Numerical and Computational Methods in Sciences & Engineering
An International Journal

http://dx.doi.org/10.18576/ncmse/020201

Propagation of sech-type solutions for conformable fractional nonlinear Schrödinger models

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Received: 12 March 2020, Revised: 23 May 2020, Accepted: 27 May 2020
Published online: 1 July 2020

Abstract: In this paper, we apply the semi-inverse variational principle for finding some explicit formulas of exact solutions for the conformable space-time fractional nonlinear (1+1)-dimensional Schrödinger models. The method is used to obtain exact solutions for three types of space-time fractional nonlinear Schrödinger equations, namely, space-time fractional NLS^{+}, space-time fractional NLS^{-}, and space-time fractional UNLS.

Keywords: Schrödinger models; The semi-inverse variational principle; Conformable fractional derivative; Symbolic computation

1 Introduction

Erwin Schrödinger, the Austrian physicist, in 1926 formulated the linear Schrödinger equation(LSE) for the first time. The equation could describe the time evolution of wave functions of non relativistic quantum particles as well. After that, scientists investigated the ways in which solutions of LSE can be associated to classical particle motion. Nonlinear Schrödinger equations(NLSEs) have an essential role in the study of many physical problems, such as quantum field theory, superconductivity, nonlinear optics, plasma physics, oceanography, and so on. A nonlinear Schrödinger evolution equation involves a dynamical balance between linear dispersive spreading of the wave and nonlinear self-interaction of the wave. Generalizations of the physically relevant with nonlinear and dispersive parameters have presented searching for the interplay between these two effects [1,2,3,4,5].

The equation has the same central importance to quantum mechanics as Newton's laws of motion have for the large-scale phenomena of classical mechanics [6]. The general form of NLSE is given by

$$i\,u_t + u_{xx} + 2\,\lambda\,|u|^2\,u = 0, \tag{1}$$

where $i = \sqrt{-1}$, $\lambda \neq 0$ is a real number and $u = u(x,t)$ is a complex-valued function, x shows the non-dimensional distance along the fiber while t shows time in dimensionless form [7,8].

There are many different methods which investigated finding solitary wave solutions of NLSEs. Ebaid and Khaled [9] applied a direct algebraic method and obtained the Jacobian-elliptic and Weierstrass-elliptic function solutions of the NLS model. Later on, Taghizadeh et al. [10] by using the first integral method obtained the traveling wave solutions of the NLS model. Islam [11] adopted the enhanced $(\frac{G'}{G})$-expansion function method to obtain the exact traveling wave solutions of the NLS model. Hafez [12] derived new general solutions of the NLS model by using the novel $(\frac{G'}{G})$-expansion function method.

Fractional partial differential equations (FPDEs), as generalization of the classical integer order partial differential equations, appear in many applications of the applied sciences for modelling problems in many different areas, such as finance, fluid flow, phase transition, stratified diffusions, damping law, rheology, diffusion processes, quantum, mechanics, viscoelasticity, quantum, chemistry, and so on. FPDEs are powerful instruments to describe real world problems more accurately than the classical integer-order ones. There are many different FDEs which are direct extensions of the integer-order differential equations. Fractional Riccati equation, fractional Bernoulli equation, and fractional Biswas-Milovic

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equation, just show some of these extensions. It must be noted that, even though FPDEs have a 300 years history, but their applications return to the last two decades.

Recently, Darvishi et al. [13] first introduced the space-time fractional version of the variety of Schrödinger models and then determined optical soliton solutions for each one. They applied the sine-cosine method, to obtain exact solutions of the space-time fractional (1+1)-dimensional NLS models. In [13], the conformable fractional derivative which is recently proposed by Khalil et al. [14] is applied. This fractional derivative covers the deficiencies of the previous fractional derivatives. One of this deficiencies is applying the chain rule. The fractional differentiation and some of its properties are presented as follow:

Definition 1. Suppose that $\alpha \in (0,1]$, and function $f(t)$ has defined for all positive t, the αth order of the conformable derivative of f with respect to t is defined as

$$D_t^{\alpha}(f(t)) = \lim_{\tau \to 0} \frac{f(t+\tau t^{1-\alpha}) - f(t)}{\tau},$$

f is called an α-conformable differentiable function.

Theorem 1. The followings are hold for α-differentiable functions f, and g, for all positive t, and $\alpha \in (0,1]$:

$$(i) \qquad D_t^{\alpha}(af+bg) = aD_t^{\alpha}(f) + bD_t^{\alpha}(g), \text{for all } a,b \in \mathbb{R}.$$

$$(ii) \qquad D_t^{\alpha}(t^{\beta}) = \beta t^{\beta-\alpha}, \text{for all } \beta \in \mathbb{R}.$$

$$(iii) \qquad D_t^{\alpha}(fg) = gD_t^{\alpha}(f) + fD_t^{\alpha}(g).$$

$$(iv) \qquad D_t^{\alpha}\left(\tfrac{f}{g}\right) = \tfrac{gD_t^{\alpha}(f) - fD_t^{\alpha}(g)}{g^2}.$$

Theorem 2. For differentiable and α-conformable differentiable function f and for differentiable function g defined in the range of f, we have

$$D_t^{\alpha}(f \circ g) = t^{1-\alpha} g(t)^{\alpha-1} g'(t) D_t^{\alpha}(f(t))_{t=g(t)},$$

where prime denotes the classical derivative with respect to t.

In this paper, we apply the semi-inverse variational principle (SVP) to obtain new exact solutions for the nonlinear (1+1) dimensions space-time fractional Schrödinger models. The first space-time fractional NLS model, namely, NLS$^+$ model is

$$iD_t^{\alpha}u + D_x^{2\beta}u + \gamma|u|^2u = 0. \tag{2}$$

The second space-time fractional NLS model, that is NLS$^-$, has the following form

$$iD_t^{\alpha}u + D_x^{2\beta}u - \gamma|u|^2u = 0. \tag{3}$$

The last space-time fractional model is an unstable space-time fractional nonlinear Schrödinger (UNLS) model which is given by

$$iD_t^{\alpha}u + D_x^{2\beta}u + 2|u|^2u - 2\gamma u = 0. \tag{4}$$

In all models we have $0 < \alpha \leq 1$, and $0 < \beta \leq 1$.
The goal of this paper is solving these three kinds of nonlinear Schrödinger equations by the semi-inverse variational principle.
The outline of this paper is as follows. In Section 2, we present a review on semi-inverse variational principle. In Section 3, some exact solutions of space-time fractional NLS models are presented. Finally, Section 4 provides a conclusion remark about the obtained results.

2 Semi-inverse variational principle

In this section, we give the main steps of the semi-inverse variational principle [16] for solving the nonlinear time-space fractional partial differential equations. They are extensions of the method for integer order differential equations.

Step 1. Consider the general form of a time-space fractional partial differential equation with independent variables x,t and a dependent variable u as

$$P(u, D_t^\alpha u, D_x^\beta u, D_t^{2\alpha} u, \cdots) = 0, \tag{5}$$

where $0 < \alpha \leq 1$, and $0 < \beta \leq 1$. Li et al. presented a fractional complex transform to convert fractional differential equations to integer order ones as follows (cf. [15]):

$$u(x,t) = u(\xi), \qquad \xi = \frac{kx^\beta}{\beta} - \frac{wt^\alpha}{\alpha}, \tag{6}$$

where k and w are non zero arbitrary constants and $\beta = \alpha \neq 0$. Substituting (6) into (5) yields a nonlinear ordinary differential equation as

$$P(u, u', u'', u''', \dots) = 0, \tag{7}$$

where the prime denotes differentiation with respect to ξ.

Step-2: If possible, integrate (7) term by term one or more times. This yields constant(s) of integration. For simplicity, the integration constant(s) can be set to zero.

Step-3: According to He's semi-inverse method, we construct the following trial-functional

$$J(U) = \int L d\xi, \tag{8}$$

where L is an unknown function of U and its derivatives.
There exist alternative approaches to the construction of the trial-functionals, see e.g., Refs. [16,17].

Step-4: By the Ritz method, we can obtain different forms of solitary wave solutions, in the form

$$U(\xi) = p\,\mathrm{sech}^n(q\xi), \tag{9}$$

where p and q represent the amplitude and inverse widths of the soliton solutions that need to be determined. Substituting (9) into (8) and making J stationary with respect to p and q results in

$$\frac{\partial J}{\partial p} = 0, \qquad \frac{\partial J}{\partial q} = 0. \tag{10}$$

The values of p and q are obtained by solving system (10). Hence the soliton solution of (9) will be determined as well.

3 Application of Semi-inverse variational principle

The application of the semi-inverse variational principle has illustrated in this section on nonlinear Schrödinger's models. Indeed, the semi-inverse variational principle is applied to solve space-time fractional NLS$^+$, space-time fractional NLS$^-$, and space-time fractional UNLS.

3.1 The space-time fractional NLS$^+$

We first consider the following space-time fractional NLS$^+$:

$$iD_t^\alpha u + D_x^{2\beta} u + 2\gamma |u|^2 u = 0. \tag{11}$$

We use the transformations

$$u(x,t) = U(\xi)\,e^{i\theta}, \quad \xi = \frac{kx^\beta}{\beta} - \frac{wt^\alpha}{\alpha}, \; \theta = \frac{dx^\beta}{\beta} - \frac{ct^\alpha}{\alpha}, \tag{12}$$

where k, w, d and q are constants, while w is the wave speed that obtain further, U gives the soliton amplitude component and θ gives the soliton phase component. Putting (12) into (11) yields:

$$\begin{aligned} D_t^{\alpha} u &= (-wU' - icU)\, e^{i\theta}, \\ D_x^{2\beta} u &= (k^2 U'' + 2idkU' - d^2 U)\, e^{i\theta}. \end{aligned} \tag{13}$$

Setting (13) into (11), gives the following equation:

$$(-w + 2dk)\, iU' + k^2 U'' + (c - d^2)U + 2\gamma U^3 = 0. \tag{14}$$

From the imaginary part, we obtain $w = 2dk$. In this case (14) can be rewritten as

$$k^2 U'' + (c - d^2)U + 2\gamma U^3 = 0, \tag{15}$$

where c, d and k are complex constants and non-zero. According to Ref. [16], by SVP [17], we can obtain the following variational formulation:

$$J(v) = \int_0^{\infty} \left[\frac{k^2}{2} (U')^2 - \frac{c - d^2}{2} U^2 - \frac{\gamma}{2} u^4 \right] d\xi. \tag{16}$$

We assume that the bright soliton solution of (11) has the following form

$$u(\xi) = p\,\mathrm{sech}(q\xi), \tag{17}$$

where p and q are unknown constants to be further determined.
By substituting (17) into (16) we obtain

$$\begin{aligned} J &= \int_0^{\infty} \left[\left(-\frac{1}{2} p^2 k^2 q^2 - \frac{1}{2} \gamma p^4 \right) \mathrm{sech}^4(q\xi) \right] d\xi + \int_0^{\infty} \left[\left(\frac{1}{2} p^2 k^2 q^2 - \frac{1}{2} p^2 c + \frac{1}{2} p^2 d^2 \right) \mathrm{sech}^2(q\xi) \right] d\xi \\ &= \frac{1}{q} \left(-\frac{1}{2} p^2 k^2 q^2 - \frac{1}{2} \gamma p^4 \right) \int_0^{\infty} \mathrm{sech}^4(\theta)\, d\theta + \frac{1}{q} \left(\frac{1}{2} p^2 k^2 q^2 - \frac{1}{2} p^2 c + \frac{1}{2} p^2 d^2 \right) \int_0^{\infty} \mathrm{sech}^2(\theta)\, d\theta \\ &= \frac{2}{3q} \left(-\frac{1}{2} p^2 k^2 q^2 - \frac{1}{2} \gamma p^4 \right) + \frac{1}{q} \left(\frac{1}{2} p^2 k^2 q^2 - \frac{1}{2} p^2 c \frac{1}{2} p^2 d^2 \right). \end{aligned}$$

To find constants p and q, we solve the following equations

$$\begin{aligned} \frac{\partial J}{\partial p} &= -\frac{p}{3q} \left(-k^2 q^2 + 4\gamma p^2 + 3c - 3d^2 \right), \\ \frac{\partial J}{\partial q} &= \frac{p^2}{6q^2} \left(k^2 q^2 + 2\gamma p^2 + 3c - 3d^2 \right), \end{aligned} \tag{18}$$

or

$$\begin{aligned} -k^2 q^2 + 4\gamma p^2 + 3c - 3d^2 &= 0, \\ k^2 q^2 + 2\gamma p^2 + 3c - 3d^2 &= 0, \end{aligned} \tag{19}$$

from (18) or (19), we can easily obtain the following relations:

$$p = \pm \frac{\sqrt{-\gamma(c - d^2)}}{\gamma} \quad , \quad q = \pm \frac{\sqrt{d^2 - c}}{k}. \tag{20}$$

Therefore by (17), we obtain the exact sech-type solutions for (15) and then the exact bright soliton solutions of (11) can be written as

$$u(x,t) = \pm \frac{\sqrt{-\gamma(c-d^2)}}{\gamma}\, \mathrm{sech} \left[\pm \frac{\sqrt{d^2 - c}}{k} \left(\frac{kx^{\beta}}{\beta} - \frac{2dkt^{\alpha}}{\alpha} \right) \right] \times e^{\frac{i\left(dx^{\beta}\alpha - ct^{\alpha}\beta\right)}{\beta\alpha}}, \tag{21}$$

where k, d and c are arbitrary constants.
We have plotted the exact bright soliton solutions of (11) as presented in (21) for $\beta = 0.5$, which have been depicted in Fig. 1, for various values of $\alpha = 0.6, 0.7, 0.8, 0.9, 1$.

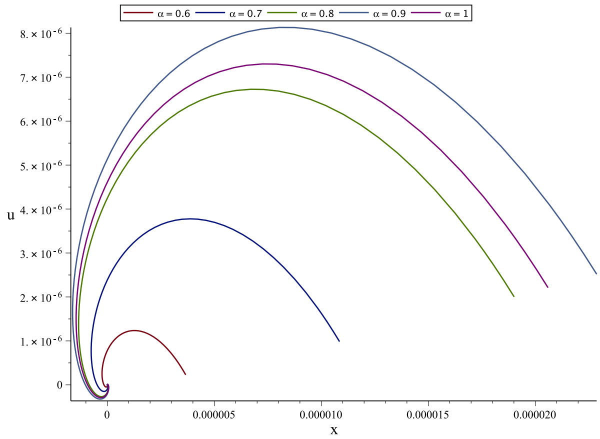

Fig. 1: Plots with respect to varying α for bright soliton solution (21) with $\beta = 0.5$ and $t = 2$.

3.2 The space-time fractional NLS$^-$

Now, we consider the following space-time fractional NLS$^-$:

$$iD_t^{\alpha}u + D_x^{2\beta}u - 2\gamma|u|^2u = 0. \tag{22}$$

Using the transformations

$$u(x,t) = U(\xi)e^{i\theta},\ \xi = \tfrac{kx^{\beta}}{\beta} - \tfrac{wt^{\alpha}}{\alpha},\ \theta = \tfrac{dx^{\beta}}{\beta} - \tfrac{ct^{\alpha}}{\alpha}, \tag{23}$$

yields $w = 2dk$ and changes Eq. (22) to the following ODE

$$k^2U'' + (c - d^2)U - 2\gamma U^3 = 0, \tag{24}$$

where c, d and k are complex constants and non-zero. By SVP, the following variational formulae is obtained:

$$J(v) = \int_0^{\infty}\left[\frac{k^2}{2}(U')^2 - \frac{c-d^2}{2}U^2 + \frac{\gamma}{2}u^4\right]d\xi. \tag{25}$$

We assume that the bright soliton solutions of (22) has the following form

$$u(\xi) = p\,\text{sech}(q\xi), \tag{26}$$

where p and q are unknown constants to be determined further.
Substituting (26) into (25) yields

$$\begin{aligned}
J &= \int_0^{\infty}\left[\left(-\frac{1}{2}p^2k^2q^2 + \frac{1}{2}\gamma p^4\right)\text{sech}^4(q\xi)\right]d\xi + \int_0^{\infty}\left[\left(\frac{1}{2}p^2k^2q^2 - \frac{1}{2}p^2c + \frac{1}{2}p^2d^2\right)\text{sech}^2(q\xi)\right]d\xi \\
&= \frac{1}{q}\left(-\frac{1}{2}p^2k^2q^2 + \frac{1}{2}\gamma p^4\right)\int_0^{\infty}\text{sech}^4(\theta)\,d\theta + \frac{1}{q}\left(\frac{1}{2}p^2k^2q^2 - \frac{1}{2}p^2c + \frac{1}{2}p^2d^2\right)\int_0^{\infty}\text{sech}^2(\theta)\,d\theta \\
&= \frac{2}{3q}\left(-\frac{1}{2}p^2k^2q^2 + \frac{1}{2}\gamma p^4\right) + \frac{1}{q}\left(\frac{1}{2}p^2k^2q^2 - \frac{1}{2}p^2c + \frac{1}{2}p^2d^2\right).
\end{aligned}$$

To find constants p and q, we solve the following equations

$$\begin{aligned}
\frac{\partial J}{\partial p} &= \frac{p}{3q}\left(k^2q^2 + 4\gamma p^2 - 3c + 3d^2\right), \\
\frac{\partial J}{\partial q} &= -\frac{p^2}{6q^2}\left(-k^2q^2 + 2\gamma p^2 - 3c + 3d^2\right),
\end{aligned} \tag{27}$$

or

$$k^2q^2 + 4\gamma p^2 - 3c + 3d^2 = 0,$$
$$-k^2q^2 + 2\gamma p^2 - 3c + 3d^2 = 0, \tag{28}$$

from (27) or (28), the following relations are obtained:

$$p = \pm\frac{\sqrt{\gamma(c-d^2)}}{\gamma} \quad , \quad q = \pm\frac{\sqrt{d^2-c}}{k}. \tag{29}$$

Therefore by (26), we obtain the exact sech-type solutions for (24) and then the exact bright soliton solutions of (22) as follows

$$u(x,t) = \pm\frac{\sqrt{-\gamma(c-d^2)}}{\gamma}\operatorname{sech}\left[\pm\frac{\sqrt{d^2-c}}{k}\left(\frac{kx^\beta}{\beta} - \frac{2dkt^\alpha}{\alpha}\right)\right] \times \mathrm{e}^{\frac{i\left(dx^\beta\alpha - ct^\alpha\beta\right)}{\beta\alpha}}, \tag{30}$$

where k, d and c are constants.
The exact solutions (30) have plotted in Fig. 2, for $\beta = 0.5$, and $\alpha = 0.6, 0.7, 0.8, 0.9, 1$. They are bright soliton solutions

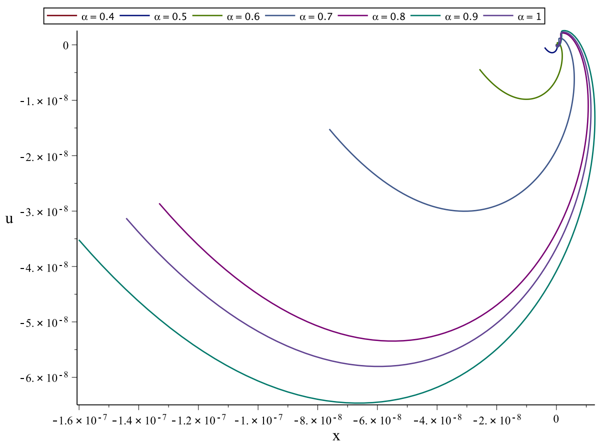

Fig. 2: Plots with respect to varying α for bright soliton solution (30) with $\beta = 0.5$ and $t = 2$.

for (22).

3.3 The space-time fractional UNLS

Finally, we consider the following space-time fractional UNLS:

$$iD_t^\alpha u + D_x^{2\beta} u + 2|u|^2 u - 2\gamma u = 0. \tag{31}$$

To change (31) to an ODE, we use the following transformations

$$u(x,t) = U(\xi)\mathrm{e}^{i\theta}, \ \xi = \frac{kx^\beta}{\beta} - \frac{wt^\alpha}{\alpha}, \ \theta = \frac{dx^\beta}{\beta} - \frac{qt^\alpha}{\alpha}, \tag{32}$$

which yields $w = 2dk$ and gives

$$k^2U'' + (q - d^2 - 2\gamma)U + 2U^3 = 0, \tag{33}$$

where c, d and k are non-zero complex constants. We obtain the following variational formulae by SVP:

$$J(v) = \int_0^\infty \left[\frac{k^2}{2}(U')^2 - \frac{c-d^2-2\gamma}{2}U^2 - \frac{1}{2}u^4\right] d\xi. \tag{34}$$

To obtain bright soliton solutions for (31), we assume that the bright soliton solution of (31) has the following form

$$u(\xi) = p\,\mathrm{sech}(q\xi), \tag{35}$$

where p and q are unknown constants to be determined further.
Substituting (35) into (34) gives

$$\begin{aligned} J &= \int_0^{\infty}\left[\left(-\frac{1}{2}p^2k^2q^2-\frac{1}{2}p^4\right)\mathrm{sech}^4(q\xi)\right]d\xi+\int_0^{\infty}\left[\left(\frac{1}{2}p^2\left(k^2q^2-c+d^2\right)+\gamma p^2\right)\mathrm{sech}^2(q\xi)\right]d\xi \\ &= \frac{1}{q}\left(-\frac{1}{2}p^2k^2q^2-\frac{1}{2}p^4\right)\int_0^{\infty}\mathrm{sech}^4(\theta)\,d\theta+\frac{1}{q}\left(\frac{1}{2}p^2\left(k^2q^2-c+d^2\right)+\gamma p^2\right)\int_0^{\infty}\mathrm{sech}^2(\theta)\,d\theta \\ &= \frac{2}{3q}\left(-\frac{1}{2}p^2k^2q^2-\frac{1}{2}p^4\right)+\frac{1}{q}\left(\frac{1}{2}p^2k^2q^2-\frac{1}{2}p^2c+\frac{1}{2}p^2d^2+\gamma p^2\right). \end{aligned}$$

To find constants p and q, we have to solve the following equations

$$\begin{aligned} \frac{\partial J}{\partial p} &= \frac{p}{3q}\left(k^2q^2-4p^2-3c+3d^2+6\gamma\right), \\ \frac{\partial J}{\partial q} &= -\frac{p^2}{6q^2}\left(-k^2q^2-2p^2-3c+3d^2+6\gamma\right), \end{aligned} \tag{36}$$

or

$$\begin{aligned} k^2q^2-4p^2-3c+3d^2+6\gamma &= 0, \\ -k^2q^2-2p^2-3c+3d^2+6\gamma &= 0. \end{aligned} \tag{37}$$

The solutions of (36) or (37) are:

$$p = \pm\sqrt{d^2+2\gamma-c} \quad , \quad q = \pm\frac{\sqrt{d^2+2\gamma-c}}{k}. \tag{38}$$

Hence from values in (38) and by (35), we obtain the exact sech-type solutions for (33) and then the exact bright soliton solutions of (31) as follows

$$u(x,t) = \pm\sqrt{d^2+2\gamma-c}\,\mathrm{sech}\left[\pm\frac{\sqrt{d^2+2\gamma-c}}{k}\left(\frac{kx^{\beta}}{\beta}-\frac{2dkt^{\alpha}}{\alpha}\right)\right] \times e^{\frac{i\left(dx^{\beta}\alpha-ct^{\alpha}\beta\right)}{\beta\alpha}}, \tag{39}$$

where k, d and c are constants.
Fig. 3 shows the exact bright soliton solutions (39) for UNLS equation (31). In this figure, we have $\beta = 0.5$, and values of α are $\alpha = 0.6, 0.7, 0.8, 0.9, 1$.

4 Conclusions

The semi-inverse variational principle is implemented to derive the sech-type solutions for space-time fractional NLS models that are considered in (1+1) dimensions. The semi-inverse variational principle provides a powerful mathematical tool to obtain more new exact sech-type and bright soliton solutions of many complex nonlinear models in physical and mathematical sciences.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

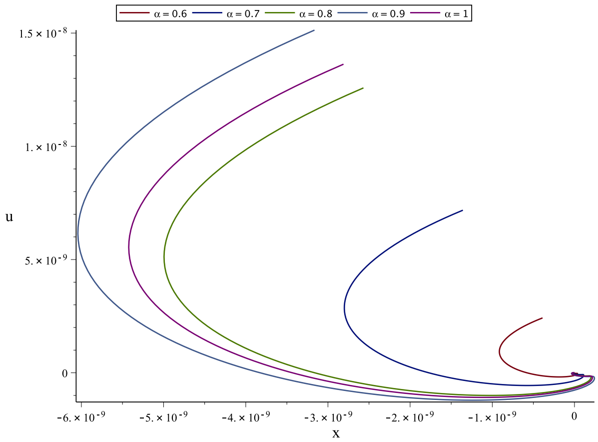

Fig. 3: Plots with respect to varying α for bright soliton solution (39) with $\beta = 0.5$ and $t = 2$.

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