

On differential geometry of timelike Smarandache curves in Minkowski 3-space

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Received: 22 Feb. 2020, Revised: 13 June 2020, Accepted: 17 June 2020

Published online: 1 July 2020

Abstract: In this paper, we acquaint a special timelike Smarandache curves \mathcal{L} reference to the Darboux frame of a timelike curve χ in Minkowski 3-space \mathfrak{R}_1^3 . We derive the Frenet invariants of \mathcal{L} and also give some geometric properties when the curve χ is a geodesic, an asymptotic and a principal curve. Finally, we give an example to illustrate these curves.

Keywords: Minkowski space-time, Smarandache curves, Darboux frame, geodesic curvature.

1 Introduction

In Smarandache geometry, a regular non-null curve in Minkowski 3-space, whose position vector is collected by the Frenet frame vectors of other regular non-null curve, is said to be Smarandache curve [1]. Recently in Euclidean and Minkowski space-times, special Smarandache curves according to different types of frames have been studied by some authors [2, 3, 8, 10].

In this paper, we introduce a special timelike Smarandache curves recording to the Darboux frame of a curve χ on timelike surface M in Minkowski 3-space \mathfrak{R}_1^3 . In Section 2, we give the basic concepts of Minkowski 3-space and Darboux frame that will be used throughout the paper. Section 3 is devoted to the study of special four timelike Smarandache curves Tn , Tg , gn , and Tgn -Smarandache curves by considering the relationship with invariants $\kappa_n(\sigma)$, $\kappa_g(\sigma)$ and $\tau_g(\sigma)$ of χ in Minkowski 3-space \mathfrak{R}_1^3 . From that point, we give some properties of these curves when χ is a geodesic, an asymptotic, or a principal curve. Finally, we illustrate these curves with an example.

2 Preliminaries

The Minkowski 3-space is three-dimensional Euclidean space provided with the Lorentzian inner product

$$\mathfrak{F} = -(du_1)^2 + (du_2)^2 + (du_3)^2,$$

where $u = (u_1, u_2, u_3)$ is a rectangular coordinate system of \mathfrak{R}_1^3 . Any vector v in \mathfrak{R}_1^3 can be characterized as follows: the vector v is called spacelike, lightlike or timelike if $\mathfrak{F}(v, v) > 0$ and $v = 0$, $\mathfrak{F}(v, v) = 0$ and $v \neq 0$ or $\mathfrak{F}(v, v) < 0$ respectively. The norm of a vector $v \in \mathfrak{R}_1^3$ is given by $\|v\| = \sqrt{|\mathfrak{F}(v, v)|}$. Similarly, any arbitrary curve $\chi = \chi(\sigma) : I \rightarrow \mathfrak{R}_1^3$ where σ is pseudo-arclength parameter, is called a spacelike curve if $\mathfrak{F}(\chi'(\sigma), \chi'(\sigma)) > 0$, lightlike if $\mathfrak{F}(\chi'(\sigma), \chi'(\sigma)) = 0$ and $\chi'(s) \neq 0$ and timelike if $\mathfrak{F}(\chi'(\sigma), \chi'(\sigma)) < 0$ and for all $\sigma \in I$.

For any unit speed timelike curve χ with Frenet-Serret frame $\{T, N, B\}$, Frenet-Serret formulas of the curve χ can be given as [4, 5, 6]:

$$\begin{pmatrix} T'(\sigma) \\ N'(\sigma) \\ B'(\sigma) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(\sigma) & 0 \\ \kappa(\sigma) & 0 & \tau(\sigma) \\ 0 & -\tau(\sigma) & 0 \end{pmatrix} \begin{pmatrix} T(\sigma) \\ N(\sigma) \\ B(\sigma) \end{pmatrix}, \tag{1}$$

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where $-\mathfrak{F}(T, T) = \mathfrak{F}(N, N) = \mathfrak{F}(B, B) = 1$ and $\mathfrak{F}(T, N) = \mathfrak{F}(T, B) = \mathfrak{F}(N, B) = 0$.

Definition 1. A spacelike (timelike) surface in the Minkowski 3-space is a surface M in \mathfrak{R}_1^3 whose the induced metric is a positive definite Riemannian metric (Lorentz metric). In other words, the normal vector on the spacelike (timelike) surface is a timelike (spacelike) vector [6].

Let $\Psi : \mathcal{V} \subset \mathfrak{R}_\infty^3 \rightarrow \mathfrak{R}_\infty^3$, $\Psi(\mathcal{V}) = \mathcal{M}$ and $\beta : I \subset \mathfrak{R} \rightarrow \mathcal{V}$ be a timelike embedding and a regular curve, respectively. Then we define a curve $\chi(\sigma) = \Psi(\beta(\sigma))$ on the surface M , and since Ψ is a timelike embedding, we have a unit spacelike normal vector field n along the surface M defined by [7]

$$n = \frac{\Psi_x \times \Psi_y}{\|\Psi_x \times \Psi_y\|}. \quad (2)$$

Hence we have a pseudo-orthonormal frame $\{T, g, n\}$ which is called the Darboux frame along the curve χ where $g(\sigma) = T(\sigma) \times n(\sigma)$ is a unit vector. The corresponding Frenet-Serret formulae of χ read

$$\begin{pmatrix} T'(\sigma) \\ g'(\sigma) \\ n'(\sigma) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g(\sigma) & \kappa_n(\sigma) \\ \kappa_g(\sigma) & 0 & -\tau_g(\sigma) \\ \kappa_n(\sigma) & \tau_g(\sigma) & 0 \end{pmatrix} \begin{pmatrix} T(\sigma) \\ g(\sigma) \\ n(\sigma) \end{pmatrix}, \quad (3)$$

where $\kappa_g(\sigma) = \mathfrak{F}(T'(\sigma), g(\sigma))$, $\kappa_n(\sigma) = \mathfrak{F}(T'(\sigma), n(\sigma))$ and $\tau_g(\sigma) = \mathfrak{F}(g'(\sigma), n(\sigma))$ are the geodesic curvature, the asymptotic curvature, and the principal curvature of χ on the surface M in \mathfrak{R}_1^3 , respectively, and σ is arc-length parameter of χ . Also, the curve χ is classified by $\kappa_g(\sigma)$, $\kappa_n(\sigma)$ and $\tau_g(\sigma)$ as postdates:

- χ is a geodesic curve if and only if $\kappa_g(\sigma) = 0$,
- χ is an asymptotic curve if and only if $\kappa_n(\sigma) = 0$,
- χ is a principal curve if and only if $\tau_g(\sigma) = 0$.

The pseudosphere with center at the origin and of radius $r = 1$ in the Minkowski 3-space \mathfrak{R}_1^3 is a quadric defined by

$$S_1^2 = \{u \in \mathfrak{R}_1^3 : \mathfrak{F}(u, u) = 1\}.$$

3 Special timelike Smarandache curves in \mathfrak{R}_1^3

In this section, we define a special timelike Smarandache curves reference to the Darboux frame in Minkowski 3-space \mathfrak{R}_1^3 . Additionally, we obtain the Frenet invariants of these curves and give some properties when the curve χ is a geodesic curve or an asymptotic curve or a principal curve.

Definition 2.[9] Let $\chi = \chi(\sigma)$ be a timelike curve in \mathfrak{R}_1^3 with the moving Darboux frame $\{T, g, n\}$. Then the timelike Tg -Smarandache curves of χ is defined by

$$\mathcal{Z}(\vartheta(\sigma)) = \frac{1}{\sqrt{2}}(T(\sigma) + g(\sigma)). \quad (4)$$

Theorem 1. Let $\chi = \chi(\sigma)$ be a timelike curve in \mathfrak{R}_1^3 with the moving Darboux frame $\{T, g, n\}$. If $\chi(\sigma)$ is a geodesic curve ($\kappa_g = 0$) with $\tau_g > \kappa_n$, then the natural curvature functions of timelike Tg -Smarandache curves satisfied the following equations,

$$\begin{aligned} \kappa_{\mathcal{Z}}(\vartheta(\sigma)) &= \frac{\sqrt{2(\tau_g^2 - \kappa_n^2)}}{(\kappa_n - \tau_g)}, \\ \tau_{\mathcal{Z}}(\vartheta(\sigma)) &= \frac{\sqrt{2}\kappa_n(\kappa_n - \tau_g)(\tau_g' - \kappa_n'\tau_g) - \kappa_n\tau_g(\kappa_n' - \tau_g')}{(\kappa_n^2 + \tau_g^2)(\kappa_n - \tau_g)^4}. \end{aligned} \quad (5)$$

Proof. Let $\mathcal{Z} = \mathcal{Z}(\vartheta)$ be a Tg -timelike Smarandache curves reference to the timelike curve $\chi = \chi(\sigma)$ in Minkowski 3-space \mathfrak{R}_1^3 . From Eqns. (3) and (5), we get

$$\mathcal{Z}'(\vartheta) = \frac{d\mathcal{Z}}{d\vartheta} \frac{d\vartheta}{d\sigma} = \frac{1}{\sqrt{2}}(\kappa_g T(\sigma) + \kappa_g g(\sigma) + (\kappa_n - \tau_g)n(\sigma)), \quad (6)$$

and

$$T_{\mathcal{X}}(\vartheta) = \frac{1}{(\kappa_n - \tau_g)} \left(\kappa_g T(\sigma) + \kappa_g g(\sigma) + (\kappa_n - \tau_g) n(\sigma) \right), \tag{7}$$

where

$$\frac{d\vartheta}{d\sigma} = \frac{\kappa_n - \tau_g}{\sqrt{2}}. \tag{8}$$

Now

$$\frac{dT_{\mathcal{X}}}{d\vartheta} = \frac{\sqrt{2}}{(\kappa_n - \tau_g)^3} \left(\varepsilon_1(\sigma) T(\sigma) + \varepsilon_2(\sigma) g(\sigma) + \varepsilon_3(\sigma) n(\sigma) \right), \tag{9}$$

where

$$\begin{cases} \varepsilon_1(\sigma) = (\kappa_n - \tau_g) \left[\kappa_g^2 + \kappa_g' + \kappa_n(\kappa_n - \tau_g) \right] - \kappa_g(\kappa_n' - \tau_g'), \\ \varepsilon_2(\sigma) = (\kappa_n - \tau_g) \left[\kappa_g^2 + \kappa_g' + \tau_g(\kappa_n - \tau_g) \right] - \kappa_g(\kappa_n' - \tau_g'), \\ \varepsilon_3(\sigma) = \kappa_g(\kappa_n - \tau_g)^2. \end{cases}$$

Then

$$\kappa_{\mathcal{X}}(\vartheta) = \frac{\sqrt{2(\varepsilon_2^2 + \varepsilon_3^2 - \varepsilon_1^2)}}{(\kappa_n - \tau_g)^3},$$

and

$$N_{\mathcal{X}}(\vartheta) = \frac{1}{\sqrt{\varepsilon_2^2 + \varepsilon_3^2 - \varepsilon_1^2}} \left(\varepsilon_1(\sigma) T(\sigma) + \varepsilon_2(\sigma) g(\sigma) + \varepsilon_3(\sigma) n(\sigma) \right).$$

Then, we have

$$B_{\mathcal{X}}(\vartheta) = \frac{1}{(\kappa_n - \tau_g) \sqrt{\varepsilon_2^2 + \varepsilon_3^2 - \varepsilon_1^2}} \left(\ell_1(\sigma) T(\sigma) + \ell_2(\sigma) g(\sigma) + \ell_3(\sigma) n(\sigma) \right),$$

where

$$\begin{cases} \ell_1(\sigma) = \varepsilon_2(\kappa_n - \tau_g) - \varepsilon_3 \kappa_g, \\ \ell_2(\sigma) = \varepsilon_1(\kappa_n - \tau_g) - \varepsilon_3 \kappa_g, \\ \ell_3(\sigma) = \kappa_g(\varepsilon_2 - \varepsilon_1). \end{cases}$$

From Eqn. (6), we get

$$\begin{aligned} \mathcal{X}''(\vartheta) = & \frac{1}{\sqrt{2}} \left[\left[\kappa_g^2 + \kappa_g' + \kappa_n(\kappa_n - \tau_g) \right] T(\sigma) + \left[\kappa_g^2 + \kappa_g' + \tau_g(\kappa_n - \tau_g) \right] g(\sigma) \right. \\ & \left. + \left[\kappa_n' - \tau_g' + \kappa_g(\kappa_n - \tau_g) \right] n(\sigma) \right], \end{aligned}$$

and

$$\mathcal{X}'''(\vartheta) = \frac{1}{\sqrt{2}} \left[\mu_1(\sigma) T(\sigma) + \mu_2(\sigma) g(\sigma) + \mu_3(\sigma) n(\sigma) \right],$$

where

$$\begin{cases} \mu_1(\sigma) = \kappa_g^3 + \kappa_g'' + 3\kappa_g \kappa_g' + (\kappa_n - \tau_g)(\kappa_n' + 2\kappa_g \tau_g + \kappa_n \kappa_g) + 3\kappa_n(\kappa_n' - \tau_g'), \\ \mu_2(\sigma) = \kappa_g^3 + \kappa_g'' + 3\kappa_g \kappa_g' + (\kappa_n - \tau_g)(\tau_g' + \kappa_n \kappa_g + \kappa_g \tau_g) + 2\tau_g(\kappa_n' - \tau_g'), \\ \mu_3(\sigma) = \kappa_n'' - \tau_g'' + (\kappa_n - \tau_g)(\kappa_n^2 - \tau_g^2 + \kappa_g + 2\kappa_g') + \kappa_g(\kappa_n' - \tau_g'). \end{cases}$$

Then

$$\begin{aligned} \tau_{\mathcal{X}}(\vartheta) = & \frac{\sqrt{2} \{ (\mu_1 - \mu_2) [\kappa_g(\kappa_n' - \tau_g') - \kappa_g'(\kappa_n - \tau_g)] + (\kappa_n - \tau_g)^2 (\mu_2 \kappa_n - \mu_1 \tau_g \mu_3 \kappa_g) \}}{[\kappa_g(\kappa_n' - \tau_g') - (\kappa_n - \tau_g)(\kappa_g' + \tau_g(\kappa_n - \tau_g))]^2 - [(\kappa_n - \tau_g)(\kappa_g' + \kappa_n(\kappa_n - \tau_g))^2} \\ & - \kappa_g(\kappa_n' - \tau_g')]^2 + \kappa_g^2(\kappa_n - \tau_g)^4} \end{aligned}$$

So if $\chi(\sigma)$ is a geodesic curve, then Eqn. (5) holds and the proof is complete.

Definition 3.[9] Let $\chi = \chi(\sigma)$ be a timelike curve in \mathfrak{R}_1^3 with the moving Darboux frame $\{T, \mathbf{g}, \mathbf{n}\}$. Then the timelike Tn -Smarandache curves of χ is defined by

$$\mathcal{X}(\vartheta(\sigma)) = \frac{1}{\sqrt{2}}(T(\sigma) + \mathbf{n}(\sigma)). \quad (10)$$

Theorem 2. Let $\chi = \chi(\sigma)$ be a timelike curve in \mathfrak{R}_1^3 with the moving Darboux frame $\{T, \mathbf{g}, \mathbf{n}\}$. If $\chi(\sigma)$ is an asymptotic line ($\kappa_n = 0$) with $\tau_g \geq \kappa_g$, then the natural curvature functions of Tn -timelike Smarandache curves satisfied the following equation

$$\begin{aligned} \kappa_{\mathcal{X}}(\vartheta) &= \frac{\sqrt{2(\tau_g^2 - \kappa_g^2)}}{(\kappa_g + \tau_g)}, \\ \tau_{\mathcal{X}}(\vartheta) &= \frac{\sqrt{2}(\kappa_g \tau_g' - \kappa_g' \tau_g)}{(\kappa_g + \tau_g)(\kappa_g^2 + \tau_g^2)}. \end{aligned} \quad (11)$$

Proof. Let $\mathcal{X} = \mathcal{X}(\vartheta)$ be a timelike Tn -Smarandache curves reference to the timelike curve $\chi = \chi(\sigma)$ in Minkowski 3-space \mathfrak{R}_1^3 . Then from Eqn. (10), we get

$$\mathcal{X}'(\vartheta) = \frac{1}{\sqrt{2}}(\kappa_n T(\sigma) + (\kappa_g + \tau_g)\mathbf{g}(\sigma) + \kappa_n \mathbf{n}(\sigma)), \quad (12)$$

and

$$T_{\mathcal{X}}(\vartheta) = \frac{1}{\kappa_g + \tau_g}(\kappa_n T(\sigma) + (\kappa_g + \tau_g)\mathbf{g}(\sigma) + \kappa_n \mathbf{n}(\sigma)), \quad (13)$$

where

$$\frac{d\vartheta}{d\sigma} = \frac{\kappa_g + \tau_g}{\sqrt{2}}. \quad (14)$$

Also, one can see that

$$\frac{dT_{\mathcal{X}}}{d\vartheta} = \frac{\sqrt{2}}{(\kappa_g + \tau_g)^3}(\gamma_1(\sigma)T(\sigma) + \gamma_2(\sigma)\mathbf{g}(\sigma) + \gamma_3(\sigma)\mathbf{n}(\sigma)),$$

where

$$\begin{aligned} \gamma_1(\sigma) &= (\kappa_g + \tau_g)[\kappa_n^2 + \kappa_n' + \kappa_g(\kappa_g + \tau_g)] - \kappa_n(\kappa_g' + \tau_g'), \\ \gamma_2(\sigma) &= \kappa_n(\kappa_g + \tau_g)^2, \\ \gamma_3(\sigma) &= (\kappa_g + \tau_g)[\kappa_n^2 + \kappa_n' - \tau_g(\kappa_g + \tau_g)] - \kappa_n(\kappa_g' + \tau_g'). \end{aligned}$$

Then,

$$\kappa_{\mathcal{X}}(\vartheta) = \frac{\sqrt{2(\gamma_2^2 + \gamma_3^2 - \gamma_1^2)}}{(\kappa_g + \tau_g)^3},$$

and

$$N_{\mathcal{X}}(\vartheta) = \frac{1}{\sqrt{\gamma_2^2 + \gamma_3^2 - \gamma_1^2}}(\gamma_1(\sigma)T(\sigma) + \gamma_2(\sigma)\mathbf{g}(\sigma) + \gamma_3(\sigma)\mathbf{n}(\sigma)).$$

So

$$B_{\mathcal{X}}(\vartheta) = \frac{1}{(\kappa_g + \tau_g)\sqrt{\gamma_2^2 + \gamma_3^2 - \gamma_1^2}}(\delta_1(\sigma)T(\sigma) + \delta_2(\sigma)\mathbf{g}(\sigma) + \delta_3(\sigma)\mathbf{n}(\sigma)),$$

where

$$\begin{cases} \delta_1(\sigma) = \gamma_2 \kappa_n - \gamma_3(\kappa_g + \tau_g), \\ \delta_2(\sigma) = (\gamma_1 - \gamma_3)\kappa_n, \\ \delta_3(\sigma) = \gamma_2 \kappa_n - \gamma_1(\kappa_g + \tau_g). \end{cases}$$

Now, from Eqn. (12) we get

$$\mathcal{L}''(\vartheta) = \frac{1}{\sqrt{2}} \left[[\kappa_n^2 + \kappa_n' + \kappa_g(\kappa_g + \tau_g)]T(\sigma) + [\kappa_g' + \tau_g' + \kappa_n(\kappa_g + \tau_g)]\mathbf{g}(\sigma) + [\kappa_n^2 - \tau_g(\kappa_g + \tau_g)]\mathbf{n}(\sigma) \right],$$

and

$$\mathcal{L}'''(\vartheta) = \frac{1}{\sqrt{2}} \left[v_1(\sigma)T(\sigma) + v_2(\sigma)\mathbf{g}(\sigma) + v_3(\sigma)\mathbf{n}(\sigma) \right],$$

where

$$\begin{cases} v_1(\sigma) = \kappa_n^3 + \kappa_n'' + 2\kappa_n\kappa_n' + \kappa_g'(\kappa_g + \tau_g) + 2\kappa_g(\kappa_g' + \tau_g'), \\ v_2(\sigma) = \kappa_n^3\tau_g + \kappa_g'' + \tau_g'' + (\kappa_g + \tau_g)(\kappa_n' + \kappa_g^2 - \tau_g^2) + \kappa_g(\kappa_n^2 + \kappa_n'), \\ v_3(\sigma) = \kappa_n^3 + 3\kappa_n\kappa_n' - \tau_g'(\kappa_g + \tau_g) - 2\tau_g(\kappa_g' + \tau_g'). \end{cases}$$

Then

$$\tau_{\mathcal{L}}(\vartheta) = \sqrt{2} \left\{ \frac{v_2\kappa_n\kappa_n' + (\kappa_g + \tau_g)(2v_2\kappa_n\kappa_g - v_3\kappa_n') + (v_1 + v_3)\kappa_n(\kappa_g' + \tau_g') - (v_3\kappa_g + v_1\tau_g)(\kappa_g + \tau_g)^2}{[\tau_g(\kappa_g + \tau_g)^2 + \kappa_n(\kappa_g' + \tau_g')]^2 + [\kappa_n\kappa_n' + 2\kappa_n\kappa_g(\kappa_g + \tau_g)]^2 + [\kappa_n(\kappa_g' + \tau_g')]^2 - (\kappa_g + \tau_g)(\kappa_n' + \kappa_g(\kappa_g + \tau_g))^2} \right\}$$

So, if $\chi(\sigma)$ is an asymptotic line, then Eqn. (11) holds and the proof is complete.

Definition 4.[9] Let $\chi = \chi(\sigma)$ be a timelike curve in \mathfrak{R}_1^3 with the moving frame $\{T, \mathbf{g}, \mathbf{n}\}$. Then the timelike gn-Smarandache curves of χ is defined by

$$\mathcal{L}(\vartheta(\sigma)) = \frac{1}{\sqrt{2}} (\mathbf{g}(\sigma) + \mathbf{n}(\sigma)). \tag{15}$$

Theorem 3. Let $\chi = \chi(\sigma)$ be a timelike curve in \mathfrak{R}_1^3 with the moving frame $\{T, \mathbf{g}, \mathbf{n}\}$. If $\chi(\sigma)$ is a principal line ($\tau_g = 0$) with $\kappa_n + \kappa_g \neq 0$, then curvature and torsion of timelike gn-Smarandache curves satisfied the following equations,

$$\begin{aligned} \kappa_{\mathcal{L}}(\vartheta) &= \frac{\sqrt{2(\kappa_n^2 + \kappa_g^2)}}{\kappa_n + \kappa_g}, \\ \tau_{\mathcal{L}}(\vartheta) &= \sqrt{2} \left\{ \frac{(3\kappa_g - \kappa_n)(\kappa_n' - \kappa_g')(\kappa_n^2 - \kappa_g^2) + 3\kappa_n\kappa_g(\kappa_n + \kappa_g)(\kappa_n' + \kappa_g')}{(\kappa_n + \kappa_g)[\kappa_n^2 + \kappa_g^2(\kappa_n + \kappa_g)^2]} \right\}. \end{aligned} \tag{16}$$

Proof. Let $\mathcal{L} = \mathcal{L}(\vartheta)$ be a timelike gn-Smarandache curves reference to the timelike curve $\chi = \chi(\sigma)$ in Minkowski 3-space \mathfrak{R}_1^3 . Then from Eqn. (15), we get

$$\mathcal{L}'(\vartheta) = \frac{d\mathcal{L}}{d\vartheta} \frac{d\vartheta}{ds} = \frac{1}{\sqrt{2}} \left((\kappa_n + \kappa_g)T(\sigma) + \tau_g\mathbf{g}(\sigma) - \tau_g\mathbf{n}(\sigma) \right), \tag{17}$$

and

$$T_{\mathcal{L}}(\vartheta) = \frac{1}{\sqrt{2\tau_g^2 - (\kappa_n + \kappa_g)^2}} \left((\kappa_n + \kappa_g)T(\sigma) + \tau_g\mathbf{g}(\sigma) - \tau_g\mathbf{n}(\sigma) \right), \tag{18}$$

where

$$\frac{d\vartheta}{ds} = \frac{\sqrt{2\tau_g^2 - (\kappa_n + \kappa_g)^2}}{\sqrt{2}}. \tag{19}$$

Then

$$\frac{dT_{\mathcal{L}}}{d\vartheta} = \frac{\sqrt{2}}{(2\tau_g^2 - (\kappa_n + \kappa_g)^2)^2} \left(\eta_1(\sigma)T(\sigma) + \eta_2(\sigma)\mathbf{g}(\sigma) + \eta_3(\sigma)\mathbf{n}(\sigma) \right), \tag{20}$$

where

$$\begin{aligned}\eta_1(\sigma) &= [2\tau_g^2 - (\kappa_n + \kappa_g)^2] [\kappa'_n + \kappa'_g - \tau_g(\kappa_n - \kappa_g)] - (\kappa_n + \kappa_g) [2\tau_g \tau'_g - (\kappa_n + \kappa_g)(\kappa'_n + \kappa'_g)], \\ \eta_2(\sigma) &= [2\tau_g^2 - (\kappa_n + \kappa_g)^2] [\tau'_g - \tau_g^2 + \kappa_g(\kappa_n + \kappa_g)] - \tau_g [2\tau_g \tau'_g - (\kappa_n + \kappa_g)(\kappa'_n + \kappa'_g)], \\ \eta_3(\sigma) &= [2\tau_g^2 - (\kappa_n + \kappa_g)^2] [\kappa_g(\kappa_n + \kappa_g) - \tau'_g - \tau_g^2] + \tau_g [2\tau_g \tau'_g - (\kappa_n + \kappa_g)(\kappa'_n + \kappa'_g)].\end{aligned}$$

Then

$$\kappa_{\mathcal{Z}}(\vartheta) = \frac{\sqrt{2(\eta_2^2 + \eta_3^2 - \eta_1^2)}}{(2\tau_g^2 - (\kappa_n + \kappa_g))^2},$$

and

$$N_{\mathcal{Z}}(\vartheta) = \frac{1}{\sqrt{\eta_2^2 + \eta_3^2 - \eta_1^2}} (\eta_1 T + \eta_2 \mathbf{g} + \eta_3 \mathbf{n}).$$

So

$$B_{\mathcal{Z}}(\vartheta) = \frac{v_1(\sigma)T(\sigma) + v_2(\sigma)\mathbf{g}(\sigma) + v_3(\sigma)\mathbf{n}(\sigma)}{\sqrt{2\tau_g^2 - (\kappa_n + \kappa_g)^2} \sqrt{\eta_2^2 + \eta_3^2 - \eta_1^2}},$$

where

$$\begin{cases} v_1(\sigma) = -(\eta_2 + \eta_3)\tau_g, \\ v_2(\sigma) = -\eta_1\tau_g - \eta_3(\kappa_n + \kappa_g), \\ v_3(\sigma) = \eta_2(\kappa_n + \kappa_g) - \eta_1\tau_g. \end{cases}$$

From Eqn. (17), we have

$$\begin{aligned}\mathcal{Z}''(\vartheta) &= \frac{1}{\sqrt{2}} \left[[(\kappa'_n + \kappa'_g) + \tau_g(\kappa_n - \kappa_g)]T(\sigma) + [\kappa_g(\kappa_n + \kappa_g) + \tau'_g - \tau_g^2]\mathbf{g}(\sigma) \right. \\ &\quad \left. + [\kappa_n(\kappa_n + \kappa_g) - \tau'_g - \tau_g^2]\mathbf{n}(\sigma) \right],\end{aligned}$$

and

$$\mathcal{Z}'''(\vartheta) = \frac{1}{\sqrt{2}} [\omega_1(\sigma)T(\sigma) + \omega_2(\sigma)\mathbf{g}(\sigma) + \omega_3(\sigma)\mathbf{n}(\sigma)],$$

where

$$\begin{cases} \omega_1(\sigma) = \kappa''_n + \kappa''_g + (\kappa_n + \kappa_g)[\kappa_n^2 + \kappa_g^2 + \tau_g^2] - (\kappa_n - \kappa_g)(2\tau'_g + \tau_g), \\ \omega_2(\sigma) = \tau''_g - \tau_g^3 - 3\tau'_g\tau_g + (\kappa_n + \kappa_g)(\kappa'_g + \kappa_n\tau_g) - \kappa_g\tau_g(\kappa_n - \kappa_g) + 2\kappa_g(\kappa'_n - \kappa'_g), \\ \omega_3(\sigma) = (\kappa_n + \kappa_g)(\kappa'_n - \kappa'_g\tau_g) - \kappa_n\tau_g(\kappa_n - \kappa_g) + 2\kappa_n(\kappa'_n - \kappa'_g) - \tau''_g - \tau_g^3. \end{cases}$$

Then

$$\tau_{\mathcal{Z}}(\vartheta) = \sqrt{2} \left\{ \begin{aligned} &(\kappa_n + \kappa_g)[(\omega_2 - \omega_3)(\kappa_g^2 + \tau_g^2 - \kappa_n^2) + (\omega_2 + \omega_3)(\tau'_g + \kappa_n\kappa_g)] \\ &- (\omega_2 + \omega_3)\tau_g(\kappa'_n + \kappa'_g) + (\omega_2 - \omega_3)\tau_g^2(\kappa_n - \kappa_g) + \omega_1[\tau_g(\kappa_n + \kappa_g)^2 - 2\tau_g^3] \\ &[\tau_g(\kappa_n + \kappa_g)^2 - 2\tau_g^3]^2 + [(\kappa_n + \kappa_g)(\tau_g^2 + \tau'_g - \kappa_n^2 - \kappa_n\kappa_g) - \tau_g(\kappa'_n + \kappa'_g) \\ &- \tau_g^2(\kappa_n - \kappa_g)]^2 + [(\kappa_n + \kappa_g)(\kappa_g^2 + \kappa_n\kappa_g - \tau'_g - \tau_g^2) \\ &- \tau_g(\kappa'_n + \kappa'_g) - \tau_g^2(\kappa_n - \kappa_g)]^2 \end{aligned} \right\}$$

So, if $\chi(\sigma)$ is a principal line ($\tau_g = 0$), then Eqn. (16) holds and the proof is complete.

Definition 5.[9] Let $\chi = \chi(\sigma)$ be a timelike curve in \mathfrak{R}_1^3 with the moving frame $\{T, \mathbf{g}, \mathbf{n}\}$. Then the timelike $Tg\mathbf{n}$ -Smarandache curves of χ is defined by

$$\mathcal{Z}(\vartheta(\sigma)) = \frac{1}{\sqrt{3}} (T(\sigma) + \mathbf{g}(\sigma) + \mathbf{n}(\sigma)). \quad (21)$$

Now, we can investigate the Frenet invariants of the timelike Tgn -Smarandache curves reference to the timelike curve $\chi = \chi(\sigma)$ in Minkowski 3-space \mathfrak{R}_1^3 . From Eqn. (22), we get

$$\mathcal{F}'(\vartheta) = \frac{d\mathcal{F}}{d\vartheta} \frac{d\vartheta}{ds} = \frac{1}{\sqrt{3}} \left((\kappa_n + \kappa_g)T(\sigma) + (\kappa_g + \tau_g)\mathfrak{g}(\sigma) + (\kappa_n - \tau_g)n(\sigma) \right), \tag{22}$$

and

$$T_{\mathcal{F}}(\vartheta) = \frac{(\kappa_n + \kappa_g)T(\sigma) + (\kappa_g + \tau_g)\mathfrak{g}(\sigma) + (\kappa_n - \tau_g)n(\sigma)}{\sqrt{(\kappa_g + \tau_g)^2 - (\kappa_n + \kappa_g)^2 + (\kappa_n - \tau_g)^2}}, \tag{23}$$

where

$$\frac{d\vartheta}{ds} = \sqrt{\frac{(\kappa_g + \tau_g)^2 - (\kappa_n + \kappa_g)^2 + (\kappa_n - \tau_g)^2}{3}}. \tag{24}$$

Then

$$\frac{dT_{\mathcal{F}}}{d\vartheta} = \frac{\sqrt{3} \left(\lambda_1(\sigma)T(\sigma) + \lambda_2(\sigma)\mathfrak{g}(\sigma) + \lambda_3(\sigma)n(\sigma) \right)}{\left((\kappa_g + \tau_g)^2 - (\kappa_n + \kappa_g)^2 + (\kappa_n - \tau_g)^2 \right)^2}, \tag{25}$$

where

$$\begin{aligned} \lambda_1(\sigma) = & (\kappa'_n + \kappa'_g) [(\kappa_g + \tau_g)^2 + (\kappa_n - \tau_g)^2] + (\kappa_g + \tau_g) \left\{ \kappa_g [(\kappa_n + \kappa_g)^2 + (\kappa_g + \tau_g)^2 + (\kappa_n - \tau_g)^2] - (\kappa_n + \kappa_g)(\kappa'_n + \kappa'_g) \right\} \\ & + (\kappa_n - \tau_g) \left\{ \kappa_n [(\kappa_n + \kappa_g)^2 + (\kappa_g + \tau_g)^2 + (\kappa_n - \tau_g)^2] - (\kappa_n + \kappa_g)(\kappa'_n - \tau'_g) \right\}, \end{aligned}$$

$$\begin{aligned} \lambda_2(\sigma) = & (\kappa'_g + \tau'_g) [(\kappa_g + \tau_g)^2 + (\kappa_n - \tau_g)^2] + (\kappa_n + \kappa_g) \left\{ \kappa_g [(\kappa_n + \kappa_g)^2 + (\kappa_g + \tau_g)^2 + (\kappa_n - \tau_g)^2] - (\kappa_g + \tau_g)(\kappa'_n + \kappa'_g) \right\} \\ & + (\kappa_n - \tau_g) \left\{ \tau_g [(\kappa_n + \kappa_g)^2 + (\kappa_g + \tau_g)^2 + (\kappa_n - \tau_g)^2] - (\kappa_g + \tau_g)(\kappa'_n - \tau'_g) \right\}, \end{aligned}$$

$$\begin{aligned} \lambda_3(\sigma) = & (\kappa'_n + \tau'_g) [(\kappa_g + \tau_g)^2 + (\kappa_n + \kappa_g)^2] + (\kappa_n + \tau_g) \left\{ \kappa_n [(\kappa_n + \kappa_g)^2 + (\kappa_g + \tau_g)^2 + (\kappa_n - \tau_g)^2] - (\kappa_n - \tau_g)(\kappa'_n + \kappa'_g) \right\} \\ & - (\kappa_g + \tau_g) \left\{ \tau_g [(\kappa_n + \kappa_g)^2 + (\kappa_g + \tau_g)^2 + (\kappa_n - \tau_g)^2] + (\kappa_n - \tau_g)(\kappa'_g + \tau'_g) \right\}. \end{aligned}$$

Then

$$\kappa_{\mathcal{F}}(\vartheta) = \frac{\sqrt{3(\lambda_2^2 + \lambda_3^2 - \lambda_1^2)}}{\left((\kappa_g + \tau_g)^2 - (\kappa_n + \kappa_g)^2 + (\kappa_n - \tau_g)^2 \right)^2},$$

and

$$N_{\mathcal{F}}(\vartheta) = \frac{1}{\sqrt{\lambda_2^2 + \lambda_3^2 - \lambda_1^2}} \left(\lambda_1(\sigma)T(\sigma) + \lambda_2(\sigma)\mathfrak{g}(\sigma) + \lambda_3(\sigma)n(\sigma) \right).$$

So

$$B_{\mathcal{F}}(\vartheta) = \frac{\rho_1(\sigma)T(\sigma) + \rho_2(\sigma)\mathfrak{g}(\sigma) + \rho_3(\sigma)n(\sigma)}{\sqrt{(\kappa_g + \tau_g)^2 - (\kappa_n + \kappa_g)^2 + (\kappa_n - \tau_g)^2} \sqrt{\lambda_2^2 + \lambda_3^2 - \lambda_1^2}}$$

where

$$\begin{cases} \rho_1(\sigma) = \lambda_2(\kappa_n - \tau_g) - \lambda_3(\kappa_g + \tau_g), \\ \rho_2(\sigma) = \lambda_1(\kappa_n - \tau_g) - \lambda_3(\kappa_n + \kappa_g), \\ \rho_3(\sigma) = \lambda_2(\kappa_n + \kappa_g) - \lambda_1(\kappa_g + \tau_g). \end{cases}$$

Then from Eqn. (22), we get

$$\begin{aligned} \mathcal{F}''(\vartheta) = & \frac{1}{\sqrt{3}} \left[[\kappa'_n + \kappa'_g + \kappa_g(\kappa_g + \tau_g) + \kappa_n(\kappa_n - \tau_g)]T(\sigma) + [\kappa'_g + \tau'_g + \kappa_g(\kappa_n + \kappa_g) \right. \\ & \left. + \tau_g(\kappa_n - \tau_g)]\mathfrak{g}(\sigma) + [\kappa'_n - \tau'_g + \kappa_n(\kappa_n + \kappa_g) - \tau_g(\kappa_g + \tau_g)]n(\sigma) \right], \end{aligned}$$

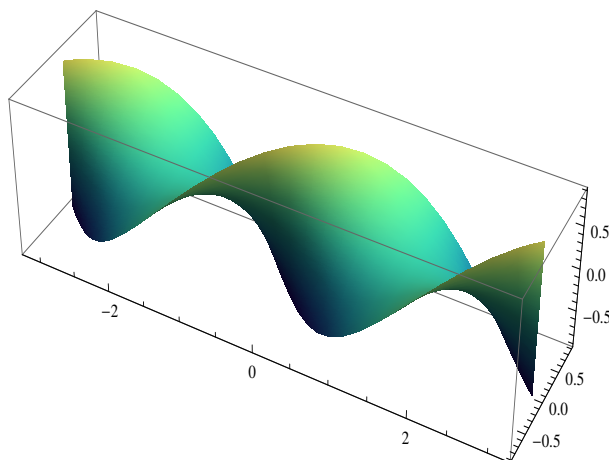


Fig. 1: The timelike surface $\Psi(\sigma, t)$.

and

$$\mathcal{L}^{e''' }(\vartheta) = \frac{1}{\sqrt{3}} \left[\xi_1(\sigma)T(\sigma) + \xi_2(\sigma)g(\sigma) + \xi_3(\sigma)n(\sigma) \right],$$

where

$$\begin{cases} \xi_1(\sigma) = \kappa_n'' + \kappa_g'' + (\kappa_n + \kappa_g)(\kappa_n^2 + \kappa_g^2) + (\kappa_g + \tau_g)(\kappa_g' - \kappa_n \tau_g) + (\kappa_n - \tau_g)(\kappa_n' + \kappa_g \tau_g) + 2\kappa_n(\kappa_n' - \tau_g') + 2\kappa_g(\kappa_g' + \tau_g'), \\ \xi_2(\sigma) = \kappa_g'' + \tau_g'' + (\kappa_g + \tau_g)(\kappa_g^2 - \tau_g^2) + (\kappa_n - \tau_g)(\tau_g' + \kappa_n \kappa_g) + (\kappa_n + \kappa_g)(\kappa_g' + \kappa_n \tau_g) + 2\kappa_g(\kappa_n' + \kappa_g') + 2\tau_g(\kappa_n' + \tau_g'), \\ \xi_3(\sigma) = \kappa_n'' - \tau_g'' + (\kappa_n - \tau_g)(\kappa_n^2 - \tau_g^2) + (\kappa_n + \kappa_g)(\kappa_n' - \kappa_g \tau_g) + (\kappa_g + \tau_g)(\kappa_n \kappa_g - \tau_g') + 2\kappa_n(\kappa_n' + \kappa_g') - 2\tau_g(\kappa_g' + \tau_g'). \end{cases}$$

Then

$$\tau_{\mathcal{L}}(\vartheta) = \sqrt{3} \left\{ \frac{\xi_1 [(\kappa_n - \tau_g)(\kappa_n' + \kappa_g') - (\kappa_n + \kappa_g)(\kappa_n' - \tau_g')] + (\kappa_n + \kappa_g)(\kappa_n - \tau_g)(\xi_3 \tau_g - \xi_1 \kappa_g) + (\kappa_g' + \tau_g') [\xi_3(\kappa_n + \kappa_g) - \xi_1(\kappa_n - \tau_g)] + (\kappa_g + \tau_g) [\xi_1(\kappa_n' - \tau_g') - \xi_3(\kappa_n' + \kappa_g')] + (\kappa_g + \tau_g) [\kappa_g(\kappa_n - \tau_g) + \kappa_n(\xi_1(\kappa_n + \kappa_g) - \xi_3(\kappa_n - \tau_g))] + (\kappa_n + \kappa_g)^2(\xi_3 \kappa_g - \xi_2 \kappa_n) + (\kappa_n - \tau_g)^2(\xi_2 \kappa_n - \xi_1 \tau_g) - (\kappa_g + \tau_g)^2(\xi_1 \tau_g + \xi_3 \kappa_g)}{[(\kappa_n + \kappa_g)(\kappa_n' - \tau_g') - (\kappa_g' + \tau_g')(\kappa_n - \tau_g) + (\kappa_n + \kappa_g)[\kappa_n(\kappa_n + \kappa_g) - \kappa_g(\kappa_n - \tau_g)] - \tau_g[(\kappa_n + \kappa_g)^2 + (\kappa_n - \tau_g)^2]} - [(\kappa_n - \tau_g)(\kappa_n' + \kappa_g') - (\kappa_n' - \tau_g')(\kappa_n + \kappa_g) + (\kappa_n + \kappa_g)[\kappa_g(\kappa_n - \tau_g) + \tau_g(\kappa_n + \kappa_g)] + \kappa_n[(\kappa_n - \tau_g)^2 - (\kappa_n + \kappa_g)^2]}^2 + [(\kappa_n + \kappa_g)(\kappa_g' + \tau_g') - (\kappa_n' + \kappa_g')(\kappa_g + \tau_g) + (\kappa_n - \tau_g)[\tau_g(\kappa_n + \kappa_g) - \kappa_n(\kappa_g + \tau_g)] + \kappa_g[(\kappa_n + \kappa_g)^2 - (\kappa_g + \tau_g)^2]}^2 \right\}$$

Let the timelike surface (see Figure 1) in the Minkowski 3-space \mathfrak{R}_1^3 is defined as

$$\begin{aligned} \Psi : \mathcal{V} \subset \mathfrak{R}^e &\rightarrow \mathfrak{R}_\infty^3, \\ (\sigma, t) &\rightarrow \Psi(\sigma, t) = \chi(\sigma) + t e(\sigma), \\ \Psi(\sigma, t) &= (\sigma, t \sin \sigma, t \cos \sigma), \end{aligned} \tag{26}$$

where $t \in (-1, 1)$. Then we get the moving Darboux frame $\{T, g, n\}$ along the curve χ as follows:

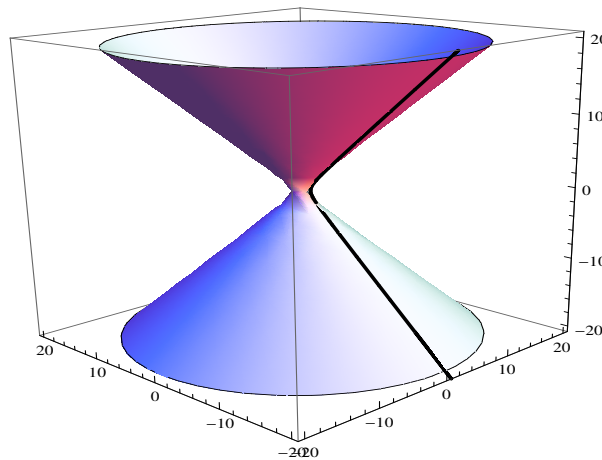


Fig. 2: The timelike $T\mathfrak{g}$ -Smarandache curves \mathcal{L} on S_1^2 .

$$\begin{aligned} T(\sigma) &= (1, 0, 0), \\ \mathfrak{g}(\sigma) &= (0, -\sin \sigma, -\cos \sigma), \\ \mathfrak{n}(\sigma) &= \frac{1}{\sqrt{1-t^2}}(-t, -\cos \sigma, \sin \sigma), \end{aligned} \tag{27}$$

where $\mathfrak{g}(\sigma)$ and $\mathfrak{n}(\sigma)$ are a unit spacelike vectors. Moreover, the geodesic curvature $\kappa_g(\sigma)$, the asymptotic curvature $\kappa_n(\sigma)$, and the principal curvature $\tau_g(\sigma)$ of the curve χ have the form

$$\begin{aligned} \kappa_n(\sigma) &= \kappa_g(\sigma) = 0, \\ \tau_g(\sigma) &= \frac{1}{\sqrt{1-t^2}}. \end{aligned} \tag{28}$$

Then, the timelike $T\mathfrak{g}$ -Smarandache curves \mathcal{L} of the curve χ is given by (see Figure 2)

$$\mathcal{L}(\vartheta(\sigma)) = \frac{1}{\sqrt{2}}(1, -\sin \sigma, -\cos \sigma). \tag{29}$$

The timelike $T\mathfrak{n}$ -Smarandache curves \mathcal{L} of the curve χ is given by (see Figure 3)

$$\mathcal{L}(\vartheta(\sigma)) = \frac{1}{\sqrt{2}}\left(1 - \frac{t}{\sqrt{1-t^2}}, -\frac{\cos \sigma}{\sqrt{1-t^2}}, \frac{\sin \sigma}{\sqrt{1-t^2}}\right). \tag{30}$$

The timelike $\mathfrak{g}\mathfrak{n}$ -Smarandache curves \mathcal{L} of the curve χ is given by (see Figure 4)

$$\mathcal{L}(\vartheta(\sigma)) = \frac{1}{\sqrt{2}}\left(-\frac{t}{\sqrt{1-t^2}}, -\frac{\cos \sigma}{\sqrt{1-t^2}} - \sin \sigma, \frac{\sin \sigma}{\sqrt{1-t^2}} - \cos \sigma\right). \tag{31}$$

The timelike $T\mathfrak{g}\mathfrak{n}$ -Smarandache curves \mathcal{L} of the curve χ is given by (see Figure 5)

$$\mathcal{L}(\vartheta(\sigma)) = \frac{1}{\sqrt{3}}\left(1 - \frac{t}{\sqrt{1-t^2}}, -\frac{\cos \sigma}{\sqrt{1-t^2}} - \sin \sigma, \frac{\sin \sigma}{\sqrt{1-t^2}} - \cos \sigma\right) \tag{32}$$

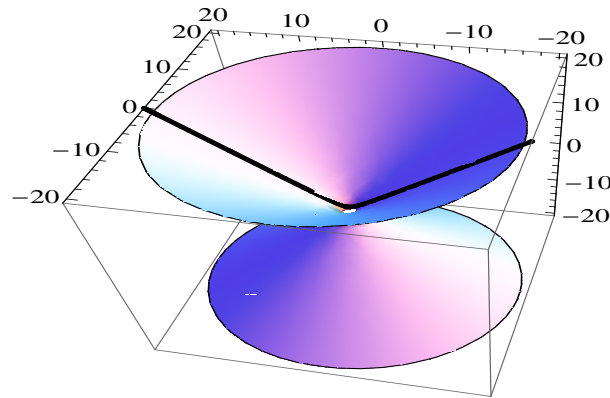


Fig. 3: The timelike Tn -Smarandache curves \mathcal{L} on S_1^2 .

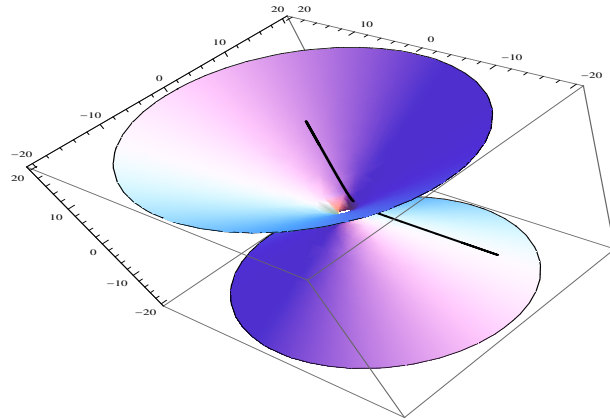


Fig. 4: The timelike gn -Smarandache curves \mathcal{L} on S_1^2 .

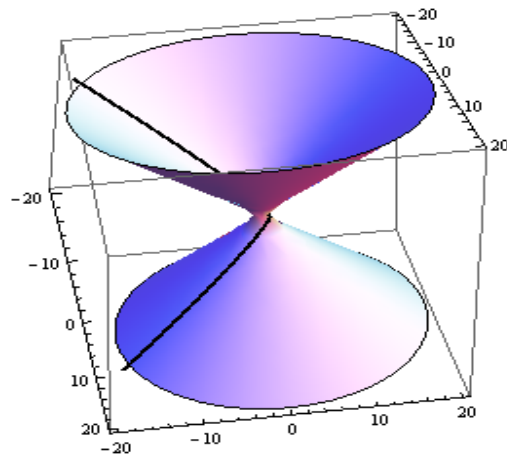


Fig. 5: The timelike Tgn -Smarandache curves \mathcal{L} on S_1^2 .

4 Conclusion

As a conclusion of our results, the special timelike Smarandache curves according to the Darboux frame of a curve χ lying fully on a timelike surface M in Minkowski 3-space \mathfrak{R}_1^3 has curvature functions related with the relationship with invariant $\kappa_n(\sigma)$, $\kappa_g(\sigma)$ and $\tau_g(\sigma)$ of χ of a curve χ when it is a geodesic, an asymptotic, or a principal curve.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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