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Existence of Solution to Dirichlet Problem for Generalized Lavrent'ev-Bitsadze Equation with a Fractional Derivative

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Abstract: In this paper, we solve the Dirichlet problem for a linear second-order partial differential equation with the Riemann-Liouville fractional derivative. When the order of fractional differentiation is an integer, the equation under consideration transforms into a mixed equation of the Lavrent'ev-Bitsadze type. Existence theorem is proved using the Fourier method and methods of special functions theory.

Keywords: Mittag-Leffler type function, generalized Lavrent'ev-Bitsadze equation with a fractional derivative, Dirichlet problem, Riemann-Liouville fractional differentiation operator.

1 Introduction

In the domain $\Omega = \{(x,y) : 0 < x < r, -a < y < b\}$, $a, b > 0$, we consider the equation

$$u_{xx}(x,y) - D^{\alpha}_{0y}u_y(x,y) = 0,\ 0 < \alpha < 1, y \neq 0, \tag{1}$$

with the Riemann-Liouville operator D^{α}_{0y} [1], [2]:

$$D^{\alpha}_{0y}v(x,y) = \begin{cases} \frac{\operatorname{sign} y}{\Gamma(-\alpha)} \int\limits_0^y |y-t|^{-\alpha-1}v(x,t)dt, & \alpha < 0, \\ v(x,y), & \alpha = 0, \\ \operatorname{sign}^n y \frac{\partial^n}{\partial y^n} D^{\alpha-n}_{0y}v(x,y), & n-1 < \alpha \leq n, n \in \mathbb{N}. \end{cases}$$

Note, as $\alpha = 1$ equation (1) transforms into a mixed equation

$$u_{xx}(x,y) - \operatorname{sign} y\, u_{yy}(x,y) = 0. \tag{2}$$

Differential equations of fractional order occur in mathematical modeling of physical processes in environmental systems with fractal geometry [1, Chap. 5]. Boundary value problems for linear partial differential equations with fractional order less than two are investigated in [3] and [4] (see also the References).

In [5], the Dirichlet problem is investigated for the generalized Laplace equation with the Caputo derivative. The Dirichlet problem for a nonlocal wave equation with the Caputo derivative is addressed in [6] and [7].

The Dirichlet problem for the Lavrent'ev-Bitsadze equation is handled in [8] and [9]. In [10], the Dirichlet problem is investigated for a mixed-type equation with a singular coefficient.

Assume $\Omega^- = \Omega \cap \{y < 0\}$, $\Omega^+ = \Omega \cap \{y > 0\}$. The function $u(x,y)$ belonging to the class $u(x,y) \in C(\bar{\Omega})$, $D^{\alpha-1}_{0y}u_y(x,y) \in C(\bar{\Omega}^-) \cap C(\bar{\Omega}^+)$, $u_{xx}(x,y)$, $D^{\alpha}_{0y}u_y(x,y) \in C(\Omega^- \cup \Omega^+)$ and satisfying equation (1) in $\Omega^- \cup \Omega^+$ is called here a regular solution to equation (1) in the domain Ω.

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This paper is organized as follows: In Section Two we solve the Dirichlet problem for equation (1) in the domain Ω. First, we prove the auxiliary lemma. Next, we solve the Dirichlet problem for equation (1) in the domain Ω^- and solve in the domain Ω^+ assuming the trace of the solution on $y=0$ is known. In addition, using the conjugation conditions, we have the trace of the desired solution in the line $y=0$. Section Three is devoted to conclusion. .

The present paper aims to prove the existence theorem to the Dirichlet problem for equation (1) in the domain Ω.

2 Dirichlet problem

Here, we consider the following problem: Find the regular solution to equation (1) in Ω satisfying the conditions

$$u(0,y)=u(r,y)=0, \quad -a\le y\le b, \tag{3}$$

$$u(x,-a)=\tau_a(x), \;\; u(x,b)=\tau_b(x), \quad 0\le x\le r, \tag{4}$$

where $\tau_a(x)$ and $\tau_b(x)$ are the given continuous functions in the segment $[0,r]$,

$$\tau_a(0)=\tau_a(r)=0, \;\; \tau_b(0)=\tau_b(r)=0,$$

$$\lim_{y\to 0+} D_{0y}^{\alpha-1}u_y=\lim_{y\to 0-} D_{0y}^{\alpha-1}u_y. \tag{5}$$

We know [11], the set of real zeros of a Mittag-Leffler type function

$$E_{\rho,\mu}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\rho k+\mu)}, \quad \rho>0, \mu\in\mathbb{C},$$

is finite for all $\rho<2$, $\mu\in\mathbb{C}$. In [12], it is proved as $\mu=\rho$ and $\mu=1$ the set is not empty.

Theorem 1. *Assume $\tau_a(x)\in C^2[0,r]$, $\tau_b(x)\in C^4[0,r]$, the functions $\tau_a'''(x)$ and $\tau_b^{(V)}(x)$ are piecewise continuous on the segment $[0,r]$, $\tau_a''(0)=\tau_a''(r)=0$, $\quad\tau_b''(0)=\tau_b''(r)=0$, $\tau_b^{(IV)}(0)=\tau_b^{(IV)}(r)=0$,*

$$\frac{b^{\alpha+1}}{r^2}\ge\frac{h}{\pi^2}, \tag{6}$$

$h=\max\{t\in\mathbb{R}: E_{\alpha+1,\alpha+1}(-t)E_{\alpha+1,1}(-t)=0\}$. *The above implies the existence of a regular solution to problem (1)–(5).*

First, prove the lemma.

Lemma 1. *Let $C(y,\lambda)=\frac{|y|^{\alpha}E_{\alpha+1,\alpha+1}(\lambda|y|^{\alpha+1})}{a^{\alpha}E_{\alpha+1,\alpha+1}(\lambda a^{\alpha+1})}$, $-a<y<0$. For any $\lambda>0$ the estimates*

$$0\le C(y,\lambda)\le 1, \tag{7}$$

$$0\le E_{\alpha+1,1}(\lambda_n|y|^{\alpha+1})-C(y,\lambda)E_{\alpha+1,1}(\lambda_n a^{\alpha+1})\le 1. \tag{8}$$

are valid.

Indeed, the function $C(y,\lambda)$ is the solution to the ordinary fractional differential equation

$$D_{0y}^{\alpha}v'(y)+\lambda v(y)=0, \quad -a<y<0. \tag{9}$$

At the point $y\in(-a,0)$ of the maximum value of the function v, we have [1]

$$D_{0y}^{\alpha+1}v\le\frac{v(y)|y|^{-\alpha-1}}{\Gamma(-\alpha)}.$$

Thus,

$$D_{0y}^{\alpha+1}v-\frac{v(0)}{|y|^{\alpha+1}\Gamma(-\alpha)}\le\frac{v(y)}{|y|^{\alpha+1}\Gamma(-\alpha)}-\frac{v(0)}{|y|^{\alpha+1}\Gamma(-\alpha)}.$$

Since $D_{0y}^{\alpha+1}v - \frac{|y|^{-\alpha-1}v(0)}{\Gamma(-\alpha)} = D_{0y}^{\alpha}D_{0y}^{1}v$, we obtain

$$D_{0y}^{\alpha}D_{0y}^{1}v \leq \frac{|y|^{-\alpha-1}}{\Gamma(-\alpha)}(v(y)-v(0)).$$

As $v(y)-v(0)>0$, $\Gamma(-\alpha)<0$, then $D_{0y}^{\alpha}D_{0y}^{1}v = -D_{0y}^{\alpha}v'(y)<0$, i. e. for any $-a<y<0$

$$D_{0y}^{\alpha}v'(y)>0.$$

Thus, we get

$$D_{0y}^{\alpha}v'(y)+\lambda v(y)>0,$$

that contradicts (9). Consequently, the greatest positive or the smallest negative value of the function $v(y)$ is as $y=-a$ or $y=0$. On the other hand,

$$C(0,\lambda)=0, C_n(-a,\lambda)=1,$$

implies estimate (7). Similarly, we can establish the validity of estimate (8). The lemma is valid.

Proof of the theorem 1. Find a solution for problem $(1)-(4)$ in the form of

$$u(x,y)=\theta(y)u(x,y)^{+}+\theta(-y)u(x,y)^{-},$$

where $\theta(y)=0, y<0, \theta(y)=1, y\geq 0$. Fhe functions $u^{-}(x,y)$ and $u^{+}(x,y)$ are the solutions to the problems:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial}{\partial y}D_{0y}^{\alpha-1}u_y=0, \tag{10}$$

$$u(0,y)=u(r,y)=0, \quad -a\leq y\leq 0, \tag{11}$$

$$u(x,-a)=\tau_a(x), \ u(x,0)=\tau_0(x), \quad 0\leq x\leq r, \tag{12}$$

and

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial}{\partial y}D_{0y}^{\alpha-1}u_y=0, \tag{13}$$

$$u(0,y)=u(r,y)=0, \quad 0\leq y\leq b, \tag{14}$$

$$u(x,0)=\tau_0(x), \ u(x,b)=\tau_b(x), \quad 0\leq x\leq r, \tag{15}$$

respectively,

where the function $\tau_0(x)$ is as yet unknown. Assuming that $\tau_0(x)$ is known we can write out the solutions to these problems. We assume that $\tau_0(x)\in C^2[0,r]$, $\tau_0'''(x)$ is a piecewise continuous function on the interval $[0,r]$.

A formal solution of problem (10)-(12) is

$$u(x,y)^{-}=\sum_{n=1}^{\infty}u_n(x,y)^{-}=\sum_{n=1}^{\infty}\Big\{\tau_{an}C(y,\lambda_n)+$$
$$\tau_{0n}\Big[E_{\alpha+1,1}(\lambda_n|y|^{\alpha+1})-C(y,\lambda_n)E_{\alpha+1,1}(\lambda_n a^{\alpha+1})\Big]\Big\}\sin(\sqrt{\lambda}_n x), \tag{16}$$

where

$$\tau_{0n}=\frac{2}{r}\int_0^r\tau_0(\xi)\sin(\sqrt{\lambda}_n\xi)d\xi,\ \tau_{an}=\frac{2}{r}\int_0^r\tau_a(\xi)\sin(\sqrt{\lambda}_n\xi)d\xi,\ \lambda_n=\left(\frac{\pi n}{r}\right)^2.$$

Taking into account the estimates (7), (8), and the Fourier coefficient properties

$$|\tau_{an}|=o(n^{-2}), n\to\infty, \tag{17}$$

$$|\tau_{0n}|=o(n^{-3}), n\to\infty, \tag{18}$$

we obtain

$$|u_n(x,y)^-| \le |\tau_{0n}| + |\tau_{an}| < K\frac{1}{n^2}.$$

It is known from [11] and [13], as $\lambda \to \infty$, the asymptotic representations

$$a^{\alpha}E_{\alpha+1,\alpha+1}(\lambda a^{\alpha+1}) = \frac{e^{\lambda^{\frac{1}{\alpha+1}}a}}{(\alpha+1)\lambda^{\frac{\alpha}{\alpha+1}}} + O\left(\lambda^{-2}\right), E_{\alpha+1,1}(\lambda a^{\alpha+1}) = \frac{e^{\lambda^{\frac{1}{\alpha+1}}a}}{\alpha+1} + O\left(\lambda^{-1}\right). \tag{19}$$

are valid. Hence,

$$C(y,\lambda_n) = O\left(\exp(\lambda_n^{\frac{1}{\alpha+1}}|y| - a)\right), 0 < |y| < a,$$

$$E_{\alpha+1,1}(\lambda_n|y|^{\alpha+1}) - C(y,\lambda_n)E_{\alpha+1,1}(\lambda_n a^{\alpha+1}) = O(1/\lambda_n).$$

Considering these two estimates and estimates (17) and (18), we can get

$$|\lambda_n u_n(x,y)^-| \le$$

$$\lambda_n\left(|\tau_{0n}|1/\lambda_n + |\tau_{an}|e^{\lambda_n^{\frac{1}{\alpha+1}}|y|-a}\right) \le N\left(\frac{1}{n^2} + e^{\lambda_n^{\frac{1}{\alpha+1}}|y|-a}\right),$$

N is some constant. Thus, the series $\sum\limits_{n=0}^{\infty}\frac{\partial^2}{\partial x^2}u_n(x,y)^- = -\sum\limits_{n=0}^{\infty}\lambda_n u_n(x,y)^-$, $\sum\limits_{n=0}^{\infty}D_{0y}^{\alpha}\frac{\partial}{\partial y}u_n(x,y)^- = \sum\limits_{n=0}^{\infty}\lambda_n u_n(x,y)^-$ converge absolutely and uniformly with respect to any closed subset of Ω^-. The functions $u_{xx}(x,y)^-, D_{0y}^{\alpha}\frac{\partial}{\partial y}u(x,y)^-$ are continuous in Ω^- since the common terms in these series are continuous and the uniformly convergent series of continuous functions defines the continuous functions. This proves the function $u(x,y)^-$ is the regular solution to equation (10) and satisfies conditions (11) and (12). Next, we construct a formal solution for problems (13)-(15) as

$$u(x,y)^+ = \sum_{n=1}^{\infty}\Big\{\tau_{bn}S(y,\lambda_n) +$$

$$+\tau_{0n}\left[E_{\alpha+1,1}(-\lambda_n y^{\alpha+1}) - S(y,\lambda_n)E_{\alpha+1,1}(-\lambda_n b^{\alpha+1})\right]\Big\}\sin(\sqrt{\lambda_n}x), \tag{20}$$

where

$$S(y,\lambda_n) = \frac{y^{\alpha}E_{\alpha+1,\alpha+1}(-\lambda_n y^{\alpha+1})}{b^{\alpha}E_{\alpha+1,\alpha+1}(-\lambda_n b^{\alpha+1})},$$

$$b^{\gamma}E_{\gamma+1,\gamma+1}(-\lambda_n b^{\gamma+1}) \ne 0. \tag{21}$$

For Mittag-Leffler type functions of series (20), as $\lambda_n \to \infty$, we have

$$E_{\alpha+1,1}(-\lambda_n b^{\alpha+1}) = \frac{b^{-\alpha-1}}{\lambda_n\Gamma(-\alpha)} + O(1/\lambda_n^2), \tag{22}$$

$$E_{\alpha+1,\alpha+1}(-\lambda_n b^{\alpha+1}) = -\frac{b^{-2\alpha-2}}{\lambda_n^2\Gamma(-\alpha-1)} + O(1/\lambda_n^3). \tag{23}$$

Subject to (21) by asymptotic (23), we get the estimate

$$|\lambda_n^2 b^{2\alpha+2}E_{\alpha+1,\alpha+1}(-\lambda_n b^{\alpha+1})| > C.$$

By (23), replacing b by y, we obtain

$$|E_{\alpha+1,\alpha+1}(-\lambda_n y^{\alpha+1})| \le \frac{M}{1+\lambda_n^2 y^{2(\alpha+1)}}, \quad \lambda_n y^{\alpha+1} \ge 0.$$

With these two estimates, we have

$$|S(y,\lambda_n)| \le \frac{y^{\alpha}\lambda_n^2 b^{2\alpha+2}}{b^{\alpha}(1+\lambda_n^2 y^{2(\alpha+1)})}. \tag{24}$$

Denote by $z^\varepsilon = \lambda_n^{2\varepsilon} y^{2\varepsilon(\alpha+1)}$. Then, $y^\alpha \lambda^2 = z^\varepsilon \lambda^{2-2\varepsilon} y^{\alpha-2\varepsilon(\alpha+1)}$. Therefore, by (24):

$$|S(y,\lambda_n)| \leq \lambda_n^{2-2\varepsilon} y^{\alpha-2\varepsilon(\alpha+1)} \frac{z^\varepsilon}{1+z}, \quad 0 \leq \varepsilon \leq 1.$$

Due to $\sup\limits_{z>0} \frac{z^\varepsilon}{1+z} = C(\varepsilon) = (1-\varepsilon)^{1-\varepsilon}\varepsilon^\varepsilon$ obtain

$$|S(y,\lambda_n)| \leq C(\varepsilon)\lambda_n^{2(1-\varepsilon)} y^{\alpha-2\varepsilon(\alpha+1)}, \quad 0 < \varepsilon < 1. \tag{25}$$

Employing (25), we get

$$|E_{\alpha+1,1}(-\lambda_n y^{\alpha+1}) - S(y,\lambda_n)E_{\alpha+1,1}(-\lambda_n b^{\alpha+1})| \leq \frac{M_1}{1+\lambda_n y^{\alpha+1}} + \frac{|S(y,\lambda_n)|}{1+\lambda_n b^{\alpha+1}} \leq$$

$$M_1 + M_2 b^{\alpha+1} C(\varepsilon) n^{2-4\varepsilon} y^{\alpha-2\varepsilon(\alpha+1)}, \quad y \geq 0, \quad \varepsilon < \frac{\alpha}{2(\alpha+1)}.$$

Hence,

$$|u(x,y)^+| \leq |\tau_{bn}| n^{4-4\varepsilon} + |\tau_{0n}|(M_1 + M_2 b^{\alpha+1} C(\varepsilon) n^{2-4\varepsilon} y^{\alpha-2\varepsilon(\alpha+1)}),$$

Since, by the assumption

$$|\tau_{bn}| = O(n^{-5}), n \to \infty, \tag{26}$$

we get

$$|u(x,y)^+| \leq n^{-4\varepsilon-1} + (M_1 n^{-3} + M_2 b^{\alpha+1} C(\varepsilon) n^{-4\varepsilon-1} y^{\alpha-2\varepsilon(\alpha+1)}).$$

This implies absolute and uniform convergence of the series (20). Using the estimates

$$E_{\alpha+1,1}(-\lambda_n y^{\alpha+1}) - S(y,\lambda_n)E_{\alpha+1,1}(-\lambda_n b^{\alpha+1}) = O(1/\lambda_n),$$

$$S(y,\lambda_n) = O(1),$$

following from (22) and (23), we obtain

$$|\lambda_n u_n(x,y)^+| \leq \lambda_n |\tau_{bn}| K + M\lambda_n |\tau_{0n}|(1/\lambda_n) < K_1 n^{-3},$$

K_1 is some constant.

Consequently, we can see the convergence of the series $\sum\limits_{n=1}^{\infty} \frac{\partial^2}{\partial x^2} u_n(x,y)^+ = -\sum\limits_{n=1}^{\infty} \lambda_n u_n(x,y)^+$, $\sum\limits_{n=1}^{\infty} D_{0y}^{\alpha} \frac{\partial}{\partial y} u_n(x,y)^+ = -\sum\limits_{n=1}^{\infty} \lambda_n u_n(x,y)^+$.

Using conjugation condition (5), find τ_{0n}. Applying the operator $D_{0y}^{\alpha-1}\frac{\partial}{\partial y}$ to function (16) and fractional integro-differentiation of Mittag-Leffler type functions

$$D_{at}^{\gamma} |t-a|^{\mu-1} E_{1/\rho}(\lambda |t-a|^\rho; \mu) = |t-a|^{\mu-\gamma-1} E_{1/\rho}(\lambda|t-a|^\rho; \mu-\gamma), \gamma \in \mathbb{R}, \tag{27}$$

$\mu > 0 \,\text{if}\, \gamma \notin \mathbb{N} \cup \{0\}, \text{and}\, \mu \in \mathbb{R}, \,\text{if}\, \gamma \in \mathbb{N} \cup \{0\},$

we obtain

$$D_{0y}^{\alpha-1}\frac{d}{dy}C(y,\lambda_n) = -\frac{E_{\alpha+1,1}(\lambda_n|y|^{\alpha+1})}{a^\alpha E_{\alpha+1,\alpha+1}(\lambda_n a^{\alpha+1})}, D_{0y}^{\alpha-1}\frac{d}{dy}E_{\alpha+1,1}(\lambda_n|y|^{\alpha+1}) = -|y|E_{\alpha+1,2}(\lambda_n|y|^{\alpha+1}).$$

and aiming $y \to 0$, on the left-hand side of (5), we obtain

$$\lim_{y\to 0} D_{0y}^{\alpha-1} u_y^+ = \sum_{n=1}^{\infty} \left\{ \frac{\tau_{bn}}{b^\alpha E_{\alpha+1,\alpha+1}(-\lambda_n b^{\alpha+1})} - \tau_{0n} \frac{E_{\alpha+1,1}(-\lambda_n b^{\alpha+1})}{b^\alpha E_{\alpha+1,\alpha+1}(-\lambda_n b^{\alpha+1})} \right\} \sin(\sqrt{\lambda_n} x).$$

Since

$$D_{0y}^{\alpha-1}\frac{d}{dy}S(y,\lambda_n) = \frac{E_{\alpha+1,1}(-\lambda_n y^{\alpha+1})}{b^\alpha E_{\alpha+1,\alpha+1}(-\lambda_n b^{\alpha+1})},$$

$$D_{0y}^{\alpha-1}\frac{d}{dy}E_{\alpha+1,1}(-\lambda_n y^{\alpha+1}) = yE_{\alpha+1,2}(-\lambda_n y^{\alpha+1}).$$

On the right-hand side of (5), we have

$$\lim_{y\to 0-} D_{0y}^{\alpha-1}u_y^- = \sum_{n=1}^{\infty}\left\{\frac{-\tau_{an}}{a^{\alpha}E_{\alpha+1,\alpha+1}(\lambda_n a^{\alpha+1})} + \tau_{0n}\frac{E_{\alpha+1,1}(\lambda_n a^{\alpha+1})}{a^{\alpha}E_{\alpha+1,\alpha+1}(\lambda_n a^{\alpha+1})}\right\}\sin(\sqrt{\lambda_n}x).$$

Therefore,

$$\tau_{0n} = \frac{a^{\alpha}E_{\alpha+1,\alpha+1}(\lambda_n a^{\alpha+1})}{\Delta}\tau_{bn} + \frac{b^{\alpha}E_{\alpha+1,\alpha+1}(-\lambda_n b^{\alpha+1})}{\Delta}\tau_{an},$$

where

$$\Delta = a^{\alpha}E_{\alpha+1,\alpha+1}(\lambda_n a^{\alpha+1})E_{\alpha+1,\alpha+1}(-\lambda_n b^{\alpha+1}) + b^{\alpha}E_{\alpha+1,\alpha+1}(-\lambda_n b^{\alpha+1})E_{\alpha+1,1}(\lambda_n a^{\alpha+1}).$$

Since $a^{\alpha}E_{\alpha+1,\alpha+1}(\lambda_n a^{\alpha+1}) > 0$, $E_{\alpha+1,1}(\lambda_n a^{\alpha+1}) > 0$, and due to (6), (23) $E_{\alpha+1,\alpha+1}(-\lambda_n b^{\alpha+1}) < 0$, $E_{\alpha+1,\alpha+1}(-\lambda_n b^{\alpha+1}) < 0$. Then, $\Delta \neq 0$. By asymptotic formulas (19) and (23) and estimates (17) and (26), we get

$$\tau_{0n} = O(\lambda_n)\,|\tau_{bn}| + O\left(\lambda_n^{\frac{\alpha}{\alpha+1}-1}\right)|\tau_{an}| = O\left(\frac{1}{n^3}\right).$$

Substituting the expression obtained above for τ_{0n} into (16) and (20), we get the required solution. This proves the theorem 1.

3 Conclusion

In this paper, the Dirichlet problem for a linear second-order partial differential equation with a fractional derivative is solved in a rectangular domain using the Fourier method. We proved values b and r that guarantee the existence of a solution in the whole domain Ω. In [14], we have proved the uniqueness of the solution.

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