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# Generalized Fractional Sturm-Liouville and Langevin Equations Involving Caputo Derivative with Nonlocal Conditions

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**Abstract:** The present paper addresses, generalized fractional Sturm-Liouville and Langevin equations with nonlocal conditions. Sufficient conditions of the existence and uniqueness for solutions of such equations investigated by applying fixed point theorems and Pachpatte's integral inequality. An example is presented to illustrate the main results.

**Keywords:** Fractional differential equations, Sturm-Liouville equation, Langevin equation, fixed point theorems.

## 1 Introduction

The study of fractional calculus has recently gained great momentum and has emerged as a significant research area. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes, see for instance the contributions [1]-[10] and references therein.

In 1908, Langevin first formulated Langevin equation which was found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [11]. Various generalizations of Langevin equation have been applied to illustrate dynamical processes in fractal medium. One generalization is the generalized Langevin equation [12]-[14] which incorporates the fractal and memory properties with a dissipative memory kernel into the Langevin equation. For more details we refer the reader to the references [15]-[19].

The Sturm-Liouville problem has applied in several applications of various areas, such as engineering and mathematics.For some recent evolutions on such equation see for examples [20,21,22,23] .The classical Sturm-Liouville problem for a linear differential equation of second order is a boundary value problem as follows:

$$\begin{cases} -\frac{d}{dt}[p(t)\frac{dx}{dt}]+v(t)x=\lambda r(t)x, \quad t\in[a,b],\\ a_1x(a)+a_2x'(a)=0,\\ b_1x(b)+b_2x'(b)=0.\end{cases}$$

Byszewski [24] and Deng [25] remarked that the nonlocal conditions can be applied in physics with better effect than the standard initial value problem to describe some physical phenomena. Nonlocal conditions were initiated by Byszewski [26] when he investigated the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems.
The present paper aims to investigate the existence and uniqueness of solutions for the following nonlocal generalized fractional Sturm-Liouville and Langevin equations

$$\begin{cases} {}^cD^{\beta}([p(t)^cD^{\alpha}+r(t)]x(t))=f\big(t,x(t)\big), t\in[0,T],\ \beta,\alpha\in(0,1],\\ x(0)+g(x)=x_0\in\mathbb{R},\ \ {}^cD^{\alpha}x(T)+h(x)=x_1\in\mathbb{R}, \end{cases} \tag{1}$$

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where ${}^cD^{\beta}, {}^cD^{\alpha}$ are the Caputo fractional derivatives, $p \in C(J,\mathbb{R})$ with $|p| \geq K > 0$, $r \in C(J,\mathbb{R}), g, h : C(J,\mathbb{R}) \to \mathbb{R}$ are continuous functions and $f \in C(J \times \mathbb{R}, \mathbb{R})$ is continuous function.

We note that:

–If $r(t) \equiv 0$ for all $t \in J$, the problem (1) is reduced to the fractional Sturm-Liouvilly equation of the form

$$\begin{cases} {}^cD^{\beta}(p(t)\, {}^cD^{\alpha}x(t)) = f\big(t, x(t)\big), t \in [0,T],\ \beta, \alpha \in (0,1], \\ x(0) + g(x) = x_0 \in \mathbb{R},\ \ {}^cD^{\alpha}x(T) + h(x) = x_1 \in \mathbb{R}. \end{cases} \tag{2}$$

–If $p(t) \equiv 1$ and $r(t) \equiv \lambda, \lambda \in \mathbb{R}$ for $t \in J$, the problem (1) is reduced to fractional Langevin equation of the form

$$\begin{cases} {}^cD^{\beta}({}^cD^{\alpha} + \lambda)x(t) = f\big(t, x(t)\big), t \in [0,T],\ \beta, \alpha \in (0,1], \\ x(0) + g(x) = x_0 \in \mathbb{R},\ \ {}^cD^{\alpha}x(T) + h(x) = x_1 \in \mathbb{R}. \end{cases} \tag{3}$$

This paper is organized as follows: Section Two addresses we set forth some preliminaries and hypotheses. In Section Three, we prove our main results by applying well known Banach contraction principle, Schaefer's fixed point theorem and Pachpatte's inequality. Section Four presents an illustrative example.

## 2 Preliminaries and hypotheses

This section handles the notations, preliminaries and hypotheses which satisfy our main results.

Let $C(J,\mathbb{R})$ be a Banach space of all continuous functions from $J$ to $\mathbb{R}$ endowed with the norm defined by

$$\|x\| = \sup\{\|x(t)\| : t \in J\}.$$

Define the constants $p_* = \inf_{t\in J} |p(t)|,\ p^* = \sup_{t\in J} |p(t)|$ and $r^* = \sup_{t\in J} |r(t)|$.

**Definition 1.** *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a suitable function $h$ is defined as*

$$I^{\alpha}_{a+}h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)ds,$$

*where $a \in \mathbb{R}$ and $\Gamma$ is the well known Gamma function.*

**Definition 2.** *The Caputo fractional derivative of order $\alpha > 0$ For a suitable function $h$ on the interval $[a,b]$, is given by*

$$({}^cD^{\alpha}_{a+}h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

*where $n = [\alpha]+1, [\alpha]$ denotes the integer part of $\alpha$.*

**Lemma 1.** *([3, 10]) Assume $\alpha > 0$, the differential equation ${}^cD^{\alpha}h(t) = 0$, has the following general solution $h(t) = c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1}$, where $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$, where $n = [\alpha]+1$.*

**Lemma 2.** *([3, 10]) Assume $\alpha > 0$,*

$$I^{\alpha}({}^cD^{\alpha}h)(t) = h(t) + c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1},$$

*for some $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$, where $n = [\alpha]+1$.*

For simplicity, we will use the following notation:

$$I^{\alpha}(f_x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s))ds$$

In order to establish the existence of solutions for nonlocal fractional differential equation (1), we present the following important lemma.

**Lemma 3.** *Let $0 < \beta, \alpha \leq 1$. A function $x \in C(J, \mathbb{R})$ is solution of the fractional integral equation*

$$\begin{aligned} x(t) =& I^{\alpha}\left(\frac{1}{p}I^{\beta}(f_x)\right)(t) - I^{\alpha}\left(\frac{r}{p}x\right)(t) \\ &- \left[I^{\beta}(f_x)(T) + p(T)h(x) - r(T)x(T) - p(T)x_1\right] I^{\alpha}\left(\frac{1}{p}\right)(t) - g(x) + x_0. \end{aligned} \tag{4}$$

*If and only if $x$ is a solution of the following nonlocal fractional differential equation*

$$\begin{cases} {}^{c}D^{\beta}([p(t)^{c}D^{\alpha} + r(t)]x(t)) = f_x(t), t \in [0,T], \beta, \alpha \in (0,1], \\ x(0) + g(x) = x_0, \ \ {}^{c}D^{\alpha}x(T) + h(x) = x_1, \end{cases} \tag{5}$$

*Proof.* Suppose that $x$ satisfies nonlocal fractional differential equation (5), then taking fractional integration of order $\beta$ to both sides of (5) and using Lemma 2 as well as definition of Riemann-Liouville fractional integral, we get

$$p(t)^{c}D^{\alpha}x(t) + r(t)x(t) = I^{\beta}(f_x)(t) - c_0,$$

where $c_0 \in \mathbb{R}$, which yields

$${}^{c}D^{\alpha}x(t) = \left(\frac{1}{p}I^{\beta}(f_x)\right)(t) - \left(\frac{r}{p}x\right)(t) - \frac{c_0}{p(t)}. \tag{6}$$

Applying the nonlocal condition ${}^{c}D^{\alpha}x(T) + h(x) = x_1$, we have

$$c_0 = I^{\beta}(f_x)(T) + p(T)h(x) - r(T)x(T) - p(T)x_1. \tag{7}$$

Again taking fractional integration of order $\alpha$ to both sides of (6), we get

$$x(t) = I^{\alpha}\left(\frac{1}{p}I^{\beta}(f_x)\right)(t) - I^{\alpha}\left(\frac{r}{p}x\right)(t) - c_0\, I^{\alpha}\left(\frac{1}{p}\right)(t) - c_1. \tag{8}$$

Now, applying the nonlocal condition $x(0) + g(x) = x_0$, we have

$$c_1 = g(x) - x_0. \tag{9}$$

Putting the values of $c_0$ and $c_1$ in (8), we obtain

$$\begin{aligned} x(t) =& I^{\alpha}\left(\frac{1}{p}I^{\beta}(f_x)\right)(t) - I^{\alpha}\left(\frac{r}{p}x\right)(t) \\ &- \left[I^{\beta}(f_x)(T) + p(T)h(x) - r(T)x(T) - p(T)x_1\right] I^{\alpha}\left(\frac{1}{p}\right)(t) - g(x) + x_0. \end{aligned} \tag{10}$$

Conversely, suppose that $x$ satisfies the equation (4), therefore, we have, $x(0) + g(x) = x_0$, ${}^{c}D^{\alpha}x(T) + h(x) = x_1$ and the nonlocal fractional differential equation (5) is also satisfied for $t \in [0,T]$.

**Lemma 4.** *(Schaefer's fixed point theorem) Let $F : X \to X$ be a completely continuous operator. If the set $E(F) = \{x \in X : x = \gamma Fx$ for some $\gamma \in [0,1]\}$ is bounded, $F$ has fixed points.*

**Theorem 1.** *(Pachpatte's inequality, [27]) Let $u(t) \in C(I = [\theta, \phi], \mathbb{R}_+)$, let $a(t,s), b(t,s) \in C(I \times I, \mathbb{R}_+)$, assume that $a(t,s), b(t,s)$ are nondecreasing in $t$, for each $s \in I$ and suppose that*

$$u(t) \leq c + \int_{\theta}^{t} a(t,s)u(s)ds + \int_{\theta}^{\phi} b(t,s)u(s)ds, \ for\ t \in I,$$

*where $c$ is a constant. If*

$$q(t) = \int_{\theta}^{\phi} b(t,s)\exp\left(\int_{\theta}^{s} a(s,\sigma)d\sigma\right)ds < 1,$$

*then*

$$u(t) \leq \frac{c}{1-q(t)}\exp\left(\int_{\theta}^{t} a(t,s)ds\right), \ for\ t \in I.$$

In order to discuss the main results, we list the following set hypotheses:

- (H1)There exists a constant $N_f > 0$, such that

$$|f(t,x(t)) - f(t,y(t))| \leq N_f |x(t) - y(t)|,$$

for each $t \in J$ and $x, y \in \mathbb{R}$.
- (H2)There exist constants $N_g, N_h > 0$, such that $|g(x) - g(y)| \leq N_g \|x - y\|$, $|h(x) - h(y)| \leq N_h \|x - y\|$, for each $x, y \in C(J, \mathbb{R})$.
- (H3)There exists a constant $a_f > 0$ such that $|f(t,x(t))| \leq a_f(1 + |x(t)|)$, where $x, y \in \mathbb{R}$ and there exist the constants $a_g, a_h > 0$ such that $|g(x)| \leq a_g(1 + \|x\|)$, $|h(x)| \leq a_h(1 + \|x\|)$, for arbitrary $x \in C(J, \mathbb{R})$.

## 3 Main result

In this section, we discuss the main results related to nonlocal fractional Sturm-Liouville and Langevin equations given in (1).

Define the operator $F : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ as follows:

$$\begin{aligned}(F(x))(t) = & I^{\alpha}\left(\frac{1}{p} I^{\beta}(f_x)\right)(t) - I^{\alpha}\left(\frac{r}{p} x\right)(t) \\ & - \left[I^{\beta}(f_x)(T) + p(T)h(x) - r(T)x(T) - p(T)x_1\right] I^{\alpha}\left(\frac{1}{p}\right)(t) - g(x) + x_0,\end{aligned} \tag{11}$$

and let the constants

$$\omega_1 = \frac{N_f}{p_* \beta \Gamma(\beta)} T^{\beta+\alpha} \left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} + \frac{1}{\alpha\Gamma(\alpha)}\right] + \frac{2r^* T^{\alpha}}{p_* \alpha \Gamma(\alpha)} + \frac{p^* N_h}{\alpha \Gamma(\alpha) p_*} T^{\alpha} + N_g \tag{12}$$

$$\omega_2 = \frac{L}{p_* \beta \Gamma(\beta)} T^{\beta+\alpha} \left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} + \frac{1}{\alpha\Gamma(\alpha)}\right] + \frac{p^* T^{\alpha}}{p_* \alpha \Gamma(\alpha)} [|h(0)| + |x_1|] + |g(0)| + |x_0|. \tag{13}$$

**Theorem 2.***Assume that the hypotheses (H1) and (H2) hold. If*

$$\omega_1 < 1. \tag{14}$$

*Then, the nonlocal fractional differential equation (1) has a unique solution on $J$.*

*Proof.*The fixed point of the operator $F$ defined in (11) is the solution of the nonlocal fractional differential equation (1). We shall use the Banach contraction principle to prove that $F$ has a fixed point.
Define, $\sup_{t \in J} |f(t,0)| = L < \infty$ and let $B_R = \{x \in C(J,X) : \|x\| \leq R\}$, where

$$R \geq \frac{\omega_2}{1 - \omega_1}.$$

In two steps, the proof will follows
**Step 1.** $F(x) \in B_R$ for every $x \in B_R$.

For $x \in B_R$ and all $t \in J$, using (H1) and (H2), we have

$$
\begin{aligned}
|(F(x))(t)| \leq\ & I^{\alpha}\left(\frac{1}{|p|} I^{\beta}(|f_x|)\right)(t) + I^{\alpha}\left(\frac{|r|}{|p|}|x|\right)(t) \\
& + \left[I^{\beta}(|f_x|)(T) + |p(T)|\ |h(x)| + |r(T)||x(T)| + |p(T)|\ |x_1|\right] I^{\alpha}\left(\frac{1}{|p|}\right)(t) \\
& + |g(x)| + |x_0| \\
\leq\ & I^{\alpha}\left(\frac{1}{|p|} I^{\beta}(|f_x - f_0| + |f_0|)\right)(t) + I^{\alpha}\left(\frac{|r|}{|p|}|x|\right)(t) \\
& + \left[I^{\beta}(|f_x - f_0| + |f_0|)(T) + |p(T)|\ (|h(x) - h(0)| + |h(0)|) + |r(T)||x(T)|\right. \\
& \left. + |p(T)|\ |x_1|\right] I^{\alpha}\left(\frac{1}{|p|}\right)(t) + |g(x) - g(0)| + |g(0)| + |x_0| \\
\leq\ & \frac{(N_f R + L)\Gamma(\beta+1)}{p_* \beta \Gamma(\beta)\Gamma(\beta+\alpha+1)}\, t^{\beta+\alpha} + \frac{r^* R}{p_* \alpha \Gamma(\alpha)}\, t^{\alpha} + \frac{(N_f R + L)}{p_* \beta \Gamma(\beta) \alpha \Gamma(\alpha)}\, T^{\beta}\, t^{\alpha} \\
& + \frac{p^*\,(N_h R + |h(0)|)}{p_* \alpha \Gamma(\alpha)}\, t^{\alpha} + \frac{r^* R}{p_* \alpha \Gamma(\alpha)}\, t^{\alpha} + \frac{p^*\ |x_1|}{p_* \alpha \Gamma(\alpha)}\, t^{\alpha} + N_g R + |g(0)| + |x_0| \\
\leq\ & \frac{N_f R}{p_* \beta \Gamma(\beta)}\, T^{\beta+\alpha}\left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} + \frac{1}{\alpha\Gamma(\alpha)}\right] \\
& + \frac{2 r^* R T^{\alpha}}{p_* \alpha \Gamma(\alpha)} + \frac{p^*\ N_h R}{\alpha \Gamma(\alpha) p_*}\, T^{\alpha} + N_g R \\
& + \frac{L}{p_* \beta \Gamma(\beta)}\, T^{\beta+\alpha}\left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} + \frac{1}{\alpha\Gamma(\alpha)}\right] \\
& + \frac{p^* T^{\alpha}}{p_* \alpha \Gamma(\alpha)}\,[|h(0)| + |x_1|] + |g(0)| + |x_0| \\
\leq\ & \omega_1\, R + \omega_2 \leq R.
\end{aligned}
$$

Thus, $\|F(x)\| \leq R$ and we conclude that for all $x \in B_R$, $F(x) \in B_R$, that is, $F : B_R \to B_R$.

**Step 2.** The operator $F$ is contraction mapping on $B_R$.
For $x,y\in B_R$ and any $t\in J$, using (H1) and (H2) , we have

$$\begin{aligned}
&|(F(x))(t)-(F(y))(t)|\\
&\le I^{\alpha}\left(\frac{1}{|p|}I^{\beta}(|f_x-f_y|)\right)(t)+I^{\alpha}\left(\frac{|r|}{|p|}|x-y|\right)(t)\\
&\quad+\Big[I^{\beta}(|f_x-f_y|)(T)+|p(T)|\;|h(x)-h(y)|\\
&\quad+|r(T)||x(T)-y(T)|\Big]\;I^{\alpha}\left(\frac{1}{|p|}\right)(t)+|g(x)-g(y)|\\
&\le \frac{1}{p_*}I^{\alpha}(I^{\beta}|f_x-f_y|)(t)+\frac{r^*}{p_*}\|x-y\|I^{\alpha}(1)(t)\\
&\quad+\Big[I^{\beta}(|f_x-f_y|)(T)+p^*\;|h(x)-h(y)|+r^*\|x-y\|\Big]\frac{1}{p_*}\;I^{\alpha}(1)(t)\\
&\quad+|g(x)-g(y)|\\
&\le \frac{N_f\|x-y\|\Gamma(\beta+1)}{p_*\beta\Gamma(\beta)\Gamma(\beta+\alpha+1)}\;t^{\beta+\alpha}+\frac{r^*}{p_*\alpha\Gamma(\alpha)}\|x-y\|\;t^{\alpha}\\
&\quad+\Big[\frac{N_f\|x-y\|}{\beta\Gamma(\beta)}T^{\beta}+p^*\;N_h\|x-y\|+r^*\|x-y\|\Big]\frac{1}{\alpha\Gamma(\alpha)p_*}\;t^{\alpha}+N_g\|x-y\|\\
&\le \frac{N_f\|x-y\|\Gamma(\beta+1)}{p_*\beta\Gamma(\beta)\Gamma(\beta+\alpha+1)}\;T^{\beta+\alpha}+\frac{r^*}{p_*\alpha\Gamma(\alpha)}\|x-y\|\;T^{\alpha}\\
&\quad+\frac{N_f\|x-y\|}{p_*\;\alpha\beta\Gamma(\alpha)\Gamma(\beta)}T^{\beta+\alpha}+\frac{p^*\;N_h}{\alpha\Gamma(\alpha)p_*}\|x-y\|\;T^{\alpha}+\frac{r^*}{\alpha\Gamma(\alpha)p_*}\|x-y\|\;T^{\alpha}+N_g\|x-y\|\\
&\le \Big[\frac{N_f\Gamma(\beta+1)}{p_*\beta\Gamma(\beta)\Gamma(\beta+\alpha+1)}\;T^{\beta+\alpha}+\frac{2r^*}{p_*\alpha\Gamma(\alpha)}\;T^{\alpha}\\
&\quad+\frac{N_f}{p_*\;\alpha\beta\Gamma(\alpha)\Gamma(\beta)}T^{\beta+\alpha}+\frac{p^*\;N_h}{\alpha\Gamma(\alpha)p_*}\;T^{\alpha}+N_g\Big]\|x-y\|\\
&\le \Big[\frac{N_f\;T^{\beta+\alpha}}{p_*\beta\Gamma(\beta)}\left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}+\frac{1}{\alpha\Gamma(\alpha)}\right)+\frac{2r^*}{p_*\alpha\Gamma(\alpha)}\;T^{\alpha}\\
&\quad+\frac{p^*\;N_h}{\alpha\Gamma(\alpha)p_*}\;T^{\alpha}+N_g\Big]\|x-y\|\\
&\le \omega_1\;\|x-y\|.
\end{aligned}$$

Hence, we get

$$\|F(x)-F(y)\|\le\omega_1\;\|x-y\|.$$

Since $\omega_1<1$, so $F$ is contraction. In view of Banach fixed theorem, the operator $F$ has a unique fixed point which is solution of the nonlocal fractional differential equation (1).

If $r(t)\equiv 0$ for $t\in J$, we have $r^*=0$ and obtain the following result:

**Corollary 1.** *Suppose that the hypotheses (H1) and (H2) hold. If*

$$\frac{N_f}{p_*\beta\Gamma(\beta)}\;T^{\beta+\alpha}\left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}+\frac{1}{\alpha\Gamma(\alpha)}\right]+\frac{p^*\;N_h}{\alpha\Gamma(\alpha)p_*}\;T^{\alpha}+N_g<1.$$

*Then the problem (2) has a unique solution on $J$.*

If $p(t)\equiv 1$ and $r(t)\equiv\lambda$ for $t\in J$, and $\lambda\in\mathbb{R}$, we have $p_*=p^*=1, r^*=|\lambda|$. We also have the following result:

**Corollary 2.** *Suppose that the hypotheses (H1) and (H2) hold. If*

$$\frac{N_f}{\beta\Gamma(\beta)}\;T^{\beta+\alpha}\left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}+\frac{1}{\alpha\Gamma(\alpha)}\right]+\frac{2|\lambda|T^{\alpha}}{\alpha\Gamma(\alpha)}+\frac{N_h}{\alpha\Gamma(\alpha)}\;T^{\alpha}+N_g<1.$$

*Then the problem (3) has a unique solution on $J$.*

Our second main result is based on the well known Schaefer's fixed point theorem.

**Theorem 3.** *Assume that the hypothesis (H3) holds. If*

$$Q = \int_0^T \frac{a_f T^\alpha}{p_* \alpha \Gamma(\alpha)\Gamma(\beta)} \frac{1}{1 - a_g - \frac{2r^* T^\alpha}{p_* \alpha\Gamma(\alpha)} - \frac{a_h p^* T^\alpha}{p_* \alpha \Gamma(\alpha)}} (T-s)^{\beta-1}$$

$$\times \exp\left( \int_0^s \frac{a_f}{p_* \Gamma(\alpha+\beta)} \frac{1}{1 - a_g - \frac{2r^* T^\alpha}{p_* \alpha\Gamma(\alpha)} - \frac{a_h p^* T^\alpha}{p_* \alpha\Gamma(\alpha)}} (s-\sigma)^{\alpha+\beta-1} d\sigma \right) ds. < 1,$$

*where*

$$a_g + \frac{2r^* T^\alpha}{p_* \alpha \Gamma(\alpha)} + \frac{a_h p^* T^\alpha}{p_* \alpha \Gamma(\alpha)} \neq 1.$$

*Then the nonlocal fractional problem (1) has at least one solution on J.*

*Proof.* By transforming the nonlocal fractional differential equation (1) into a fixed point problem, let the operator $F : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ as defined in (11).

We will subdivide the proof into four steps.

**Step 1.** $F$ is continuous operator.
Assume the sequence $\{x_n\}$ convergent to $x$ in $C(J,\mathbb{R})$. Then for all $t \in J$, we get

$$\begin{aligned}
|(F(x_n))(t) - (F(x))(t)| &\leq I^\alpha \left( \frac{1}{|p|} I^\beta (|f_{x_n} - f_x|) \right)(t) + I^\alpha \left( \frac{|r|}{|p|} |x_n - x| \right)(t) \\
&\quad + \Big[ I^\beta(|f_{x_n} - f_x|)(T) + |p(T)|\; |h(x_n) - h(x_n)| \\
&\quad + |r(T)||x_n(T) - x_n(T)| \Big] I^\alpha \left( \frac{1}{|p|} \right)(t) + |g(x_n) - g(x)| \\
&\leq \frac{\Gamma(\beta+1)\|f(.,x_n(.)) - f(.,x(.))\|}{\beta\,\Gamma(\beta)\Gamma(\beta+\alpha+1)p_*} t^{\beta+\alpha} + \frac{r^*\|x_n - x\|}{p_*\alpha\Gamma(\alpha)} t^\alpha \\
&\quad + \Big[ \frac{\|f(.,x_n(.)) - f(.,x(.))\|}{\beta\,\Gamma(\beta)} T^\beta + p^*\; |h(x_n) - h(x)| \\
&\quad + r^*\|x_n - x\| \Big] \frac{1}{\alpha\Gamma(\alpha)p_*}\, t^\alpha + |g(x_n) - g(x)| \\
&\leq \frac{\Gamma(\beta+1)\|f(.,x_n(.)) - f(.,x(.))\|}{\beta\,\Gamma(\beta)\Gamma(\beta+\alpha+1)p_*} T^{\beta+\alpha} + \frac{r^*\|x_n - x\|}{p_*\alpha\Gamma(\alpha)} T^\alpha \\
&\quad + \frac{1}{\alpha\Gamma(\alpha)p_*} \frac{\|f(.,x_n(.)) - f(.,x(.))\|}{\beta\,\Gamma(\beta)} T^{\beta+\alpha} + p^* \frac{|h(x_n) - h(x)|}{\alpha\Gamma(\alpha)p_*}\, T^\alpha \\
&\quad + \frac{r^*\|x_n - x\|}{\alpha\Gamma(\alpha)p_*}\, T^\alpha + |g(x_n) - g(x)|.
\end{aligned}$$

Taking supremum, we get

$$\begin{aligned}
\|Fx_n - Fx\| &\leq \frac{\Gamma(\beta+1)\|f(.,x_n(.)) - f(.,x(.))\|}{\beta\,\Gamma(\beta)\Gamma(\beta+\alpha+1)p_*} T^{\beta+\alpha} + \frac{r^*\|x_n - x\|}{p_*\alpha\Gamma(\alpha)} T^\alpha \\
&\quad + \frac{1}{\alpha\Gamma(\alpha)p_*} \frac{\|f(.,x_n(.)) - f(.,x(.))\|}{\beta\,\Gamma(\beta)} T^{\beta+\alpha} + p^* \frac{|h(x_n) - h(x)|}{\alpha\Gamma(\alpha)p_*}\, T^\alpha \\
&\quad + \frac{r^*\|x_n - x\|}{\alpha\Gamma(\alpha)p_*}\, T^\alpha + |g(x_n) - g(x)|,
\end{aligned}$$

since $f, g, h$ are continuous functions, which yield that

$$\|Fx_n - Fx\| \to 0 \text{ as } n \to \infty.$$

Thus, $F$ is continuous operator.
**Step 2.** The operator $F$ maps bounded sets into itself in $C(J,\mathbb{R})$.
It is sufficient to prove that for any $\eta > 0$, there exists a $\ell > 0$ such that for each $x \in B_\eta = \{x \in C(J,\mathbb{R}) : \|x\| \le \eta\}$, we have $\|Fx\| \le l$. For all $t \in J$, by using (H3), we obtain

$$\begin{aligned}
|(F(x))(t)| \le\, & I^\alpha\left(\frac{1}{|p|}I^\beta(|f_x|)\right)(t) + I^\alpha\left(\frac{|r|}{|p|}|x|\right)(t)\\
& + \left[I^\beta(|f_x|)(T) + |p(T)|\ |h(x)| + |r(T)||x(T)| + |p(T)|\ |x_1|\right] I^\alpha\left(\frac{1}{|p|}\right)(t)\\
& + |g(x)| + |x_0|\\
\le\, & I^\alpha\left(\frac{1}{|p|}I^\beta(a_f(1+|x|))\right)(t) + I^\alpha\left(\frac{|r|}{|p|}|x|\right)(t)\\
& + \Big[I^\beta(a_f(1+|x|))(T) + |p(T)|\ a_h(1+\|x\|) + |r(T)||x(T)|\\
& + |p(T)|\ |x_1|\Big] I^\alpha\left(\frac{1}{|p|}\right)(t) + a_g(1+\|x\|) + |x_0|\\
\le\, & \frac{(a_f(1+\|x\|))\Gamma(\beta+1)}{p_*\beta\Gamma(\beta)\Gamma(\beta+\alpha+1)}\,T^{\beta+\alpha} + \frac{r^*\|x\|}{p_*\alpha\Gamma(\alpha)}\,T^\alpha + \frac{(a_f(1+\|x\|))}{p_*\beta\Gamma(\beta)\alpha\Gamma(\alpha)}\,T^{\beta+\alpha}\\
& + \frac{p^*a_h(1+\|x\|)}{p_*\alpha\Gamma(\alpha)}\,T^\alpha + \frac{r^*\|x\|}{p_*\alpha\Gamma(\alpha)}\,T^\alpha + \frac{p^*\ |x_1|}{p_*\alpha\Gamma(\alpha)}\,T^\alpha + a_g(1+\|x\|) + |x_0|\\
\le\, & \frac{a_f(1+\eta)}{p_*\beta\Gamma(\beta)}\,T^{\beta+\alpha}\left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} + \frac{1}{\alpha\Gamma(\alpha)}\right]\\
& + \frac{2r^*\eta T^\alpha}{p_*\alpha\Gamma(\alpha)} + \frac{p^*a_h(1+\eta)}{p_*\alpha\Gamma(\alpha)}\,T^\alpha + \frac{p^*\ |x_1|}{p_*\alpha\Gamma(\alpha)}\,T^\alpha + a_g(1+\eta) + |x_0|\\
\le\, & \ell,
\end{aligned}$$

where

$$\begin{aligned}
\ell :=& \frac{a_f(1+\eta)}{p_*\beta\Gamma(\beta)}\,T^{\beta+\alpha}\left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} + \frac{1}{\alpha\Gamma(\alpha)}\right]\\
& + \frac{2r^*\eta T^\alpha}{p_*\alpha\Gamma(\alpha)} + \frac{p^*a_h(1+\eta)}{p_*\alpha\Gamma(\alpha)}\,T^\alpha + \frac{p^*\ |x_1|}{p_*\alpha\Gamma(\alpha)}\,T^\alpha + a_g(1+\eta) + |x_0|.
\end{aligned}$$

Therefore, we get

$\|(F(x))(t)\| \le \ell$ and hence $\|Fx\| \le \ell$.
**Step 3.** $F$ maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.

Let $0 \le t_1 \le t_2 \le T, x \in B_\eta$. Using (H3), again we get

$$
\begin{aligned}
&|(F(x))(t_2) - (F(x))(t_1)| \\
&= \left| I^\alpha \left( \frac{1}{p} I^\beta (f_x) \right)(t_2) - I^\alpha \left( \frac{r}{p} x \right)(t_2) \right. \\
&\quad - \left[ I^\beta (f_x)(T) + p(T)h(x) - r(T)x(T) - p(T)x_1 \right] I^\alpha \left( \frac{1}{p} \right)(t_2) \\
&\quad - I^\alpha \left( \frac{1}{p} I^\beta (f_x) \right)(t_1) + I^\alpha \left( \frac{r}{p} x \right)(t_1) \\
&\quad \left. + \left[ I^\beta (f_x)(T) + p(T)h(x) - r(T)x(T) - p(T)x_1 \right] I^\alpha \left( \frac{1}{p} \right)(t_1) \right| \\
&\le \left| I^\alpha \left( \frac{1}{p} I^\beta (f_x) \right)(t_2) - I^\alpha \left( \frac{1}{p} I^\beta (f_x) \right)(t_1) \right| + \left| I^\alpha \left( \frac{r}{p} x \right)(t_2) - I^\alpha \left( \frac{r}{p} x \right)(t_1) \right| \\
&\quad + \left[ \frac{a_f (1+\eta)}{\beta \Gamma(\beta)} T^\beta + p^* a_h (1+\eta) + r^* \eta + p^* |x_1| \right] . \left| I^\alpha \left( \frac{1}{p} \right)(t_2) - I^\alpha \left( \frac{1}{p} \right)(t_1) \right| \\
&\le \frac{a_f (1+\eta)}{p_* \Gamma(\alpha+\beta)} \left| \int_0^{t_2} (t_2 - s)^{\alpha+\beta-1} ds - \int_0^{t_1} (t_1 - s)^{\alpha+\beta-1} ds \right| \\
&\quad + \frac{r^* \eta}{p_* \Gamma(\alpha)} \left| \int_0^{t_2} (t_2 - s)^{\alpha-1} ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} ds \right| \\
&\quad + \frac{1}{p_* \Gamma(\alpha)} \left[ \frac{a_f (1+\eta)}{\beta \Gamma(\beta)} T^\beta + p^* a_h (1+\eta) + r^* \eta + p^* |x_1| \right] \\
&\quad . \left| \int_0^{t_2} (t_2 - s)^{\alpha-1} ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} ds \right| \\
&\le \frac{a_f (1+\eta)}{p_* \Gamma(\alpha+\beta)} \left| \frac{t_2^{\alpha+\beta}}{\alpha+\beta} - \frac{t_1^{\alpha+\beta}}{\alpha+\beta} \right| + \frac{r^* \eta}{p_* \Gamma(\alpha)} \left| \frac{t_2^\alpha}{\alpha} - \frac{t_1^\alpha}{\alpha} \right| \\
&\quad + \frac{1}{p_* \Gamma(\alpha)} \left[ \frac{a_f (1+\eta)}{\beta \Gamma(\beta)} T^\beta + p^* a_h (1+\eta) + r^* \eta + p^* |x_1| \right] . \left| \frac{t_2^\alpha}{\alpha} - \frac{t_1^\alpha}{\alpha} \right| .
\end{aligned}
$$

Its clear that, if $t_2 \to t_1$, then right-hand side of the above-mentioned inequality tends to zero. Since $x$ is an arbitrary in $B_\eta$, hence $F$ is equicontinuous operator.

In view of steps 1-3 together with the well known Arzelá-Ascoli theorem, we can deduce that $F$ is continuous and completely continuous.

**Step 4.** A priori bounds.

Now, we prove that the set $E(F) = \{x \in C(J, \mathbb{R}) : x = \gamma F x \text{ for some } \gamma \in [0,1]\}$, is bounded.

Let $x \in E(F)$, then $x = \gamma F x$ for some $\gamma \in [0,1]$. Thus, for each $t \in J$, we have

$$
\begin{aligned}
x(t) = \gamma &\left( I^\alpha \left( \frac{1}{p} I^\beta (f_x) \right)(t) - I^\alpha \left( \frac{r}{p} x \right)(t) \right. \\
&\left. - \left[ I^\beta (f_x)(T) + p(T)h(x) - r(T)x(T) - p(T)x_1 \right] I^\alpha \left( \frac{1}{p} \right)(t) - g(x) + x_0 \right).
\end{aligned}
$$

Using (H3), for all $t \in J$, we get

$$
\begin{aligned}
|(x)(t)| \leq\, & I^{\alpha}\left(\frac{1}{|p|} I^{\beta}(|f_x|)\right)(t) + I^{\alpha}\left(\frac{|r|}{|p|}|x|\right)(t) \\
& + \left[I^{\beta}(|f_x|)(T) + |p(T)|\ |h(x)| + |r(T)||x(T)| + |p(T)|\ |x_1|\right] I^{\alpha}\left(\frac{1}{|p|}\right)(t) \\
& + |g(x)| + |x_0| \\
\leq\, & I^{\alpha}\left(\frac{1}{|p|} I^{\beta}(a_f(1+|x(t)|))\right)(t) + I^{\alpha}\left(\frac{|r|}{|p|}|x|\right)(t) \\
& + \left[I^{\beta}(a_f(1+|x(t)|))(T) + |p(T)|\ a_h(1+\|x\|) + |r(T)||x(T)|\right. \\
& \left. + |p(T)|\ |x_1|\right] I^{\alpha}\left(\frac{1}{|p|}\right)(t) + a_g(1+\|x\|) + |x_0| \\
\leq\, & \frac{a_f T^{\alpha+\beta}}{p_*\Gamma(\alpha+\beta+1)} + \frac{a_f}{p_*\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|x(s)|ds \\
& + \frac{r^* T^{\alpha}}{p_*\alpha\Gamma(\alpha)}\|x\| + \frac{a_f T^{\alpha+\beta}}{p_*\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{a_f T^{\alpha}}{p_*\alpha\Gamma(\alpha)\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}|x(s)|ds \\
& + \frac{a_h p^* T^{\alpha}}{p_*\alpha\Gamma(\alpha)} + \frac{a_h p^* T^{\alpha}}{p_*\alpha\Gamma(\alpha)}\|x\| + \frac{r^* T^{\alpha}}{p_*\alpha\Gamma(\alpha)}\|x\| + \frac{p^*|x_1|T^{\alpha}}{p_*\alpha\Gamma(\alpha)} + a_g + a_g\|x\| + |x_0|,
\end{aligned}
$$

and hence

$$
\begin{aligned}
\|x\| \leq\, & \frac{a_f T^{\alpha+\beta}}{p_*\Gamma(\alpha+\beta+1)} + \frac{a_f}{p_*\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\|x\|ds \\
& + \frac{2r^* T^{\alpha}}{p_*\alpha\Gamma(\alpha)}\|x\| + \frac{a_f T^{\alpha+\beta}}{p_*\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{a_f T^{\alpha}}{p_*\alpha\Gamma(\alpha)\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}\|x\|ds \\
& + \frac{a_h p^* T^{\alpha}}{p_*\alpha\Gamma(\alpha)} + \frac{a_h p^* T^{\alpha}}{p_*\alpha\Gamma(\alpha)}\|x\| + \frac{p^*|x_1|T^{\alpha}}{p_*\alpha\Gamma(\alpha)} + a_g + a_g\|x\| + |x_0|,
\end{aligned}
$$

which yields that

$$
\begin{aligned}
\|x\| \leq\, & \frac{a_g + |x_0| + \frac{a_f T^{\alpha+\beta}}{p_*\Gamma(\alpha+\beta+1)} + \frac{a_f T^{\alpha+\beta}}{p_*\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{a_h p^* T^{\alpha}}{p_*\alpha\Gamma(\alpha)} + \frac{p^*|x_1|T^{\alpha}}{p_*\alpha\Gamma(\alpha)}}{1 - a_g - \frac{2r^*T^{\alpha}}{p_*\alpha\Gamma(\alpha)} - \frac{a_h p^* T^{\alpha}}{p_*\alpha\Gamma(\alpha)}} \\
& + \frac{a_f}{p_*\Gamma(\alpha+\beta)} \frac{1}{1 - a_g - \frac{2r^*T^{\alpha}}{p_*\alpha\Gamma(\alpha)} - \frac{a_h p^* T^{\alpha}}{p_*\alpha\Gamma(\alpha)}} \int_0^t (t-s)^{\alpha+\beta-1}\|x\|ds \\
& + \frac{a_f T^{\alpha}}{p_*\alpha\Gamma(\alpha)\Gamma(\beta)} \frac{1}{1 - a_g - \frac{2r^*T^{\alpha}}{p_*\alpha\Gamma(\alpha)} - \frac{a_h p^* T^{\alpha}}{p_*\alpha\Gamma(\alpha)}} \int_0^T (T-s)^{\beta-1}\|x\|ds.
\end{aligned}
$$

Applying Pachpatte's integral inequality Theorem 1, we get

$$
\|x\| \leq \frac{M}{1-Q}\exp\left(\int_0^t \frac{a_f}{p_*\Gamma(\alpha+\beta)} \frac{1}{1 - a_g - \frac{2r^*T^{\alpha}}{p_*\alpha\Gamma(\alpha)} - \frac{a_h p^* T^{\alpha}}{p_*\alpha\Gamma(\alpha)}}(t-s)^{\alpha+\beta-1}ds\right) := \rho,
$$

where

$$M=\frac{a_g+|x_0|+\dfrac{a_fT^{\alpha+\beta}}{p_*\Gamma(\alpha+\beta+1)}+\dfrac{a_fT^{\alpha+\beta}}{p_*\Gamma(\alpha+1)\Gamma(\beta+1)}+\dfrac{a_hp^*T^\alpha}{p_*\alpha\Gamma(\alpha)}+\dfrac{p^*|x_1|T^\alpha}{p_*\alpha\Gamma(\alpha)}}{1-a_g-\frac{2r^*T^\alpha}{p_*\alpha\Gamma(\alpha)}-\dfrac{a_hp^*T^\alpha}{p_*\alpha\Gamma(\alpha)}},$$

$$Q=\int_0^T\frac{a_fT^\alpha}{p_*\alpha\Gamma(\alpha)\Gamma(\beta)}\frac{1}{1-a_g-\frac{2r^*T^\alpha}{p_*\alpha\Gamma(\alpha)}-\dfrac{a_hp^*T^\alpha}{p_*\alpha\Gamma(\alpha)}}(T-s)^{\beta-1}$$
$$\times\exp\left(\int_0^s\frac{a_f}{p_*\Gamma(\alpha+\beta)}\frac{1}{1-a_g-\frac{2r^*T^\alpha}{p_*\alpha\Gamma(\alpha)}-\dfrac{a_hp^*T^\alpha}{p_*\alpha\Gamma(\alpha)}}(s-\sigma)^{\alpha+\beta-1}d\sigma\right)ds.$$

Thus, there exists a constant $\rho>0$ such that $\|x\|\le\rho$. This shows that the set $E(F)$ is bounded.

According to well known Schaefer's fixed point theorem, we deduce that $F$ has a fixed point which is solution of nonlocal fractional differential equation (1).

If $r(t)\equiv 0$ for $t\in J$, $r^*=0$ and we have the following result.

**Corollary 3.** *Assume that the hypothesis (H3) holds. If*

$$Q_1=\int_0^T\frac{a_fT^\alpha}{p_*\alpha\Gamma(\alpha)\Gamma(\beta)}\frac{1}{1-a_g-\dfrac{a_hp^*T^\alpha}{p_*\alpha\Gamma(\alpha)}}(T-s)^{\beta-1}$$
$$\times\exp\left(\int_0^s\frac{a_f}{p_*\Gamma(\alpha+\beta)}\frac{1}{1-a_g-\dfrac{a_hp^*T^\alpha}{p_*\alpha\Gamma(\alpha)}}(s-\sigma)^{\alpha+\beta-1}d\sigma\right)ds<1,$$

*where*

$$a_g+\frac{a_hp^*T^\alpha}{p_*\alpha\Gamma(\alpha)}\neq 1.$$

*Then the nonlocal fractional differential equation* (2) *has at least one solution on J.*

If $p(t)\equiv 1$ and $r(t)\equiv\lambda$ for $t\in J$, and $\lambda\in\mathbb{R}$, then we have $p_*=p^*=1, r^*=|\lambda|$ and we obtain the following result.

**Corollary 4.** *Assume that the hypothesis (H3) holds. If*

$$Q_2=\int_0^T\frac{a_fT^\alpha}{\alpha\Gamma(\alpha)\Gamma(\beta)}\frac{1}{1-a_g-\frac{2|\lambda|T^\alpha}{\alpha\Gamma(\alpha)}-\dfrac{a_hT^\alpha}{\alpha\Gamma(\alpha)}}(T-s)^{\beta-1}$$
$$\times\exp\left(\int_0^s\frac{a_f}{\Gamma(\alpha+\beta)}\frac{1}{1-a_g-\frac{2|\lambda|T^\alpha}{\alpha\Gamma(\alpha)}-\dfrac{a_hT^\alpha}{\alpha\Gamma(\alpha)}}(s-\sigma)^{\alpha+\beta-1}d\sigma\right)ds<1,$$

*where*

$$a_g+\frac{2|\lambda|T^\alpha}{\alpha\Gamma(\alpha)}+\frac{a_hT^\alpha}{\alpha\Gamma(\alpha)}\neq 1.$$

*Then the nonlocal fractional differential equation* (3) *has at least one solution on J.*

## 4 Example

Throughout this section, we discuss an example to illustrate the usefulness of our main results.
Consider the following nonlocal fractional differential equation

$$\begin{cases} {}^cD^{\frac{1}{4}}\left([(\sqrt{t}+10)\ {}^cD^{\frac{2}{3}}+e^{-2t}]x(t)\right)=\frac{e^{-t}+|x(t)|}{(9+e^t)(1+|x(t)|)}, t\in J=[0,1],\\ x(0)+\sum_{j=1}^{m}\lambda_j x(t_j)=0, \lambda_j>0, 0<t_1<t_2<\cdots<t_m<1,\\ {}^cD^{\frac{2}{3}}x(1)+\frac{1}{5}\frac{|x(t)|}{(1+|x(t)|)}=0. \end{cases} \tag{15}$$

Here, $\beta=\frac{1}{4}, \alpha=\frac{2}{3}, p(t)=\sqrt{t}+10, r(t)=e^{-2t}, T=1, f\left(t,x(t)\right)=\frac{e^{-t}+|x(t)|}{(9+e^t)(1+|x(t)|)}$, $g(x)=\sum_{j=1}^{m}\lambda_j x(t_j), \quad h(x)=\frac{1}{5}\frac{|x(t)|}{(1+|x(t)|)}$, and we obtain

$$\begin{aligned} &\left|f\left(t,x(t)\right)-f\left(t,y(t)\right)\right|\\ &=\left|\frac{e^{-t}+|x(t)|}{(9+e^t)(1+|x(t)|)}-\frac{e^{-t}+|y(t)|}{(9+e^t)(1+|y(t)|)}\right|\\ &=\left|\frac{(e^{-t}+|x(t)|)(1+|y(t)|)-(e^{-t}+|y(t)|)(1+|x(t)|)}{(9+e^t)(1+|x(t)|)(1+|y(t)|)}\right|\\ &=\left|\frac{(|x(t)|-|y(t)|)(1-e^{-t})}{(9+e^t)(1+|x(t)|)(1+|y(t)|)}\right|\\ &\le\frac{1-e^{-1}}{10}\Big||x(t)|-|y(t)|\Big|\\ &\le\frac{1-e^{-1}}{10}\Big|x(t)-y(t)\Big|. \end{aligned}$$

$$\begin{aligned} \left|g(x)-g(y)\right| &\le \sum_{j=1}^{m}\lambda_j \max_{t_j\in J}\|x(t_j)-y(t_j)\|.\\ \left|h(x)-h(y)\right| &\le \frac{1}{5}\|x-y\|. \end{aligned}$$

Thus, we have $r^*=1, p^*=11, p_*=10$, $N_f=\frac{1-e^{-1}}{10}, N_g=\sum_{j=1}^{m}\lambda_j$, and $N_h=\frac{1}{5}$.

From nonlocal fractional differential equation (15), we observe that the hypotheses of Theorem 2 can be establish by choosing a sufficiently small values of $\lambda_j$ such that

$$\begin{aligned} \omega_1 &= \frac{\frac{1-e^{-1}}{10}\ (1)^{2/3+1/4}}{(10)(1/4)\Gamma(1/4)}\left[\frac{\Gamma(1/4+1)}{\Gamma(1/4+2/3+1)}+\frac{1}{(2/3)\Gamma(2/3)}\right]\\ &\quad+\frac{(2)\ (1)\ (1)^{2/3}}{(10)(2/3)\Gamma(2/3)}+\frac{(11)(\frac{1}{5})\ (1)^{2/3}}{(10)(2/3)\Gamma(2/3)}+\sum_{j=1}^{m}\lambda_j\\ &\approx 0.4786+\sum_{j=1}^{m}\lambda_j<1, \end{aligned}$$

so nonlocal fractional differential equation (15) has a unique solution on $[0,1]$.

## 5 Conclusion

In this paper we have proved the existence and uniqueness solution of generalized nonlocal fractional Sturm-Liouville and Langevin equations 1. When $r(t)\equiv 0$ for all $t\in J$, we get the nonlocal fractional Sturm-Liouvilly equation and if $p(t)\equiv 1$, $r(t)\equiv\lambda, \lambda\in\mathbb{R}$ for all $t\in J$, we obtain the nonlocal fractional Langevin equation. To prove our results, we have applied the Banach fixed point theorem, Schaefer's fixed point theorem and Pachpatte's integral inequality.

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