanh^2(\lambda x)\right) - \frac{2048}{3}\lambda^9\tanh(\lambda x)\operatorname{sech}^2(\lambda x)t\\ &+ \frac{262144}{9}\lambda^{16}\left(\cosh(2\lambda x)-2\right)\operatorname{sech}^4(\lambda x)t^2\\ &- \frac{134217728}{81}\lambda^{23}\left(\cosh(2\lambda x)-5\right)\tanh(\lambda x)\operatorname{sech}^4(\lambda x)t^3 + \cdots,\end{aligned} \tag{53}$$

which represent the first four terms series expansion of the exact solution Eq. (49).

*Remark.*The result obtained in Example 1 and 2 by the two methods are in consistent with the result obtained by q-HAM [16].

*Example 3.*Consider the 7TK-K equation [49]

$$\mathcal{D}_t^\alpha u + 2016u^3u_x + 630u_x^3 + 2268uu_xu_{xx} + 504u^2u_{xxx} + 252u_{xx}u_{xxx} + 147u_xu_{xxxx} + 42uu_{xxxxx} + u_{xxxxxxx} = 0, \tag{54}$$

with the initial condition

$$u(x,0) = f(x) = \frac{\lambda^2}{3}\left(1 - \frac{3}{2}\tanh^2(\lambda x)\right). \tag{55}$$

The exact solution of Eq. (54) for a special case when $\alpha = 1$ is

$$u(x,t) = \frac{\lambda^2}{3}\left(1 - \frac{3}{2}\tanh^2\left(\lambda\left[x + \frac{4\lambda^6}{3}t\right]\right)\right), \tag{56}$$

where λ is an arbitrary parameter.

FRDTM solution:

From Eq. (32) with $\mathcal{S}_1 = 2016$, $\mathcal{S}_2 = 630$, $\mathcal{S}_3 = 2268$, $\mathcal{S}_4 = 504$, $\mathcal{S}_5 = 252$, $\mathcal{S}_6 = 147$, and $\mathcal{S}_7 = 42$, we obtain

$$\begin{aligned} U_{k+1}(x) = -\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\Bigg[&2016\sum_{i=0}^{k}\sum_{l=0}^{i}\sum_{j=0}^{l}U_jU_{(l-j)}U_{(i-l)}\frac{\partial U_{(k-i)}}{\partial x} \\ &+ 630\sum_{i=0}^{k}\sum_{l=0}^{i}\frac{\partial U_l}{\partial x}\frac{\partial U_{(i-l)}}{\partial x}\frac{\partial U_{(k-i)}}{\partial x} + 2268\sum_{i=0}^{k}\sum_{l=0}^{i}U_l\frac{\partial U_{(i-l)}}{\partial x}\frac{\partial^2 u_{(k-i)}}{\partial x^2} \\ &+ 504\sum_{i=0}^{k}\sum_{l=0}^{i}U_lU_{(i-l)}\frac{\partial^3 U_{(k-i)}}{\partial x^3} + 252\sum_{i=0}^{k}\frac{\partial^2 U_i}{\partial x^2}\frac{\partial^3 u_{(k-i)}}{\partial x^3} \\ &+ 147\sum_{i=0}^{k}\frac{\partial U_i}{\partial x}\frac{\partial^4 U_{(k-i)}}{\partial x^4} + 42\sum_{i=0}^{k}U_i\frac{\partial^5 U_{(k-i)}}{\partial x^5} + \frac{\partial^7 U_k}{\partial x^7}\Bigg]. \end{aligned} \tag{57}$$

From initial condition Eq. (55), we write

$$U_0 = \frac{\lambda^2}{3}\left(1 - \frac{3}{2}\tanh^2(\lambda x)\right). \tag{58}$$

Using Eq. (57) with initial condition Eq. (58) for $k = 0, 1, 2, \cdots$, we obtain the following successive solutions:

$$U_0 = \frac{\lambda^2}{3}\left(1 - \frac{3}{2}\tanh^2(\lambda x)\right),$$

$$U_1 = -\frac{4\lambda^9\tanh(\lambda x)\operatorname{sech}^2(\lambda x)}{3\Gamma(\alpha+1)},$$

$$U_2 = \frac{16\lambda^{16}\big(\cosh(2\lambda x) - 2\big)\operatorname{sech}^4(\lambda x)}{9\Gamma(2\alpha+1)},$$

$$\begin{aligned} U_3 =& \frac{2\lambda^{23}\big(771164\cosh(4\lambda x) - 6758516\cosh(2\lambda x)\big)\tanh(\lambda x)\operatorname{sech}^{10}(\lambda x)}{27\Gamma(3\alpha+1)} \\ &- \frac{2\lambda^{23}\big(\cosh(8\lambda x) + 15116\cosh(6\lambda x) - 7424005\big)\tanh(\lambda x)\operatorname{sech}^{10}(\lambda x)}{27\Gamma(3\alpha+1)} \\ &- \frac{560\lambda^{23}\Gamma(2\alpha+1)\big(51\cosh(4\lambda x) - 447\cosh(2\lambda x)\big)\tanh(\lambda x)\operatorname{sech}^{10}(\lambda x)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} \\ &+ \frac{560\lambda^{23}\Gamma(2\alpha+1)\big(\cosh(6\lambda x) - 491\big)\tanh(\lambda x)\operatorname{sech}^{10}(\lambda x)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}. \end{aligned}$$

In this way, the remaining terms can be obtained.

q-HATM solution:

From Eq. (37) with $\mathcal{S}_1 = 2016$, $\mathcal{S}_2 = 630$, $\mathcal{S}_3 = 2268$, $\mathcal{S}_4 = 504$, $\mathcal{S}_5 = 252$, $\mathcal{S}_6 = 147$, and $\mathcal{S}_7 = 42$, we obtain

$$\begin{aligned}\mathcal{R}_k\big(\mathbf{u}_{k-1}\big) =& \mathscr{L}\left[u_{k-1}(x,t)\right] - \left(1-\frac{\Upsilon_k^*}{n}\right)\frac{1}{s}\left\{\frac{\lambda^2}{3}\left(1-\frac{3}{2}\tanh^2(\lambda x)\right)\right\} \\ &+ \frac{1}{s^\alpha}\mathscr{L}\Bigg[2016\sum_{i=0}^{k-1}\sum_{l=0}^{i}\sum_{j=0}^{l}u_j u_{(l-j)}u_{(i-l)}\frac{\partial u_{(k-1-i)}}{\partial x} \\ &+ 630\sum_{i=0}^{k-1}\sum_{l=0}^{i}\frac{\partial u_l}{\partial x}\frac{\partial u_{(i-l)}}{\partial x}\frac{\partial u_{(k-1-i)}}{\partial x} + 2268\sum_{i=0}^{k-1}\sum_{l=0}^{i}u_l\frac{\partial u_{(i-l)}}{\partial x}\frac{\partial^2 u_{(k-1-i)}}{\partial x^2} \\ &+ 504\sum_{i=0}^{k-1}\sum_{l=0}^{i}u_l u_{(i-l)}\frac{\partial^3 u_{(k-1-i)}}{\partial x^3} + 252\sum_{i=0}^{k-1}\frac{\partial^2 u_i}{\partial x^2}\frac{\partial^3 u_{(k-1-i)}}{\partial x^3} \\ &+ 147\sum_{i=0}^{k-1}\frac{\partial u_i}{\partial x}\frac{\partial^4 u_{(k-1-i)}}{\partial x^4} + 42\sum_{i=0}^{k-1}u_i\frac{\partial^5 u_{(k-1-i)}}{\partial x^5} + \frac{\partial^7 u_{(k-1)}}{\partial x^7}\Bigg].\end{aligned} \tag{59}$$

Considering Eq. (38) using Eq. (59), we obtain the sequential solutions:

$$u_0 = \frac{\lambda^2}{3}\left(1-\frac{3}{2}\tanh^2(\lambda x)\right),$$

$$u_1 = \frac{4\lambda^9\hbar\tanh(\lambda x)\operatorname{sech}^2(\lambda x)}{3\Gamma(\alpha+1)}t^\alpha,$$

$$u_2 = (n+\hbar)\,u_1 + \frac{16\lambda^{16}\hbar^2\big(\cosh(2\lambda x)-2\big)\operatorname{sech}^4(\lambda x)}{9\Gamma(2\alpha+1)}t^{2\alpha},$$

$$\begin{aligned}u_3 =& (n+\hbar)\,u_2 + \frac{16\lambda^{16}\hbar^2(n+\hbar)\big(\cosh(2\lambda x)-2\big)\operatorname{sech}^4(\lambda x)}{9\Gamma(2\alpha+1)}t^{2\alpha} \\ &- \frac{2\lambda^{23}\hbar^3\big(771164\cosh(4\lambda x)-6758516\cosh(2\lambda x)\big)\tanh(\lambda x)\operatorname{sech}^{10}(\lambda x)}{27\Gamma(3\alpha+1)}t^{3\alpha} \\ &+ \frac{2\lambda^{23}\hbar^3\big(\cosh(8\lambda x)+15116\cosh(6\lambda x)-7424005\big)\tanh(\lambda x)\operatorname{sech}^{10}(\lambda x)}{27\Gamma(3\alpha+1)}t^{3\alpha} \\ &+ \frac{560\lambda^{23}\hbar^3\Gamma(2\alpha+1)\big(51\cosh(4\lambda x)-447\cosh(2\lambda x)\big)\tanh(\lambda x)\operatorname{sech}^{10}(\lambda x)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}t^{3\alpha} \\ &- \frac{560\lambda^{23}\hbar^3\Gamma(2\alpha+1)\big(\cosh(6\lambda x)-491\big)\tanh(\lambda x)\operatorname{sech}^{10}(\lambda x)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}t^{3\alpha}.\end{aligned}$$

Following the same approach, we can obtain the remaining terms. The special case when $\alpha = 1$, the FRDTM and q-HATM with $n = 1,\ \hbar = -1$ series solution reduces to

$$\begin{aligned}u(x,t) =& \frac{1}{3}\lambda^2\left(1-\frac{3}{2}\tanh^2(\lambda x)\right) - \frac{4}{3}\lambda^9 h\, t\tanh(\lambda x)\operatorname{sech}^2(\lambda x) \\ &+ \frac{8}{9}\lambda^{16}t^2\operatorname{sech}^4(\lambda x)\big(\cosh(2\lambda x)-2\big) \\ &- \frac{64}{81}\lambda^{23}t^3\tanh(\lambda x)\operatorname{sech}^4(\lambda x)\big(\cosh(2\lambda x)-5\big)+\cdots,\end{aligned} \tag{60}$$

which represent the first four terms series expansion of the exact solution Eq. (56).

5 Numerical comparison

Here, we present the numerical study to confirm the effectiveness and exactness of the two proposed techniques used in obtaining numerical solutions to the 7TFLK-dV, 7TFS-K and 7TFK-K equations. The error analysis in terms of relative error is presented. The comparison among the FRDTM, q-HATM and the exact solution has been conducted graphically and numerically. The nature of FRDTM and q-HATM solution relative to t for different values of α which revealed the solution profiles for Eqs. (40), (47) and (54) respectively has been presented. To choose the optimal value of the auxiliary parameter $\hbar$ that gives the fast convergence, we present the $\hbar$-curves for different $n,\ \lambda$, and fractional order α.

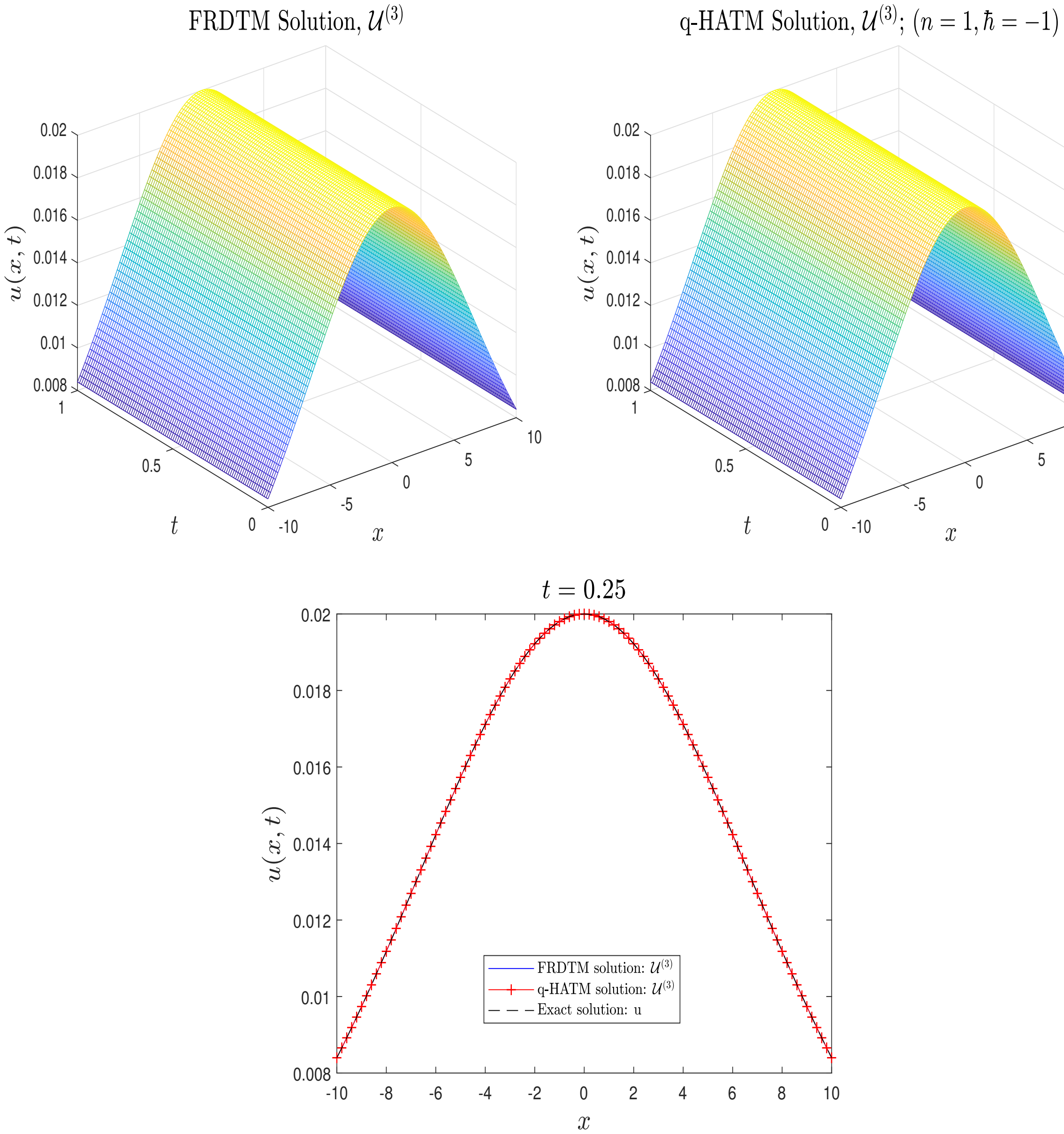

Fig. 1: The comparison of FRDTM, q-HATM and exact solution when $\alpha = 1$ and $\lambda = 0.1$ for Example 1

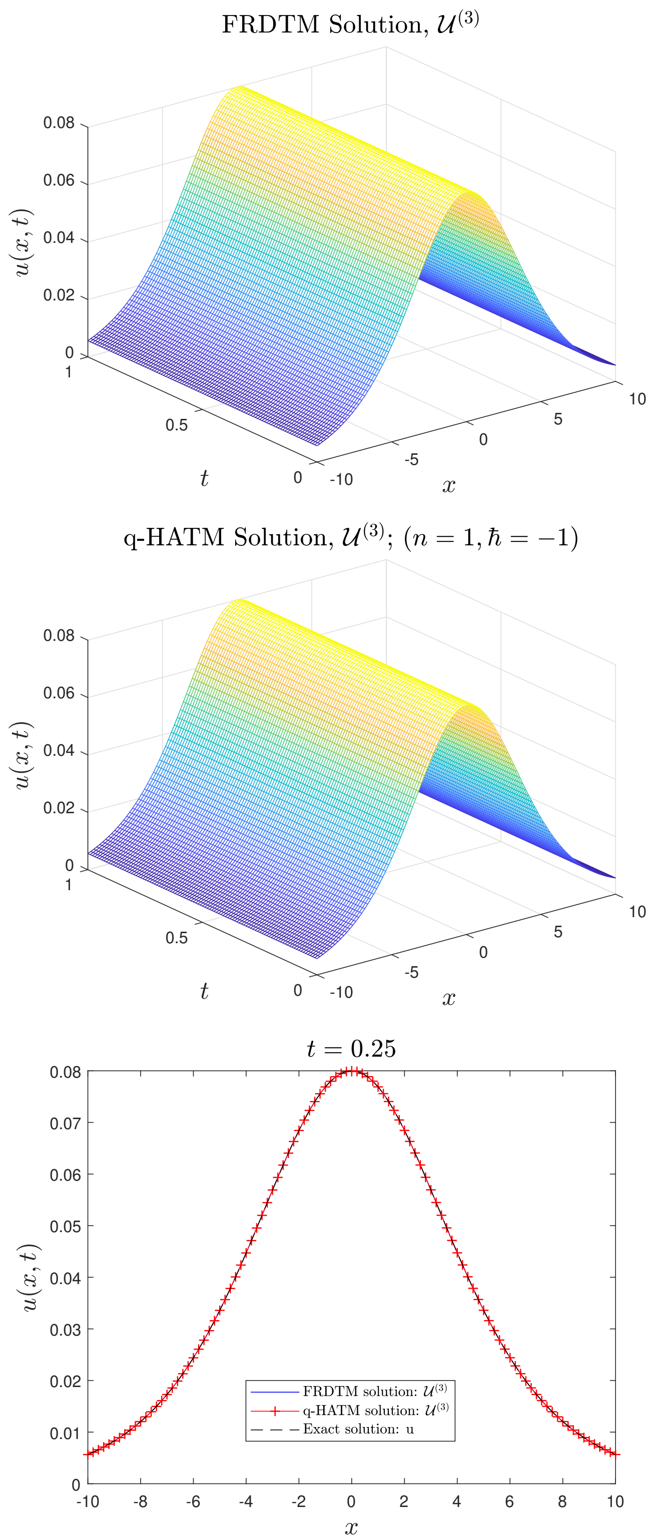

Fig. 2: The comparison of FRDTM, q-HATM and exact solution when $\alpha = 1$ and $\lambda = 0.2$ for Example 1

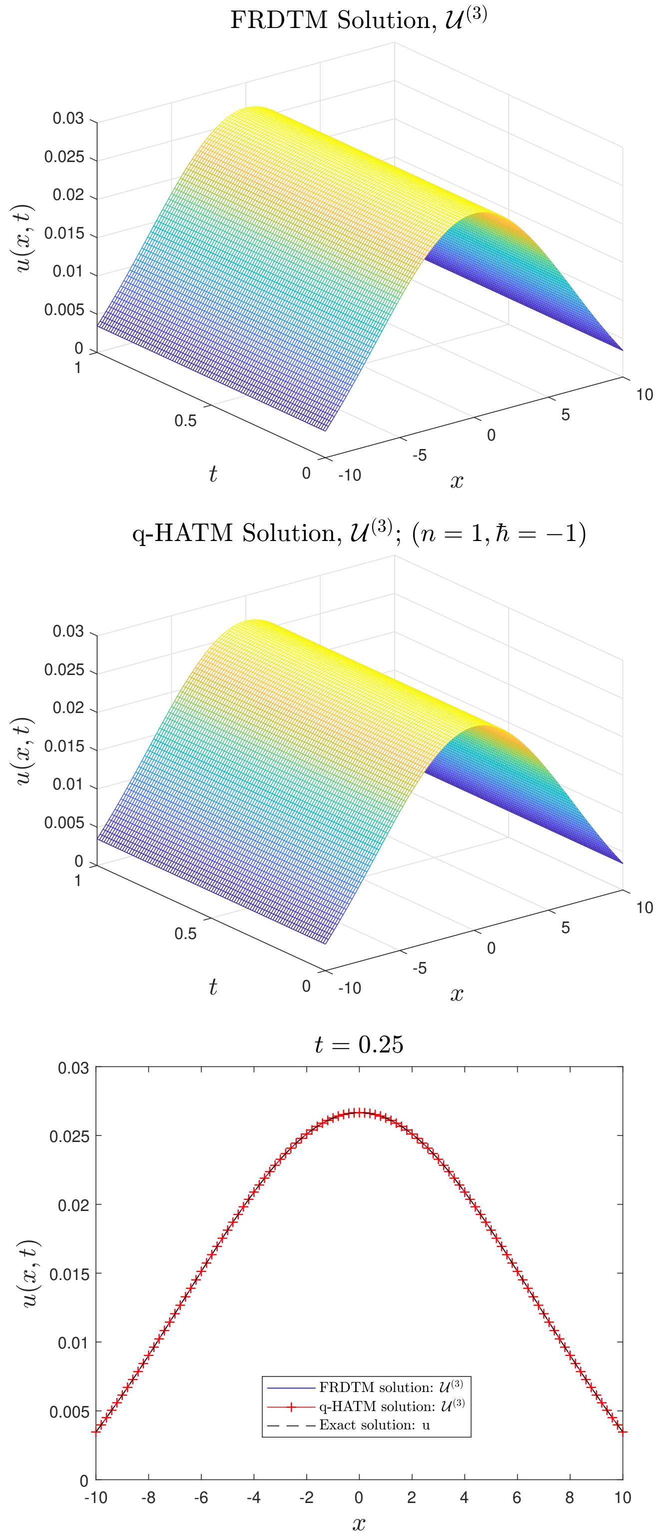

Fig. 3: The comparison of FRDTM, q-HATM and exact solution when $\alpha = 1$ and $\lambda = 0.1$ for Example 2

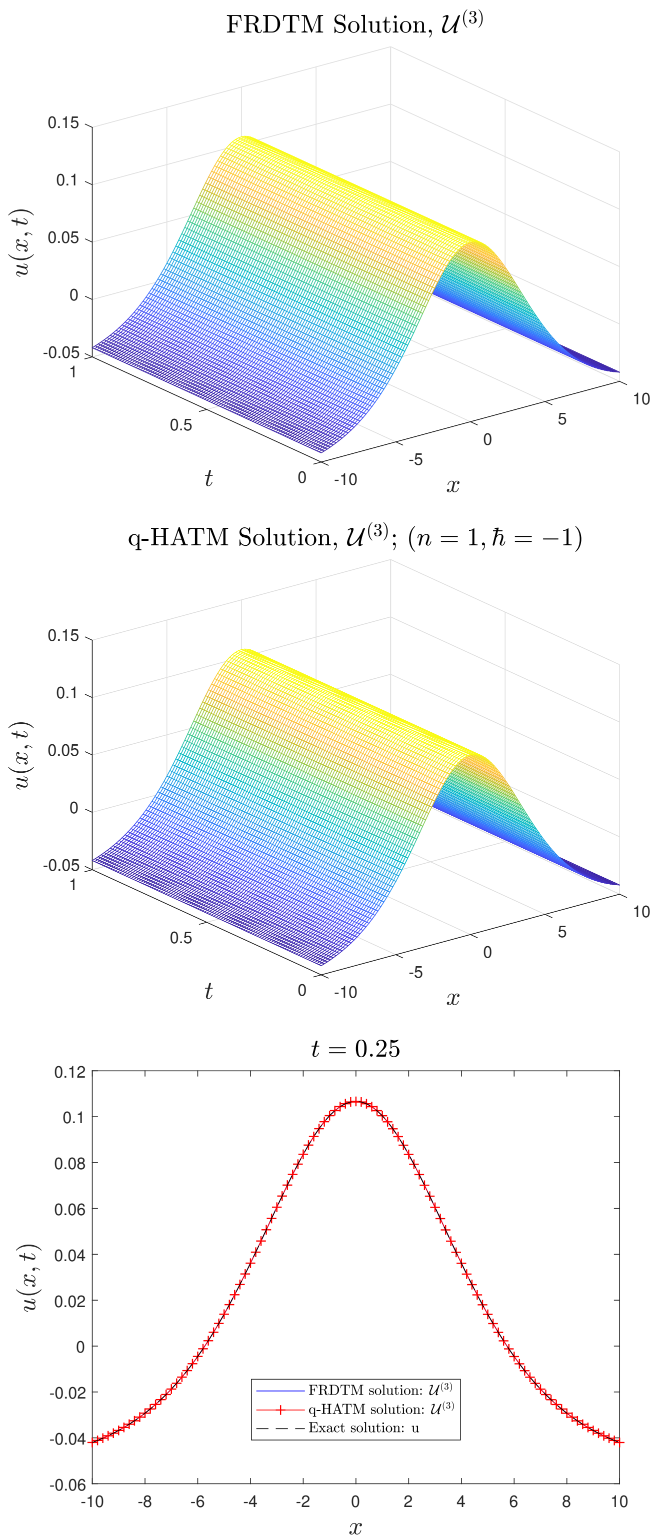

Fig. 4: The comparison of FRDTM, q-HATM and exact solution when $\alpha = 1$ and $\lambda = 0.2$ for Example 2

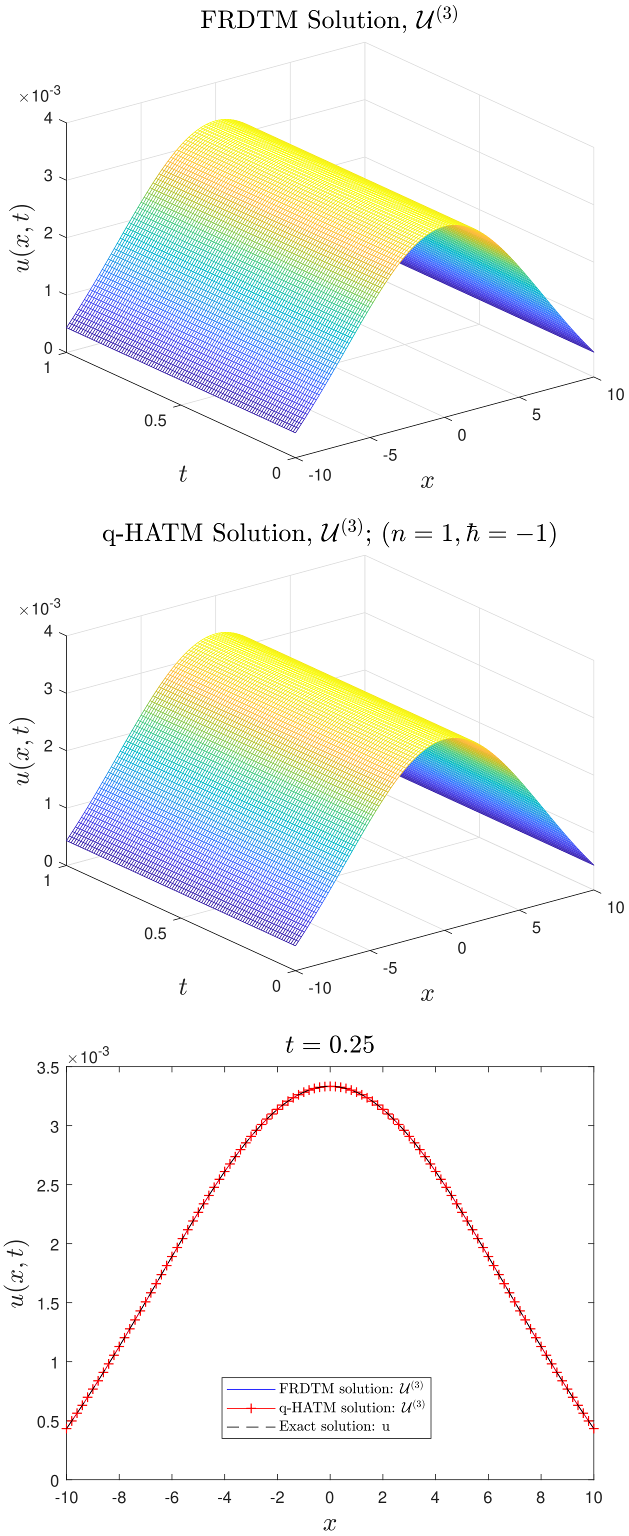

Fig. 5: The comparison of FRDTM, q-HATM and exact solution when $\alpha = 1$ and $\lambda = 0.1$ for Example 3

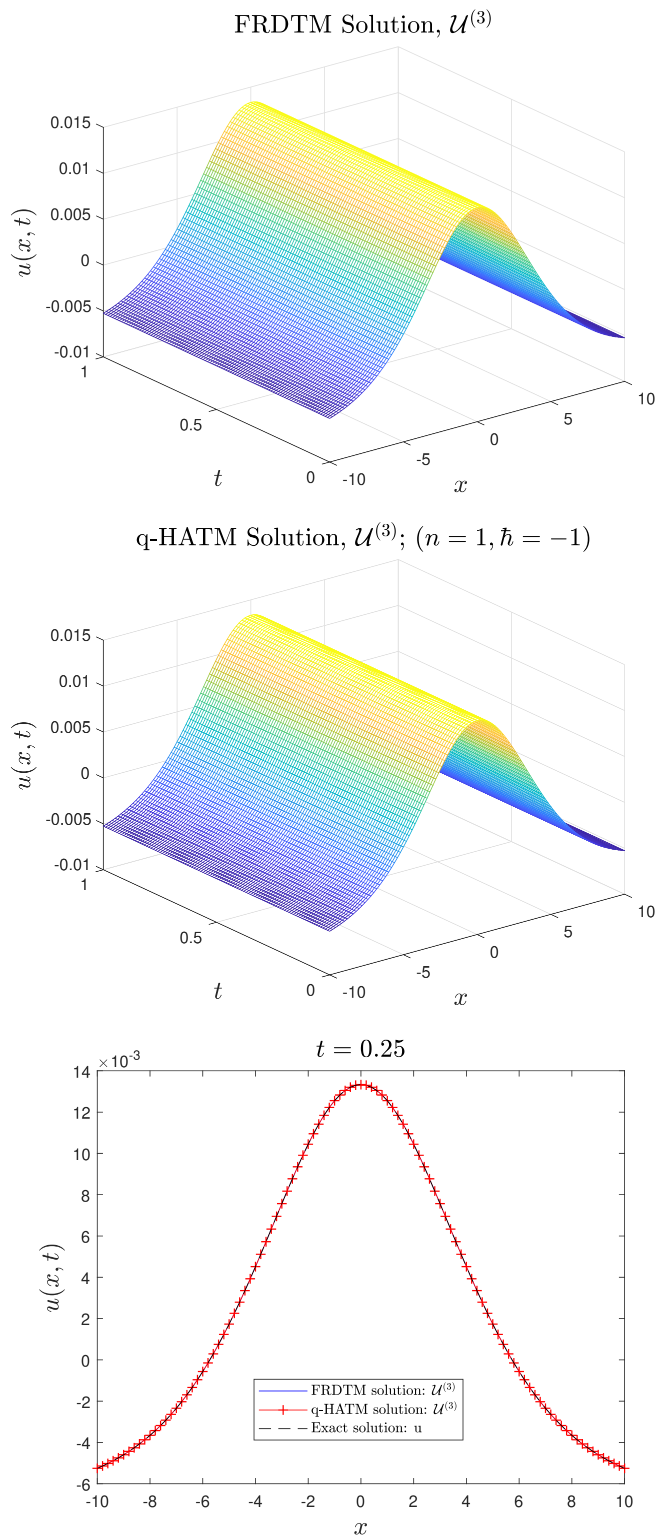

Fig. 6: The comparison of FRDTM, q-HATM and exact solution when $\alpha = 1$ and $\lambda = 0.2$ for Example 3

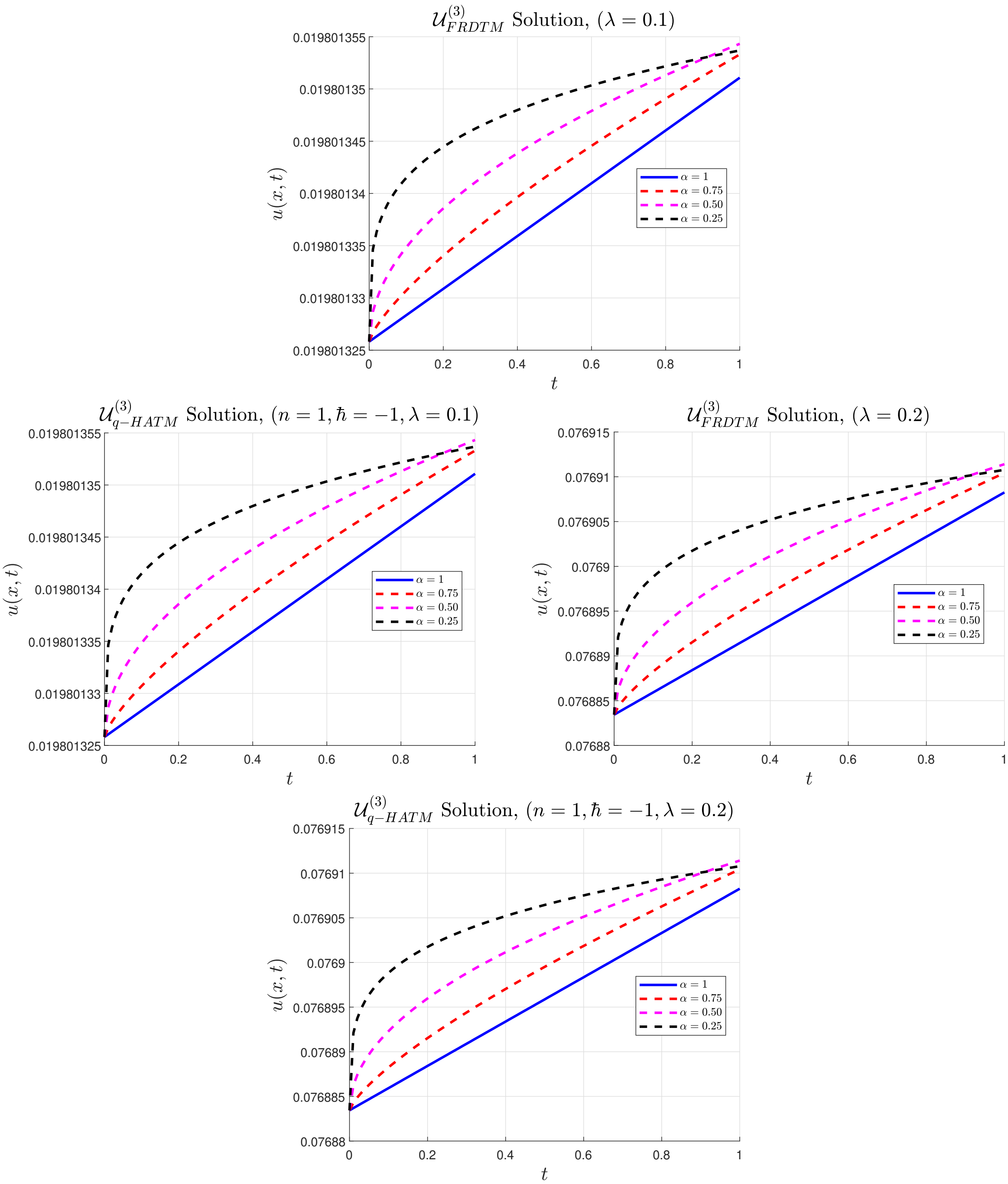

Fig. 7: The nature relative to t of FRDTM and q-HATM solution when $x = 1$ with different α for Example 1

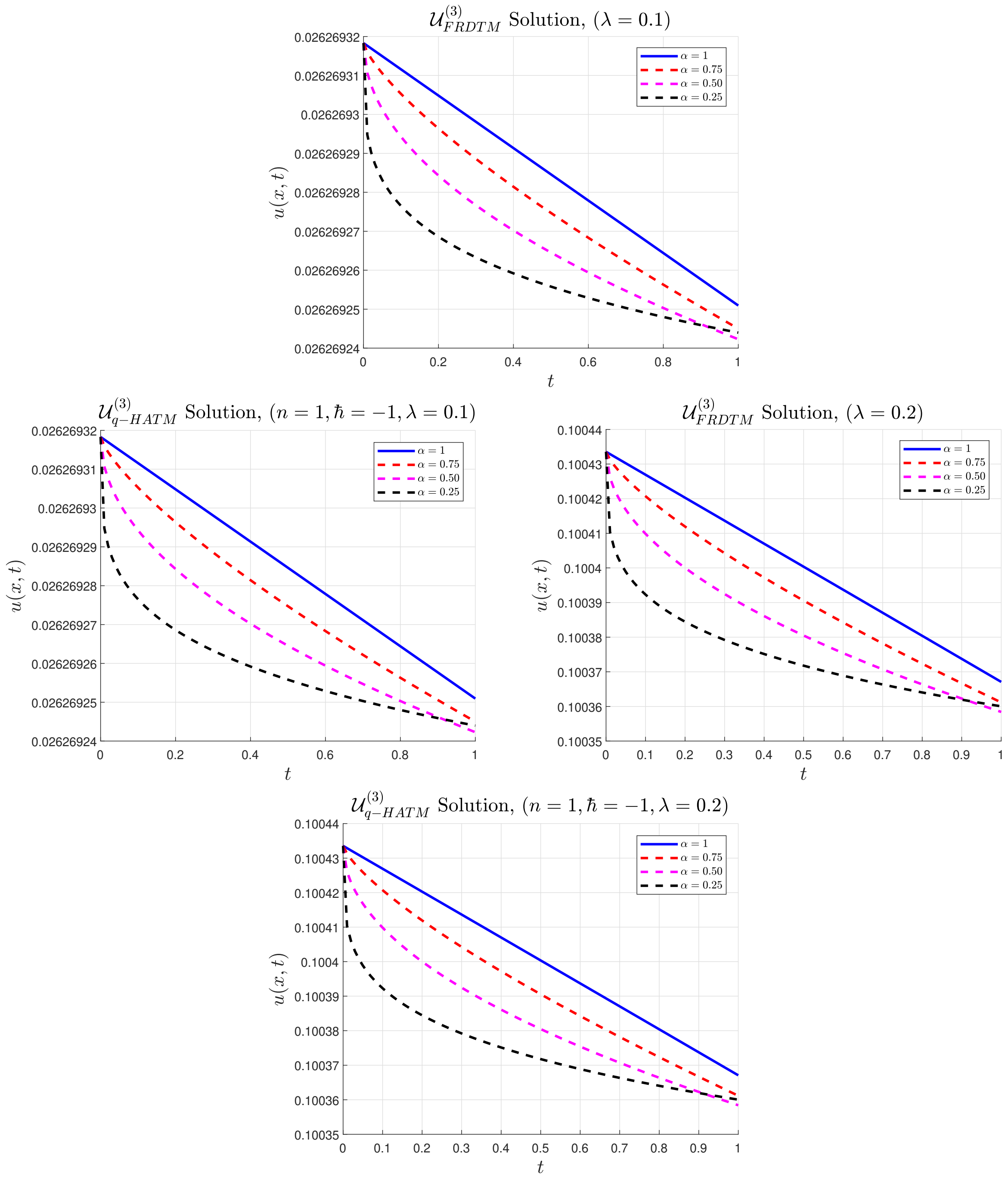

Fig. 8: The nature relative to t of FRDTM and q-HATM solution when $x = 1$ with different α for Example 2

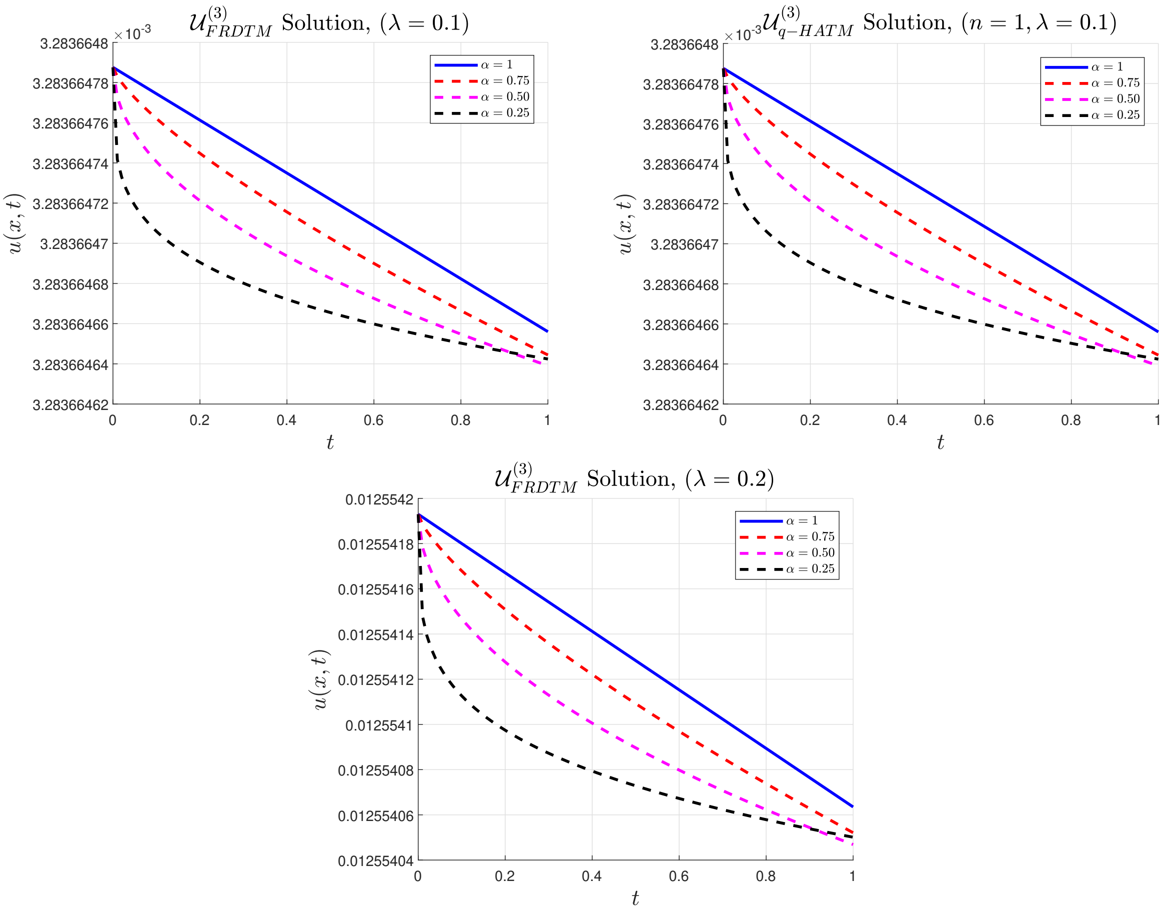

Fig. 9

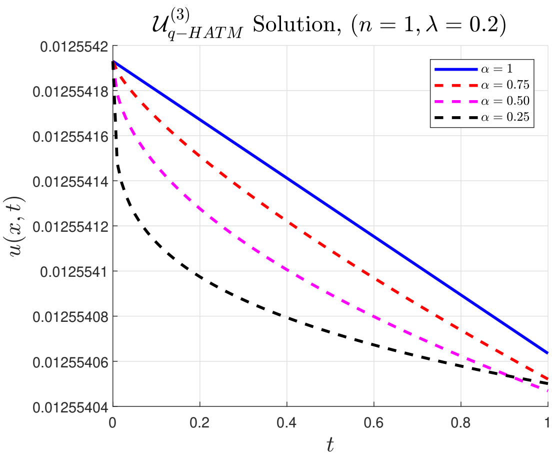

Fig. 10: The nature relative to t of FRDTM and q-HATM solution when $x = 1$ and $\hbar = -1$ with different α for Example 3

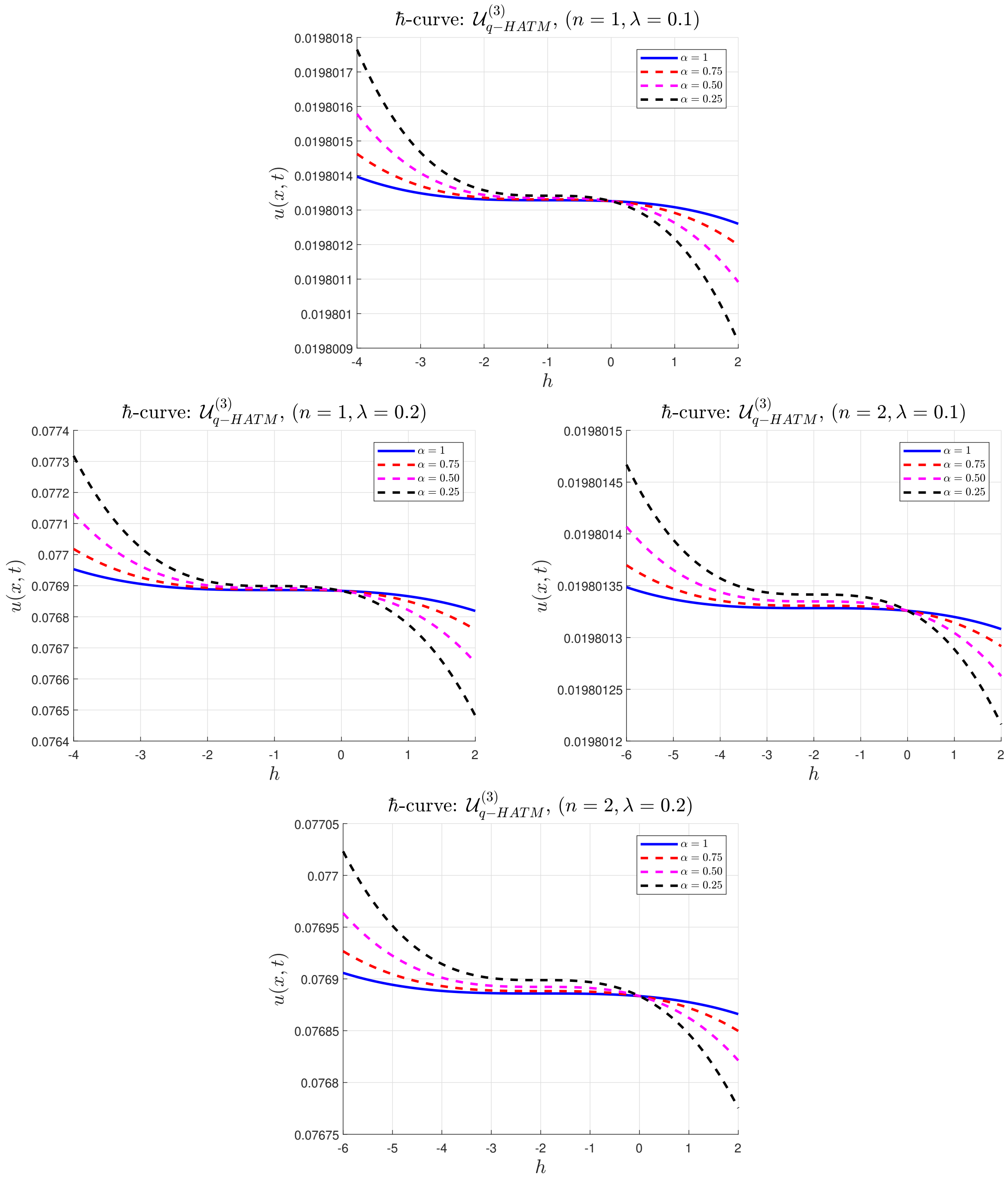

Fig. 11: $\hbar$-curves plot of $\mathcal{U}^{(3)}_{q-HATM}$ solution when $x = 1,\ t = 0.1$ with different n, λ and α for Example 1

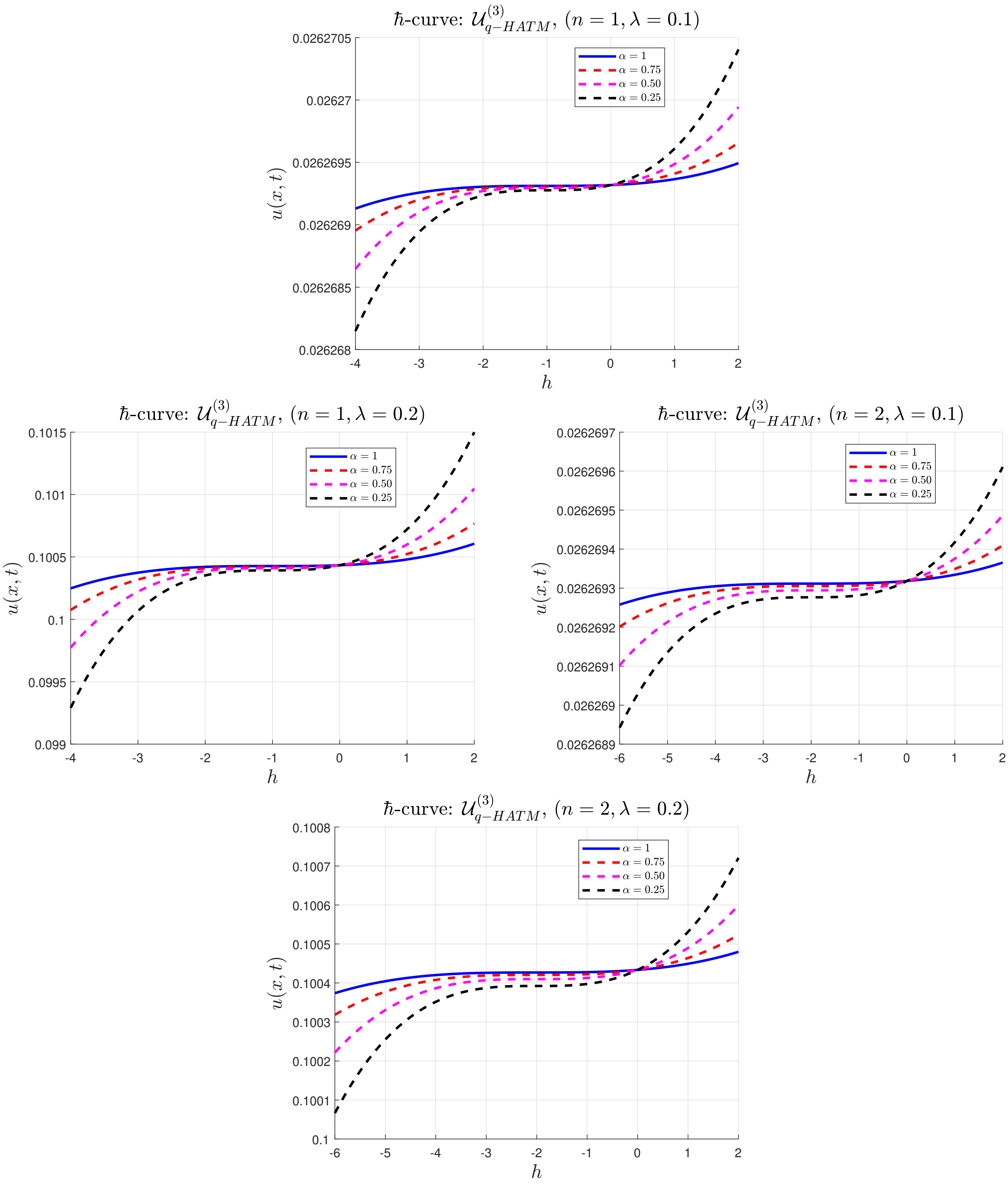

Fig. 12: $\hbar$-curves plot of $\mathcal{U}^{(3)}_{q-HATM}$ solution when $x = 1,\ t = 0.1$ with different n, λ and α for Example 2

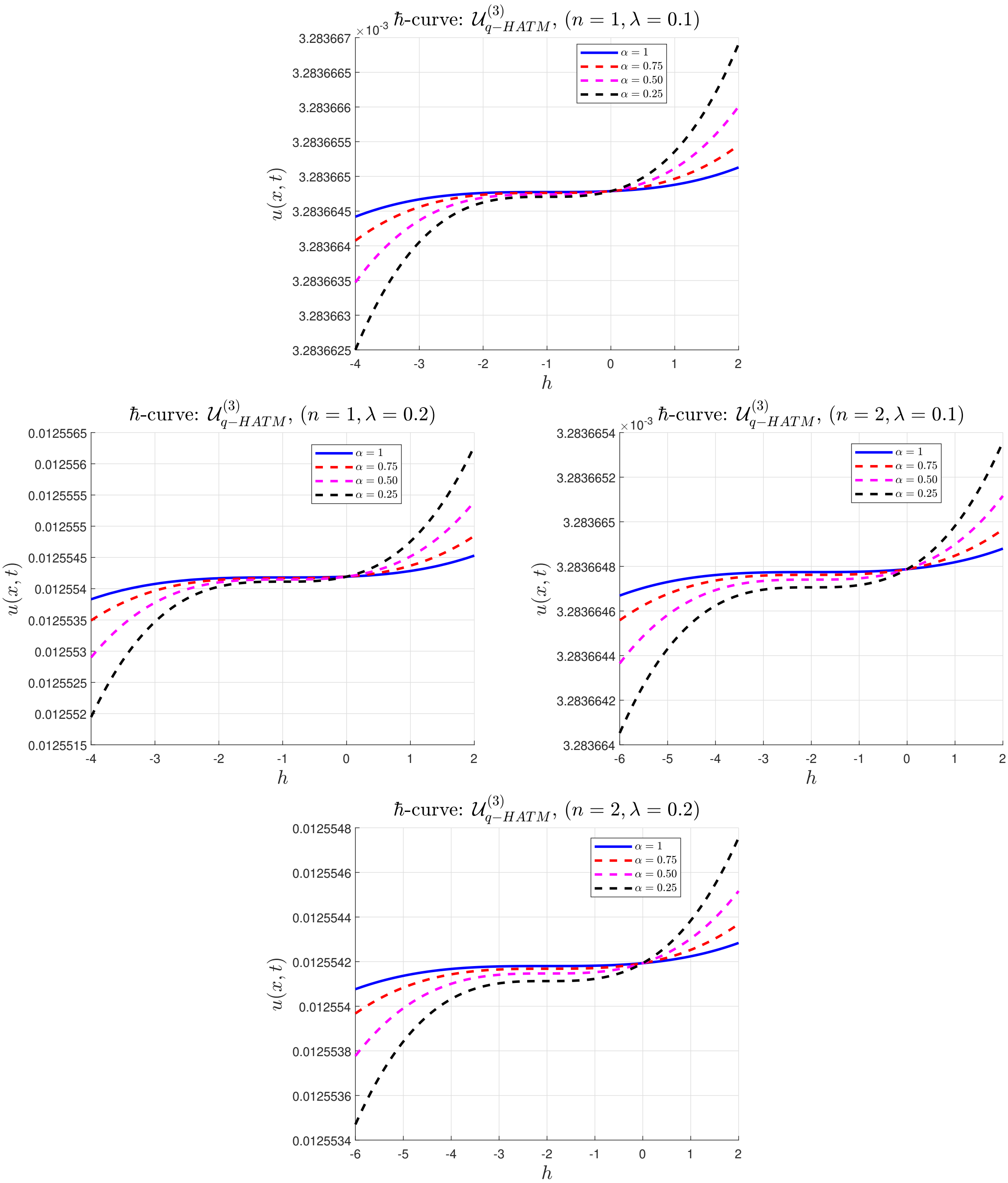

Fig. 13: $\hbar$-curves plot of $\mathcal{U}^{(3)}_{q-HATM}$ solution when $x = 1,\ t = 0.1$ with different n, λ and α for Example 3

Table 2: The relative error of FRDTM, q-HATM and the exact solution Eq. (42) when $\alpha = 1,\ n = 1,\ \hbar = -1$ and $\lambda = 0.1$ for Example 1

x	t	$\mathcal{U}^{(3)}_{FRDTM}$	$\mathcal{U}^{(3)}_{q-HATM}$	u_{Exact}	Relative Error (FRDTM)	Relative Error (q-HATM)
0.0	0.1	0.0200000000	0.0200000000	0.0200000000	$1.7347234760 \times 10^{-16}$	$1.7347234760 \times 10^{-16}$
	0.3	0.0200000000	0.0200000000	0.0200000000	0	0
	0.5	0.0200000000	0.0200000000	0.0200000000	$3.4694469520 \times 10^{-16}$	$3.4694469520 \times 10^{-16}$
1.0	0.1	0.0198013283	0.0198013283	0.0198013283	$1.7521283885 \times 10^{-16}$	$1.7521283885 \times 10^{-16}$
	0.3	0.0198013334	0.0198013334	0.0198013334	0	0
	0.5	0.0198013384	0.0198013384	0.0198013384	0	0
2.0	0.1	0.0192208645	0.0192208645	0.0192208645	0	0
	0.3	0.0192208742	0.0192208742	0.0192208742	$1.8050411812 \times 10^{-16}$	$1.8050411812 \times 10^{-16}$
	0.5	0.0192208839	0.0192208839	0.0192208839	$1.8050402692 \times 10^{-16}$	$1.8050402692 \times 10^{-16}$
3.0	0.1	0.0183027461	0.0183027461	0.0183027461	$1.8955882032 \times 10^{-16}$	$1.8955882032 \times 10^{-16}$
	0.3	0.0183027597	0.0183027597	0.0183027597	$1.8955867895 \times 10^{-16}$	$1.8955867895 \times 10^{-16}$
	0.5	0.0183027734	0.0183027734	0.0183027734	$3.7911707518 \times 10^{-16}$	$3.7911707518 \times 10^{-16}$
4.0	0.1	0.0171127840	0.0171127840	0.0171127840	$2.0274006515 \times 10^{-16}$	$2.0274006515 \times 10^{-16}$
	0.3	0.0171128007	0.0171128007	0.0171128007	0	0
	0.5	0.0171128173	0.0171128173	0.0171128173	$2.0273967075 \times 10^{-16}$	$2.0273967075 \times 10^{-16}$

Table 3: The relative error of FRDTM, q-HATM and the exact solution Eq. (42) when $\alpha = 1,\ n = 1,\ \hbar = -1$ and $\lambda = 0.2$ for Example 1

x	t	$\mathcal{U}^{(3)}_{FRDTM}$	$\mathcal{U}^{(3)}_{q-HATM}$	u_{Exact}	Relative Error (FRDTM)	Relative Error (q-HATM)
0.0	0.1	0.0799999995	0.0799999995	0.0799999995	$1.7347234876 \times 10^{-16}$	$1.7347234876 \times 10^{-16}$
	0.3	0.0799999952	0.0799999952	0.0799999952	$2.6020853711 \times 10^{-15}$	$2.6020853711 \times 10^{-15}$
	0.5	0.0799999866	0.0799999866	0.0799999866	$1.8735016684 \times 10^{-14}$	$1.8735016684 \times 10^{-14}$
1.0	0.1	0.0768859244	0.0768859244	0.0768859244	$3.6099683810 \times 10^{-16}$	$3.6099683810 \times 10^{-16}$
	0.3	0.0768908933	0.0768908933	0.0768908933	$1.8048675483 \times 10^{-15}$	$1.8048675483 \times 10^{-15}$
	0.5	0.0768958585	0.0768958585	0.0768958585	$1.3535632551 \times 10^{-14}$	$1.3535632551 \times 10^{-14}$
2.0	0.1	0.0684553638	0.0684553638	0.0684553638	$2.0272754456 \times 10^{-16}$	$2.0272754456 \times 10^{-16}$
	0.3	0.0684638840	0.0684638840	0.0684638840	$4.0540463104 \times 10^{-16}$	$4.0540463104 \times 10^{-16}$
	0.5	0.0684724021	0.0684724021	0.0684724021	$1.6214167915 \times 10^{-15}$	$1.6214167915 \times 10^{-15}$
3.0	0.1	0.0569312299	0.0569312299	0.0569312299	$1.2188203057 \times 10^{-16}$	$1.2188203057 \times 10^{-16}$
	0.3	0.0569412474	0.0569412474	0.0569412474	$1.3404664710 \times 10^{-15}$	$1.3404664710 \times 10^{-15}$
	0.5	0.0569512645	0.0569512645	0.0569512645	$9.9908106600 \times 10^{-15}$	$9.9908106600 \times 10^{-15}$
4.0	0.1	0.0447292793	0.0447292793	0.0447292793	0	0
	0.3	0.0447390118	0.0447390118	0.0447390118	$1.8611659835 \times 10^{-15}$	$1.8611659835 \times 10^{-15}$
	0.5	0.0447487450	0.0447487450	0.0447487450	$1.5816469905 \times 10^{-14}$	$1.5816469905 \times 10^{-14}$

Remark.

1. The behaviour of the obtained solutions at distinct λ is revealed in Figures 1-6 and the effect of λ can be clearly observed.
2. The solution profiles observed in Figures 7-10 is different for different α. These solution profiles help understand the nature of considered models for distinct values of fractional order. From these, one can see that the proposed problems show different behaviour and help the reader understand effect of the fractional order.
3. The $\hbar = -1$ used in the present investigation is chosen from the range of the optimal value. This range is obtained from the $\hbar$ curve plots Figures 11-13 with the aid of horizontal line guide. We could also see from these plots that $n = 2$ allows larger range of values for the optimal choice of $\hbar$.
4. In Tables 2-7, we observed that the obtained results using the two proposed techniques are in consistent with the exact solution for a special case when $\alpha = 1$ for different x and t.
5. We observe that the difference between the sets of numerical values obtained using FRDTM, q-HATM and the exact values are graphically almost indistinguishable.

6.The relative error in Tables 2-7 gives us an indication of how good the obtained numerical values by these two methods are in relative to the exact values.

Table 4: The relative error of FRDTM, q-HATM and the exact solution Eq. (49) when $\alpha = 1,\ n = 1,\ \hbar = -1$ and $\lambda = 0.1$ for Example 2

x	t	$\mathcal{U}^{(3)}_{FRDTM}$	$\mathcal{U}^{(3)}_{q-HATM}$	u_{Exact}	Relative Error (FRDTM)	Relative Error (q-HATM)
0.0	0.1	0.0266666667	0.0266666667	0.0266666667	0	0
	0.3	0.0266666667	0.0266666667	0.0266666667	0	0
	0.5	0.0266666667	0.0266666667	0.0266666667	0	0
1.0	0.1	0.0262693116	0.0262693116	0.0262693116	$1.3207224496 \times 10^{-16}$	$1.3207224496 \times 10^{-16}$
	0.3	0.0262692981	0.0262692981	0.0262692981	0	0
	0.5	0.0262692846	0.0262692846	0.0262692846	$1.3207238044 \times 10^{-16}$	$1.3207238044 \times 10^{-16}$
2.0	0.1	0.0251083730	0.0251083730	0.0251083730	$1.3817888347 \times 10^{-16}$	$1.3817888347 \times 10^{-16}$
	0.3	0.0251083471	0.0251083471	0.0251083471	0	0
	0.5	0.0251083212	0.0251083212	0.0251083212	0	0
3.0	0.1	0.0232721269	0.0232721269	0.0232721269	$1.4908164436 \times 10^{-16}$	$1.4908164436 \times 10^{-16}$
	0.3	0.0232720905	0.0232720905	0.0232720905	$1.4908187753 \times 10^{-16}$	$1.4908187753 \times 10^{-16}$
	0.5	0.0232720541	0.0232720541	0.0232720541	$1.4908211070 \times 10^{-16}$	$1.4908211070 \times 10^{-16}$
4.0	0.1	0.0208921959	0.0208921959	0.0208921959	$1.6606425508 \times 10^{-16}$	$1.6606425508 \times 10^{-16}$
	0.3	0.0208921515	0.0208921515	0.0208921515	$1.6606460790 \times 10^{-16}$	$1.6606460790 \times 10^{-16}$
	0.5	0.0208921071	0.0208921071	0.0208921071	$1.6606496072 \times 10^{-16}$	$1.6606496072 \times 10^{-16}$

Table 5: The relative error of FRDTM, q-HATM and the exact solution Eq. (49) when $\alpha = 1,\ n = 1,\ \hbar = -1$ and $\lambda = 0.2$ for Example 2

x	t	$\mathcal{U}^{(3)}_{FRDTM}$	$\mathcal{U}^{(3)}_{q-HATM}$	u_{Exact}	Relative Error (FRDTM)	Relative Error (q-HATM)
0.0	0.1	0.1066666648	0.1066666648	0.1066666648	$1.3010426303 \times 10^{-16}$	$1.3010426303 \times 10^{-16}$
	0.3	0.1066666495	0.1066666495	0.1066666495	$1.1449176785 \times 10^{-14}$	$1.1449176785 \times 10^{-14}$
	0.5	0.1066666189	0.1066666189	0.1066666189	$8.8861249813 \times 10^{-14}$	$8.8861249813 \times 10^{-14}$
1.0	0.1	0.1004269123	0.1004269123	0.1004269123	0	0
	0.3	0.1004136393	0.1004136393	0.1004136393	$8.4305783751 \times 10^{-15}$	$8.4305783751 \times 10^{-15}$
	0.5	0.1004003534	0.1004003534	0.1004003534	$6.5241960054 \times 10^{-14}$	$6.5241960054 \times 10^{-14}$
2.0	0.1	0.0835575085	0.0835575085	0.0835575085	$1.6608666367 \times 10^{-16}$	$1.6608666367 \times 10^{-16}$
	0.3	0.0835347750	0.0835347750	0.0835347750	$9.9679117882 \times 10^{-16}$	$9.9679117882 \times 10^{-16}$
	0.5	0.0835120341	0.0835120341	0.0835120341	$7.4779695869 \times 10^{-15}$	$7.4779695869 \times 10^{-15}$
3.0	0.1	0.0605057513	0.0605057513	0.0605057513	$1.1468155927 \times 10^{-16}$	$1.1468155927 \times 10^{-16}$
	0.3	0.0604790354	0.0604790354	0.0604790354	$8.0312552894 \times 10^{-15}$	$8.0312552894 \times 10^{-15}$
	0.5	0.0604523181	0.0604523181	0.0604523181	$6.0146252756 \times 10^{-14}$	$6.0146252756 \times 10^{-14}$
4.0	0.1	0.0361025183	0.0361025183	0.0361025183	$1.9219971983 \times 10^{-16}$	$1.9219971983 \times 10^{-16}$
	0.3	0.0360765700	0.0360765700	0.0360765700	$1.5002360943 \times 10^{-14}$	$1.5002360943 \times 10^{-14}$
	0.5	0.0360506244	0.0360506244	0.0360506244	$1.2453222177 \times 10^{-13}$	$1.2453222177 \times 10^{-13}$

Table 6: The relative error of FRDTM, q-HATM and the exact solution Eq. (56) when $\alpha = 1,\ n = 1,\ \hbar = -1$ and $\lambda = 0.1$ for Example 3

x	t	$\mathcal{U}^{(3)}_{FRDTM}$	$\mathcal{U}^{(3)}_{q-HATM}$	u_{Exact}	Relative Error (q-HATM)	Relative Error (q-HATM)
0.0	0.1	0.0033333333	0.0033333333	0.0033333333	0	0
	0.3	0.0033333333	0.0033333333	0.0033333333	$1.3010426070 \times 10^{-16}$	$1.3010426070 \times 10^{-16}$
	0.5	0.0033333333	0.0033333333	0.0033333333	0	0
1.0	0.1	0.0032836648	0.0032836648	0.0032836648	$1.3207221163 \times 10^{-16}$	$1.3207221163 \times 10^{-16}$
	0.3	0.0032836647	0.0032836647	0.0032836647	$1.3207221268 \times 10^{-16}$	$1.3207221268 \times 10^{-16}$
	0.5	0.0032836647	0.0032836647	0.0032836647	$1.3207221374 \times 10^{-16}$	$1.3207221374 \times 10^{-16}$
2.0	0.1	0.0031385482	0.0031385482	0.0031385482	0	0
	0.3	0.0031385482	0.0031385482	0.0031385482	0	0
	0.5	0.0031385481	0.0031385481	0.0031385481	$1.3817881778 \times 10^{-16}$	$1.3817881778 \times 10^{-16}$
3.0	0.1	0.0029090181	0.0029090181	0.0029090181	$1.4908152959 \times 10^{-16}$	$1.4908152959 \times 10^{-16}$
	0.3	0.0029090180	0.0029090180	0.0029090180	$1.4908153324 \times 10^{-16}$	$1.4908153324 \times 10^{-16}$
	0.5	0.0029090180	0.0029090180	0.0029090180	$1.4908153688 \times 10^{-16}$	$1.4908153688 \times 10^{-16}$
4.0	0.1	0.0026115272	0.0026115272	0.0026115272	$1.6606408143 \times 10^{-16}$	$1.6606408143 \times 10^{-16}$
	0.3	0.0026115271	0.0026115271	0.0026115271	0	0
	0.5	0.0026115270	0.0026115270	0.0026115270	0	0

Table 7: The relative error of FRDTM, q-HATM and the exact solution Eq. (56) when $\alpha = 1,\ n = 1,\ \hbar = -1$ and $\lambda = 0.2$ for Example 3

x	t	$\mathcal{U}^{(3)}_{FRDTM}$	$\mathcal{U}^{(3)}_{q-HATM}$	u_{Exact}	Relative Error (FRDTM)	Relative Error (q-HATM)
0.0	0.1	0.0133333333	0.0133333333	0.0133333333	0	0
	0.3	0.0133333333	0.0133333333	0.0133333333	0	0
	0.5	0.0133333333	0.0133333333	0.0133333333	0	0
1.0	0.1	0.0125541800	0.0125541800	0.0125541800	$1.3817895474 \times 10^{-16}$	$1.3817895474 \times 10^{-16}$
	0.3	0.0125541541	0.0125541541	0.0125541541	$1.3817923980 \times 10^{-16}$	$1.3817923980 \times 10^{-16}$
	0.5	0.0125541282	0.0125541282	0.0125541282	$1.3817952486 \times 10^{-16}$	$1.3817952486 \times 10^{-16}$
2.0	0.1	0.0104460869	0.0104460869	0.0104460869	$1.6606443149 \times 10^{-16}$	$1.6606443149 \times 10^{-16}$
	0.3	0.0104460425	0.0104460425	0.0104460425	0	0
	0.5	0.0104459981	0.0104459981	0.0104459981	$1.6606584278 \times 10^{-16}$	$1.6606584278 \times 10^{-16}$
3.0	0.1	0.0075648625	0.0075648625	0.0075648625	$2.2931328583 \times 10^{-16}$	$2.2931328583 \times 10^{-16}$
	0.3	0.0075648103	0.0075648103	0.0075648103	$1.1465743373 \times 10^{-16}$	$1.1465743373 \times 10^{-16}$
	0.5	0.0075647581	0.0075647581	0.0075647581	$2.2931644913 \times 10^{-16}$	$2.2931644913 \times 10^{-16}$
4.0	0.1	0.0045144113	0.0045144113	0.0045144113	$9.6065873452 \times 10^{-16}$	$9.6065873452 \times 10^{-16}$
	0.3	0.0045143607	0.0045143607	0.0045143607	$7.6853561636 \times 10^{-16}$	$7.6853561636 \times 10^{-16}$
	0.5	0.0045143100	0.0045143100	0.0045143100	$1.9213606132 \times 10^{-16}$	$1.9213606132 \times 10^{-16}$

6 Conclusion

We have successfully taken the advantages of the two reliable techniques namely the FRDTM and q-HATM, to obtain numerical solutions of the 7TFLK-dV, 7TFS-K and 7TFK-K equations. To reveal the applicability and consistency of the two methods used in this present work, we provided the error estimate in terms of relative error which are displayed in Tables 2-7. The tables and plots illustrated the two techniques used are very effective and consistent with the exact solution in a classical case when $\alpha = 1$. We revealed that the series solutions obtained by FRDTM and q-HATM both converge to the exact solution. Finally, we conclude that these two methods are accurate and easy to implement with minimal computation and can be considered powerful techniques in solving strongly nonlinear partial differential equations with non integer order.

Conflict of Interest

The authors declare that they have no conflict of interest.

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