Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/140414

Modeling and Analysis of Symptomatic and Asymptomatic Infections of Zika Virus Disease with Non-Monotonic Incidence Rate

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Received: 2 Feb. 2020, Revised: 2 Apr. 2020, Accepted: 8 Apr. 2020
Published online: 1 Jul. 2020

Abstract: In this Zika virus disease model, we have incorporated a non-monotonic incidence rate for the human to human transmission, which describes the psychological effect from the behavioural change of the susceptible individuals and the crowding effect of the infected individuals. The proposed epidemic model has seven mutually exclusive sub-classes of humans and mosquitoes which are analyzed by a suitable approach to control the Zika virus. Transmission of Zika virus is of three types, which are between human to human, human to mosquito and mosquito to human. The existence, the local stability of the disease-free, and the endemic equilibrium have been investigated in detail. Our model also exhibits backward bifurcation which suggests that merely reducing the basic reproduction number R_0 below one is not enough to make disease-free equilibrium globally stable. We presented the sensitivity analysis of some key parameters of the basic reproduction number. Later, we extended the model to the optimal control problem and obtained control strategies to reduce the symptomatic and asymptomatic infection levels. Numerical simulation is performed to support our analytical results.

Keywords: Zika Virus, Reproduction Number, Stability Analysis, Bifurcation Analysis, Sensitivity Analysis, Optimal Control.

1 Introduction

Zika virus(ZIKV) is transmitted to humans through the contact of an infected female mosquito of the Aedes genes species [1]. This mosquito-borne disease is documented in many tropical and sub-tropical regions, where Yellow fever, Chikungunya, and Dengue are also prevalent [2]. There is documented evidence of sexual transmission in nine different countries (The USA, Canada, Argentina, Chile, Italy, France, Portugal, Peru, and New Zealand). Recently, many blood transfusion cases have been also identified. Different symptoms of the ZIKV are usually mild and no specific medicine exists as a treatment. Most symptoms are visible for two to seven days [3]. The main symptoms of ZIKV are fever, fatigue, sweating, loss of appetite, joint pain, muscle pain, headache, conjunctivitis, skin rashes, and vomiting, etc. At present, no vaccine is available for the disease.

From April to May in the year of 2007, the first largest outbreak of ZIKV was identified in The Federal States of Micronesia [4]. In 2013, a large number of individuals were infected by ZIKV in French Polynesia and territories in the South Pacific. In 2015, some South American countries were affected by ZIKV, especially in Brazil and Colombia [5,6]. In 2016, the first sexual transmission of ZIKV case was detected in France. Between March 2014 and May 2015, 42 Guillain - Barre's Syndrome cases were reported in the French Polynesia outbreak, including 8 cases with microcephaly [7,8].

With all other symptoms of ZIKV including GBS [9] and microcephaly, infants were born to mothers who were infected with ZIKV during pregnancy. In 1st February 2016, the World Health Organization (WHO) declared that the ZIKV epidemic is a Public Health Emergency of International Concern [10] in response to multiple cases of microcephaly and many neurological disorders were identified in Brazil [11]. At the end of February 2016, Pan American Health Organization [12] noticed that ZIKV spread very quickly in Central and South American

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countries, while large number of confirmed cases were identified in the Caribbean (North America). According to WHO, ZIKV transmission is reported in more than 69 countries, while 13 countries have reported human to human sexual transmission of ZIKV [13].

Several mathematical epidemiological models have been formulated and described the transmission dynamics of the ZIKV disease. Daozhou et al. [1] presented a deterministic model where they computed basic reproduction number and numerical simulation through analyzing real data. Kucharski et al. [7] analyzed transmission dynamics of zika virus in Island Populations - A modelling analysis of the 2013-14 French Polynesia Outbreak. Moreno et al. [9] constructed and analyzed two-patch model for ZIKV. Agusto et al. [14] proposed a mathematical model by including vertical transmission in human population. They investigated the development of microcephaly among newborn babies through mother to child transmission. Many authors describe the dynamics of the ZIKV theoretically and mathematically. Bonyah et al. [15] constructed a deterministic model. They applied simple mass-action type incidence and used prevention, treatment, and insecticide as a control strategy to reduce the ZIKV. Srivastav et al. [16] formulated a new mathematical model for the transmission dynamics of ZIKV. In that model, they used standard mass-action type incidence and incorporated media impact for the human population to reduce the transmission. Srivastav et al. [17] proposed a complex network model. They computed basic reproduction number and sensitivity analysis to find which parameter is more sensitive to the basic reproduction number. The numerical simulation was performed using real-time data.

In our proposed model, we incorporated a non-monotone incidence rate for the human to human transmission. The significance of non-monotone incidence function is that the number of effective contacts reduces at high infective levels between infective populations and susceptible populations [18,19]. This paper is organized, as follows: Section 2 presents the formulation of mathematical model. Section 3 involves an analysis of the model and existence of equilibria. Section 4 discusses the existence of bifurcation. Section 5 presents the stability analysis of the model. Section 6 addresses the sensitivity indexes of the parameters of basic reproduction number. Section 7 demonstrates the numerical simulation of the model. Section 8 handles the optimal control model and its analysis. Section 9 demonstrates the numerical simulation results of the optimal control model. Section 10 is devoted to conclusion.

2 The model

We have proposed a new ZIKV model to investigate the dynamics of the disease transmission with a non-monotone incidence rate. The proposed model is formulated by considering both human and mosquito populations. The ZIKV transmission is possible between mosquito to human, human to mosquito and human to human. The human population has been divided into five different compartments according to the nature of the disease : Susceptible human population S_h (the individuals who can suffer from the disease, but are not yet infective) at time t, Exposed human population E_h (the individuals infected by disease, but not yet infectious) at time t, Symptomatic Infected individuals I_{sh} (number of infected individuals with symptoms and are capable of transmitting the disease) at time t, Asymptomatic Infected individuals I_{ah} (number of infected individuals without showing symptoms of the disease, but are capable of transmitting the disease) at time t, and Recovered human population R_h (the recovered individuals who has been recovered or removed from the host population by either permanent immunity or temporary immunity, or isolated or dead) at time t. Also, the vector population has been divided into two different compartments according to the nature of the disease : Susceptible mosquito S_v at time t, and Infected mosquito I_v at time t. Here, we assume that susceptible individuals after being exposed to the ZIKV infection can progress to the symptomatic infective and asymptotic infective with different transmission rates. Both types of infectious individuals recovered from the ZIKV infection spontaneously or therapeutically. However, the rates of recovery may vary from symptomatic compartment to asymptotic compartment.

In this model, we introduced a non-monotone incident function $\frac{\beta_1 S_h(I_{sh}+I_{ah})}{1+\alpha(I_{sh}+I_{ah})^2}$ for the human population to reduce the transmission, where $\beta_1(I_{sh}+I_{ah})$ is the force of infection of the ZIKV. The non-monotone incidence rate becomes the bilinear incidence rate [18] when $\alpha = 0$. Also, $\frac{1}{(I_{sh}+I_{ah})^2}$ represents the psychological effect from the behavioral change of the susceptible individuals when the number of infective individuals is very large [18,19], and the parameter α measures the psychological effect .

Consider Λ_h is the recruitment rate of human population, Λ_v is the recruitment of vector(mosquito) population, β_1 is the transmission rate between S_h and I_h, β_2 is the transmission rate between S_h and I_v, β_{sh} is the progression rate of E_h and I_{sh}, β_{ah} is the progression rate of E_h and I_{ah}, β_v is the transmission rate between I_h and S_v, μ_h is natural mortality rate of human population, μ_1 is the natural mortality rate of human population due to infection, μ_v is the natural mortality rate of vector(mosquito) population, γ_h is the recovery rate of symptomatic infective (human) population, γ_1 is the

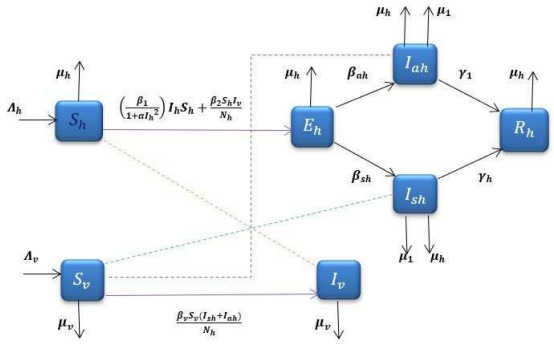

Fig. 1: Flow diagram of the model.

recovery rate of asymptomatic infective (human) population.

Keeping the above facts in mind, a mathematical model with its flow diagram given in Figure 1 is, as follows:

$$\begin{aligned}
\frac{dS_h}{dt} &= \Lambda_h - \beta_1 \frac{S_h(I_{sh}+I_{ah})}{1+\alpha(I_{sh}+I_{ah})^2} - \beta_2 \frac{S_h I_v}{N_h} - \mu_h S_h \\
\frac{dE_h}{dt} &= \beta_1 \frac{S_h(I_{sh}+I_{ah})}{1+\alpha(I_{sh}+I_{ah})^2} + \beta_2 \frac{S_h I_v}{N_h} - (\beta_{sh}+\beta_{ah}+\mu_h)E_h \\
\frac{dI_{sh}}{dt} &= \beta_{sh}E_h - (\gamma_h+\mu_h+\mu_1)I_{sh} \qquad (1) \\
\frac{dI_{ah}}{dt} &= \beta_{ah}E_h - (\gamma_1+\mu_h+\mu_1)I_{ah} \\
\frac{dR_h}{dt} &= \gamma_h I_{sh} + \gamma_1 I_{ah} - \mu_h R_h \\
\frac{dS_v}{dt} &= \Lambda_v - \beta_v \frac{S_v(I_{sh}+I_{ah})}{N_h} - \mu_v S_v \\
\frac{dI_v}{dt} &= \beta_v \frac{S_v(I_{sh}+I_{ah})}{N_h} - \mu_v I_v
\end{aligned}$$

2.1 Positivity and boundedness of the solutions

Here, we shall show that the positivity and boundedness of the population. From the system (1), we have

$$\begin{aligned}
\left(\frac{dS_h}{dt}\right)_{S_h=0} &= \Lambda_h > 0, \\
\left(\frac{dE_h}{dt}\right)_{E_h=0} &= \beta_1 \frac{S_h(I_{sh}+I_{ah})}{1+\alpha(I_{sh}+I_{ah})^2} + \beta_2 \frac{S_h I_v}{N_h} \geq 0, \\
\left(\frac{dI_{sh}}{dt}\right)_{I_{sh}=0} &= \beta_{sh}E_h \geq 0, \\
\left(\frac{dI_{ah}}{dt}\right)_{I_{ah}=0} &= \beta_{ah}E_h \geq 0, \\
\left(\frac{dR_h}{dt}\right)_{R_h=0} &= \gamma_h I_{sh} + \gamma_1 I_{ah} \geq 0, \\
\left(\frac{dS_v}{dt}\right)_{S_v=0} &= \Lambda_v > 0, \\
\left(\frac{dI_v}{dt}\right)_{I_v=0} &= \frac{\beta_v S_v(I_{sh}+I_{ah})}{N_h} \geq 0
\end{aligned}$$

Here, all the rates are non-negative, so if we start in the interior of the non-negative bounding $R_+^5 \times R_+^2$, we shall always remain in this cone keeping mind of the fact that direction of the vector field is inward on all the bounding planes. We note the change rate of the total population $N_h = S_h + E_h + I_{sh} + I_{ah} + R_h$ and $N_v = S_v + I_v$ are given by the following differential equations.

$$\begin{aligned}
\frac{dN_h}{dt} &= \Lambda_h - \mu_h N_h - \mu_1 (I_{sh}+I_{ah}) \\
\frac{dN_v}{dt} &= \Lambda_v - \mu_v N_v.
\end{aligned}$$

This gives $\limsup_{t\to\infty} \frac{dN_h}{dt} \leq \frac{\Lambda_h}{\mu_h}, lim\sup_{t\to\infty} \frac{dN_v}{dt} \leq \frac{\Lambda_v}{\mu_v}$. Therefore, all $S_h, E_h, I_{sh}, I_{ah}, R_h$ are bounded by $\frac{\Lambda_h}{\mu_h}$ and the solutions S_v, I_v are bounded by $\frac{\Lambda_v}{\mu_v}$. Hence, the biological feasible region of the system (1) is given by the following positively invariant region:

$$\Omega = \Omega_h \times \Omega_v \subset R_+^5 \times R_+^2$$

where,

$$\Omega_h = \{(S_h, E_h, I_{sh}, I_{ah}, R_h) \in R_+^5 : 0 \leq N_h \leq \frac{\Lambda_h}{\mu_h}\};$$
$$\Omega_v = \{(S_v, I_v) \in R_+^2 : 0 \leq N_v \leq \frac{\Lambda_v}{\mu_v}\}.$$

3 Analysis of the model

As $N_h = S_h + E_h + I_{sh} + I_{ah} + R_h$ and $N_v = S_v + I_v$, we consider the following form of the system for further

analysis:

$$\begin{aligned}
\frac{dN_h}{dt} &= \Lambda_h - \mu_h N_h - \mu_1 (I_{sh} + I_{ah}) \\
\frac{dE_h}{dt} &= \beta_1 \frac{(I_{sh} + I_{ah})(N_h - E_h - I_{sh} - I_{ah} - R_h)}{1 + \alpha (I_{sh} + I_{ah})^2} \\
&\quad + \frac{\beta_2 I_v}{N_h}(N_h - E_h - I_{sh} - I_{ah} - R_h) \\
&\quad - (\beta_{sh} + \beta_{ah} + \mu_h) E_h \\
\frac{dI_{sh}}{dt} &= \beta_{sh} E_h - (\gamma_h + \mu_h + \mu_1) I_{sh} \\
\frac{dI_{ah}}{dt} &= \beta_{ah} E_h - (\gamma_1 + \mu_h + \mu_1) I_{ah} \\
\frac{dR_h}{dt} &= \gamma_h I_{sh} + \gamma_1 I_{ah} - \mu_h R_h \\
\frac{dN_v}{dt} &= \Lambda_v - \mu_v N_v \\
\frac{dI_v}{dt} &= \beta_v \frac{(N_v - I_v)(I_{sh} + I_{ah})}{N_h} - \mu_v I_v.
\end{aligned} \tag{2}$$

3.1 Disease-free equilibrium point E_0

We consider the system (2) and find the disease-free equilibrium point. For our model, we have disease-free equilibrium point as $E_0=(N_h^0, E_h^0, I_{sh}^0, I_{ah}^0, R_h^0, N_v^0, I_v^0) = \left(\frac{\Lambda_h}{\mu_h}, 0, 0, 0, 0, \frac{\Lambda_v}{\mu_v}, 0\right)$.

3.2 The basic reproduction number R_0

We find the basic reproduction number R_0 by following the next generation matrix method as described in [20,21]. Following the same notations as in [20,21], we find the matrix $\mathscr{F}$ and $\mathscr{V}$, as follows:

$$\mathscr{F} = \begin{pmatrix} \frac{\beta_1 (I_{sh} + I_{ah}) P_h}{1 + \alpha (I_{sh} + I_{ah})^2} + \frac{\beta_2 P_h I_v}{N_h} \\ 0 \\ 0 \\ \beta_v \frac{(N_v - I_v)(I_{sh} + I_{ah})}{N_h} \end{pmatrix}$$

and

$$\mathscr{V} = \begin{pmatrix} (\beta_{sh} + \beta_{ah} + \mu_h) E_h \\ -\beta_{sh} E_h + (\gamma_h + \mu_h + \mu_1) I_{sh} \\ -\beta_{ah} E_h + (\gamma_1 + \mu_h + \mu_1) I_{ah} \\ \mu_v I_v \end{pmatrix}$$

where, $P_h = (N_h - E_h - I_{sh} - I_{ah} - R_h)$

F= Jacobian of $\mathscr{F}$ at $E_0 = \begin{pmatrix} 0 & \beta_1 N_h^0 & \beta_1 N_h^0 & \beta_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\beta_v N_v^0}{N_h^0} & \frac{\beta_v N_v^0}{N_h^0} & 0 \end{pmatrix}$

and, V= Jacobian of $\mathscr{V}$ at E_0

$$= \begin{pmatrix} \beta_{sh} + \beta_{ah} + \mu_h & 0 & 0 & 0 \\ -\beta_{sh} & \gamma_h + \mu_h + \mu_1 & 0 & 0 \\ -\beta_{ah} & 0 & \gamma_1 + \mu_h + \mu_1 & 0 \\ 0 & 0 & 0 & \mu_v \end{pmatrix}$$

and it follows that

$$FV^{-1} = \begin{pmatrix} \beta_1 N_h^0 D_1 (\beta_{sh} D_2 + \beta_{ah} D_3) & \beta_1 N_h^0 D_2 & \beta_1 N_h^0 & \beta_2 D_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\beta_v N_v^0 D_1 (\beta_{sh} D_2 + \beta_v D_3)}{N_h^0} & \frac{\beta_v N_v^0}{N_v^0} & \frac{\beta_v N_v^0}{N_v^0} & 0 \end{pmatrix},$$

where $D_1 = \frac{1}{\beta_{sh} + \beta_{ah} + \mu_h}, D_2 = \frac{1}{\gamma_h + \mu_h + \mu_1}, D_3 = \frac{1}{\gamma_1 + \mu_h + \mu_1}, D_4 = \frac{1}{\mu_v}$

The largest eigenvalue of FV^{-1} is called the basic reproduction number R_0 and is obtained, as follows:

$$R_0 = \frac{R_1 + \sqrt{R_1^2 + 4R_2}}{2}$$

where,
$R_1 = \beta_1 N_h^0 D_1 (\beta_{sh} D_2 + \beta_{ah} D_3)$ and
$R_2 = \frac{4\beta_2 \beta_v N_v^0 (\beta_{sh} D_2 + \beta_{ah} D_3) D_1 D_4}{N_h^0}$

Here, R_1 denotes the basic reproduction due to human to human transmission by ignoring the transmission due to vectors. Similarly, R_2 denotes the basic reproduction due to interactions with vectors in the absence of human to human transmission. The reproduction number R_0 gives the average number of infected individuals generated by the one infected in a fully susceptible population and for our model, it is given by the above expression of R_0.

3.3 Existence of endemic equilibrium

For the system (2), we get the endemic equilibrium point as $E_1 = (N_h^*, E_h^*, I_{sh}^*, I_{ah}^*, R_h^*, N_v^*, I_v^*)$
where

$$N_h^* = \frac{\Lambda_h - \mu_1 (I_{sh}^* + I_{ah}^*)}{\mu_h} = \frac{\Lambda_h}{\mu_h} - \frac{\mu_1 (d_1 + d_2)}{\mu_h} E_h^*$$

$$I_{sh}^* = \frac{\beta_{sh} E_h^*}{\gamma_h + \mu_h + \mu_1} = d_1 E_h^*$$

$$I_{ah}^* = \frac{\beta_{ah} E_h^*}{\gamma_1 + \mu_h + \mu_1} = d_2 E_h^*$$

$$R_h^* = \frac{\gamma_h I_{sh}^* + \gamma_1 I_{ah}^*}{\mu_h} = d_3 E_h^*$$

$$N_v^* = \frac{\Lambda_v}{\mu_v}$$

$$I_v^* = \frac{\beta_v \frac{\Lambda_v}{\mu_v}(d_1+d_2)E_h^*}{\beta_v(d_1+d_2)E_h^* + \mu_v\left(\frac{\Lambda_h}{\mu_h} - \frac{\mu_1(d_1+d_2)}{\mu_h}E_h^*\right)},$$

provided $\frac{\Lambda_h}{\mu_h} > \frac{\mu_1(d_1+d_2)}{\mu_h}E_h^*$

Substituting the value of $N_h^*, I_{sh}^*, I_{ah}^*, R_h^*, N_v^*, I_v^*$ in the equilibrium $\frac{dE_h^*}{dt}$ and E_h^* is the positive root of the following non-linear equation, we get

$$g(E_h) = \frac{D\beta_1(\Lambda_h - BE_h)}{1+\alpha D^2 E_h^2} - \mu_h(\beta_{sh}+\beta_{ah}+\mu_h) + \frac{\beta_2\Lambda_v\mu_h^2 D(\Lambda_h - BE_h)}{\mu_v(\Lambda_h - \mu_1 E_h D)\{\mu_h\beta_v E_h + \mu_v(\Lambda_h - \mu_1 E_h D)\}} = 0$$

$$g(0) = \frac{1}{(\beta_{sh}+\beta_{ah}+\mu_h)}(R_3 - 1) > 0, \quad for \quad R_3 = (R_1+R_2) > 1$$

$$g\left(\frac{\Lambda_h}{\mu_1}\right) = -\frac{\Lambda_h}{\mu_1}\left[\frac{D\beta_1(B-\mu_1)}{1+\alpha\left(\frac{D\Lambda_h}{\mu_1}\right)^2}\right] - \frac{\Lambda_h}{\mu_1}\left[\frac{D\beta_2\Lambda_v\mu_h^2(B-\mu_1)}{\mu_v\Lambda_h(1-d_1)\left[\frac{\mu_h\beta_v\Lambda_h D}{\mu_1} + \mu_v\Lambda_h(1-D)\right]}\right] - \mu_h(\beta_{sh}+\beta_{ah}+\mu_h) < 0$$

$$g(A) = -\left[\frac{D\beta_1(AB-\Lambda_h)}{1+\alpha DA^2}\right] - \mu_h(\beta_{sh}+\beta_{ah}+\mu_h) - \frac{\beta_2\Lambda_v\mu_h^2 D(AB-\Lambda_h)}{\mu_v(\Lambda_h - \mu_1 d_1 A)\{\mu_h\beta_v AD + \mu_v(\Lambda_h - \mu_1 AD)\}} < 0$$

Where, $A = \frac{\Lambda_h}{(\gamma_h+\mu_h+\mu_1)(\gamma_1+\mu_h+\mu_1)}, B = \mu_1 d_1 + \mu_h + \mu_h d_2 + \mu_h d_3,\ D = d_1 + d_2$

It is easy to observe that for $A < E_h < \frac{\Lambda_h}{\mu_1}$, $g(E_h)$ is always negative, i.e. there is no change of sign in $g(E_h)$. Thus, no root of $g(E_h)$ exists in the interval $A < E_h < \frac{\Lambda_h}{\mu_1}$. Hence, we can conclude that at least one root of $g(E_h) = 0$ exists in the interval $0 < E_h < A$.

$$\frac{dg(E_h)}{dE_h} = -\left[\frac{B\left[1+\alpha D^2\right] + 2\alpha D^2 P_2}{\mu_h\left[1+\alpha D^2 E_h^2\right]} + \frac{\mu_h\beta_2\Lambda_v BD}{\mu_v}\right] - \left[\frac{P_2\beta_2\Lambda_v E_h\mu_h^2\beta_v D^2\left[\Lambda_h - 2\mu_1 DE_h\right] + 2\mu_v P_1}{P_1\left[\mu_h\beta_v DE_h + \mu_v P_1\right]^2}\right] < 0$$

where $P_1 = \Lambda_h - \mu_1 DE_h$, $P_2 = (\Lambda_h - B)$

The above-mentioned expression is negative under the condition $\Lambda_h > B$ and $\Lambda_h > 2\mu_1 DE_h$, then we can say that there exists unique positive root E_h^* (say) of $\frac{dg(E_h)}{dI_h} = 0$

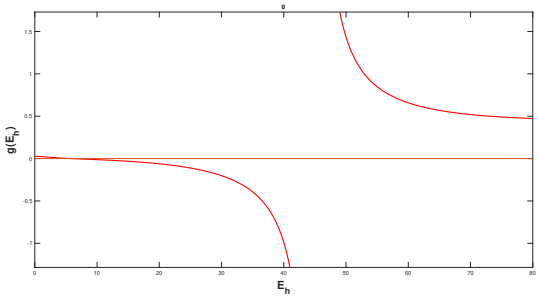

Fig. 2: Performing $g(E_h)$ with E_h for existence of one root of $g(E_h) = 0$ of the parameters as: $\beta_1 = 0.012, \beta_2 = 0.045, \beta_{sh} = 0.069, \beta_{ah} = 0.075, \beta_v = 0.8075, \Lambda_h = 2, \Lambda_v = 50, \mu_h = 0.009, \mu_v = 0.039, \mu_1 = 0.043, \gamma_h = 0.0258, \gamma_1 = 0.0236, \alpha = 0.02$

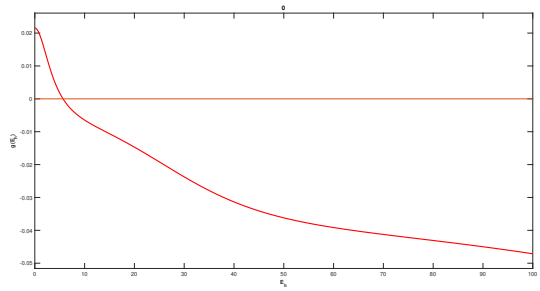

Fig. 3: Performing $g(E_h)$ with E_h for existence of one root of $g(E_h) = 0$ of the parameters as: $\beta_1 = 0.012, \beta_2 = 0.045, \beta_{sh} = 0.0069, \beta_{ah} = 0.0075, \beta_v = 0.8075, \Lambda_h = 2, \Lambda_v = 50, \mu_h = 0.009, \mu_v = 0.39, \mu_1 = 0.043, \gamma_h = 0.0258, \gamma_1 = 0.0236, \alpha = 0.02$

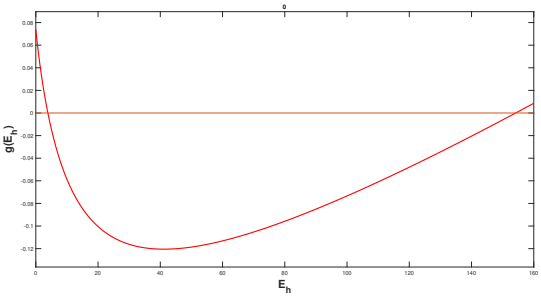

Fig. 4: Performing $g(E_h)$ with E_h for existence of one root of $g(E_h) = 0$ of the parameters as: $\beta_1 = 0.012, \beta_2 = 0.045, \beta_{sh} = 0.000069, \beta_{ah} = 0.000075, \beta_v = 0.8075, \Lambda_h = 2, \mu_h = 0.00012, \mu_v = 0.39, \mu_1 = 0.043, \gamma_h = 0.00258, \gamma_1 = 0.00236, \alpha = 0.02$

in the interval $0 < E_h < A$. Also, it is clear that if $\frac{dg(E_h)}{dI_n} < 0$ at A, then it must be negative for all E_h in the interval $0 < I_h < A$. Hence, under this condition, we get the positive equilibrium point $E_1 = (N_h^*, E_h^*, I_{sh}^*, I_{ah}^*, R_h^*, N_v^*, I_v^*)$. This fact is demonstrated in Figures 2 and 3 by choosing a suitable set of parameters. However, if $\frac{dg(E_h)}{dE_h}$ is not negative throughout the interval $0 < E_h < A$, then there is a possibility of more than one root of the given equation $g(E_h) = 0$. In general, vector-borne disease model exhibits backward bifurcation which corresponds to endemic equilibrium points for $R_0 < 1$. For our model too, we get two positive roots of $g(E_h) = 0$ for some suitable set of parameters. This fact is demonstrated in Figure 4. And corresponding to these two roots, we get two endemic equilibrium of our system (2).

4 Existence of bifurcation

In this section, we investigate the existence of Backward bifurcation for the model (2). To analyze this, we use the center manifold theory as described in Castillo-Chavez and Song [22], which is reproduced below for convenience.

Theorem 1. *Consider the following general system of ordinary differential equations with a parameter ϕ,*

$$\frac{dx}{dt} = f(x,\phi),$$
$$f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}.$$

and

$$f \in \mathbb{C}^2(\mathbb{R}^n \times \mathbb{R})$$

where 0 is the equilibrium point of the system $(i.e. f(0,\phi)) \equiv 0$ for all ϕ and
(i). $A = D_x f(0,0) = \left(\frac{\partial f_i}{\partial x_j}(0,0)\right)$ is the linearization matrix of the system around the equlibrium 0 with f evaluted at 0;
(ii). Zero is the simple eigenvalue of A and other eigenvalues of A have negative real parts;
(iii). Matrix A has a right eigenvector w and a left eigenvector v corresponding to the zero eigenvalue.

Let f_k be the kth component of f and

$$a_1 = \sum_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0,0)$$

$$b_1 = \sum_{k,i=1}^{n} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \phi}(0,0)$$

Then, the local dynamics of the system around the equilibrium point 0 is totally defined by the signs a_1 and b_1.
1. $a_1 > 0$, $b_1 > 0$. When $\phi < 0$ with $|\phi| \ll 1$, 0 is locally asymptotically stable, and there exists a positive unstable equilibrium; when $0 < \phi \ll 0$, 0 is unstable and there exists a negative and locally asymptotically stable equilibrium;
2. $a_1 < 0$, $b_1 < 0$. When $\phi < 0$ with $|\phi| \ll 1$, 0 is unstable; when $0 < \phi \ll 0$, 0 is locally asymptotically stable, and there exists a positive unstable equilibrium;
3. $a_1 > 0$, $b_1 < 0$. When $\phi < 0$ with $|\phi| \ll 1$ is unstable, and there exists a locally asymptotically stable negative equilibrium; when $0 < \phi \ll 0$, 0 is stable, and a positive unstable equilibrium appears;
4. $a_1 < 0$, $b_1 > 0$. When ϕ changes from negative to positive, 0 changes its stability from stable to unstable. Correspondingly, a negative unstable equilibrium becomes positive and locally asymptotically stable.

If $a_1 > 0$ and $b_1 > 0$, the backward bifurcation occurs at $\phi = 0$. Here, we take transmission rate β_1 as backward bifurcation parameter, so $R_0 = 1$ gives

$$\beta_1 = \frac{\mu_h}{\Lambda_h D + D_1 P_3}\left(1 - \frac{\mu_h \beta_2 \beta_v \Lambda_v D_1 D_4 P_3}{\mu_v \Lambda_h}\right)$$

where $P_3 = (\beta_{sh} D_2 + \beta_{ah} D_3)$

Then, let us consider the following change of variables

$N_h = x_1, E_h = x_2, I_{sh} = x_3, I_{ah} = x_4, R_h = x_5, N_v = x_6, I_v = x_7$. Also, using vector notation $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)^T$, our system (2) can be re-written as

$$\frac{dX}{dt} = F(x), where F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T$$

$$\begin{aligned}
x_1' &= \Lambda_h - \mu_h x_1 - \mu_1(x_3 + x_4)\\
x_2' &= \frac{\beta_1(x_1 - x_2 - x_3 - x_4 - x_5)(x_3 + x_4)}{1 + \alpha(x_3 + x_4)^2}\\
&\quad + \beta_2(x_1 - x_2 - x_3 - x_4 - x_5)\frac{x_7}{x_1} - (\beta_{sh} + \beta_{ah} + \mu_h)x_2\\
x_3' &= \beta_{sh} x_2 - (\gamma_h + \mu_h + \mu_1)x_3 \qquad (3)\\
x_4' &= \beta_{ah} x_2 - (\gamma_1 + \mu_h + \mu_1)x_4\\
x_5' &= \gamma_h x_3 + \gamma_1 x_4 - \mu_h x_5\\
x_6' &= \Lambda_v - \mu_v x_6\\
x_7' &= \beta_v \frac{(x_6 - x_7)(x_3 + x_4)}{x_1} - \mu_v x_6
\end{aligned}$$

The Jacobian of the above system at disease-free equilibrium point is given by

$$J(\beta_1) = \begin{pmatrix} -\mu_h & 0 & -\mu_1 & 0 & 0 & 0 & 0 \\ 0 & -m_1 & \beta_1 x_1 & \beta_1 x_1 & 0 & 0 & \beta_2 \\ 0 & \beta_{sh} & -m_2 & 0 & 0 & 0 & 0 \\ 0 & \beta_{ah} & 0 & -m_2 & 0 & 0 & 0 \\ 0 & 0 & \gamma_h & \gamma_1 & -\mu_h & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mu_v & 0 \\ 0 & 0 & \beta_v \frac{x_6}{x_1} & \beta_v \frac{x_6}{x_1} & 0 & \eta_v & -\mu_v \end{pmatrix}$$

Where $x_1 = \frac{\Lambda_h}{\mu_h}, x_6 = \frac{\Lambda_v}{\mu_v}, m_1 = (\beta_{sh} + \beta_{ah} + \mu_h), m_2 = (\gamma_h + \mu_h + \mu_1)$

4.1 Eigenvalues of $J_{\beta_1^*}$

It can be easily seen that the Jacobian with $\beta_1 = \beta_1^*$ of the linearized system has a simple zero eigenvalue and the other eigenvalues have negative real parts. Hence, the center manifold theorem can be used to analyze the dynamics of the syster (3) near $\beta_1 = \beta_1^*$. For the case when $R_0 = 1$, using the technique in Castillo-Chavez and Song [22], it can be shown that the matrix $J_{\beta_1^*}$ has a right eigenvector (corresponding to the zero eigenvalue) given by $w = [w_1, w_2, w_3, w_4, w_5, w_6, w_7]^T$, where

$$w_1 = -\frac{\mu_1}{\mu_h}, w_2 = \frac{\gamma_1 + \mu_h + \mu_1}{\beta_{ah}}, w_3 = 1,$$

$$w_4 = 1, w_5 = \frac{\mu_h}{\gamma_h \gamma_1}, w_6 = 0, w_7 = \frac{2\beta_v x_6}{\mu_v x_1}$$

Similarly, the matrix $J_{\beta_1^*}$ has a left eigenvector (corresponding to the zero eigenvalue) denoted by $v = [v_1.v_2, v_3, v_4, v_5, v_6, v_7]^T$, where

$$v_1 = 0, v_2 = 1, v_3 = \frac{\beta_{sh} + \beta_{ah} + \mu_h}{\beta_{sh}} - \frac{\beta_{ah}\beta_1 x_1}{\beta_{sh}(\gamma_1 + \mu_h)},$$

$$v_4 = \frac{\beta_1 x_1}{\gamma_1 + \mu_h}, v_5 = 0, v_6 = \frac{\eta_v}{\beta_2}, v_7 = \frac{\mu_v}{\beta_2}$$

4.2 Computation of a_1

For the system (3), the associated non-zero partial derivatives are given by

$$\frac{\partial^2 f_2}{\partial x_1 \partial x_3} = \beta_1 = \frac{\partial^2 f_2}{\partial x_3 \partial x_1}; \quad \frac{\partial^2 f_2}{\partial x_1 \partial x_4} = \beta_1 = \frac{\partial^2 f_2}{\partial x_4 \partial x_1};$$
$$\frac{\partial^2 f_2}{\partial x_1 \partial x_7} = \beta_2 \left(\frac{1 - x_1}{x_1^2}\right) = \frac{\partial^2 f_2}{\partial x_7 \partial x_1};$$
$$\frac{\partial^2 f_2}{\partial x_2 \partial x_3} = -\beta_1 = \frac{\partial^2 f_2}{\partial x_3 \partial x_2};$$
$$\frac{\partial^2 f_2}{\partial x_2 \partial x_4} = -\beta_1 = \frac{\partial^2 f_2}{\partial x_4 \partial x_2}; \quad \frac{\partial^2 f_2}{\partial x_2 \partial x_7} = -\frac{\beta_2}{x_1} = \frac{\partial^2 f_2}{\partial x_7 \partial x_2}$$
$$\frac{\partial^2 f_2}{\partial x_3^2} = -3\beta_1 = \frac{\partial^2 f_2}{\partial x_4^2}; \quad \frac{\partial^2 f_2}{\partial x_3 \partial x_4} = -\beta_1 = \frac{\partial^2 f_2}{\partial x_4 \partial x_3};$$
$$\frac{\partial^2 f_2}{\partial x_3 \partial x_5} = -\beta_1 = \frac{\partial^2 f_2}{\partial x_5 \partial x_3} \frac{\partial^2 f_2}{\partial x_3 \partial x_7} = -\frac{\beta_2}{x_1} = \frac{\partial^2 f_2}{\partial x_7 \partial x_3};$$
$$\frac{\partial^2 f_2}{\partial x_4 \partial x_5} = -\beta_1 = \frac{\partial^2 f_2}{\partial x_5 \partial x_4}; \quad \frac{\partial^2 f_2}{\partial x_4 \partial x_7} = -\frac{\beta_2}{x_1} = \frac{\partial^2 f_2}{\partial x_7 \partial x_4}$$
$$\frac{\partial^2 f_2}{\partial x_5 \partial x_7} = -\frac{\beta_2}{x_1} = \frac{\partial^2 f_2}{\partial x_7 \partial x_5}$$

It follows from the above expressions that

$$\begin{aligned} a_1 &= v_2 \left[2w_1 w_3 \beta_1 + 2w_1 w_4 \beta_1 + 2w_1 w_7 \beta_2 \left(\frac{1 - x_1}{x_1^2}\right)\right] \\ &- v_2 \left[2w_2 w_3 \beta_1 - 2w_2 w_4 \beta_1 - 2w_2 w_7 \frac{\beta_2}{x_1} - 3w_3^2 \beta_1\right] \\ &- v_2 \left[2w_3 w_4 \beta_1 + 2w_3 w_5 \beta_1 + 2w_3 w_7 \frac{\beta_2}{x_1} + 3w_4^2 \beta_1\right] \\ &+ v_2 \left[2w_4 w_5 \beta_1 + 2w_4 w_7 \frac{\beta_2}{x_1} + 2w_5 w_7 \frac{\beta_2}{x_1}\right] \\ &= -4\beta_1 \left[\frac{\mu_1}{\mu_h} + \frac{\gamma_1 + \mu_h + \mu_1}{\beta_{ah}} + 2\beta_1 + \frac{mu_h \beta_1}{\gamma_h + \gamma_h}\right] \\ &- 4\beta_1 \left[\frac{2\beta_v \mu_v^2 \Lambda_v}{\mu_v^2 \Lambda_v^2} + \frac{(\gamma_h + \mu_h + \mu_1)\beta_v \mu_h^2 \Lambda_v}{\beta_{ah} \mu_v^2 \Lambda_h^2}\right] \\ &- \frac{\beta_2 \beta_v \mu_h \Lambda_v}{\mu_v^2 \Lambda_h^2} \left[\left(\frac{\mu_h - \Lambda_h}{\Lambda_h}\right) - \left(\frac{\mu_h^2}{\gamma_h + \gamma_1}\right)\right] \end{aligned}$$

4.3 Computation of b_1

For the system (3), the associated non-zero partial derivatives are given by

$$\frac{\partial^2 f_2}{\partial x_3 \partial \beta_1} = x_1 = \frac{\partial^2 f_2}{\partial x_4 \partial \beta_1}$$

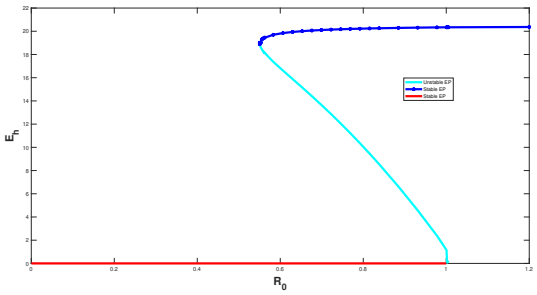

Fig. 5: Plot diagram is infective populations with reproduction number showing the backward bifurcation by considering bifurcation parameter β_1

It follows from the above expressions that

$$b_1 = v_2w_3x_1 + v_2w_4x_1 = \frac{2\Lambda_h}{\mu_h} > 0$$

Here, it is clear that the coefficient b_1 is positive and according to the theorem (4.1), the coefficient a_1 will define the phenomenon of backward bifurcation in our model. If the sign of the coefficient a_1 is positive, it implies that the model will have backward bifurcation around the disease-free equilibrium for $\beta_1 = \beta_1^*$ and the fact is demonstrated in **Figure 5**. This suggests that the disease-free is globally unstable.

5 Stability analysis

5.1 Local stability of disease-free equilibrium(DFE)

Theorem 2.*If $R_0 < 1$, the disease-free equilibrium E_0 is locally asymptotically stable, otherwise it is unstable.*

The Jacobian matrix of the system (2) at disease-free equilibrium point $E_0 = (N_h^0, 0, 0, 0, N_v^0, 0)$ is obtained, as follows:

$$J_0 = \begin{pmatrix} -\mu_h & 0 & -\mu_1 & 0 & 0 & 0 & 0 \\ 0 & -m_1 & \frac{\beta_1\Lambda_h}{\mu_v} & \frac{\beta_1\Lambda_h}{\mu_v} & 0 & 0 & \beta_2 \\ 0 & \beta_{sh} & -m_2 & 0 & 0 & 0 & 0 \\ 0 & \beta_{ah} & 0 & -m_2 & 0 & 0 & 0 \\ 0 & 0 & \gamma_h & \gamma_1 & -\mu_h & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mu_v & 0 \\ 0 & 0 & \frac{\beta_1\Lambda_v\mu_h}{\Lambda_h\mu_v} & \frac{\beta_v\Lambda_v\mu_h}{\Lambda_h\mu_v} & 0 & 0 & -\mu_v \end{pmatrix}$$

where, $m_1 = (\beta_{sh} + \beta_{ah} + \mu_h), m_2 = (\gamma_h + \mu_h + \mu_1)$

Clearly, three eigenvalues of the matrix J_0 are $-\mu_h$,$-\mu_h$ and $-\mu_v$ and the remaining four eigenvalues are the roots of the following bi-quadratic equation:

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

where,

$$a_3 = \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} + \frac{1}{D_4}$$

$$a_2 = \frac{1}{D_1D_2} + \frac{1}{D_2D_3} + \frac{1}{D_3D_1} + \frac{1}{D_1D_4} + \frac{1}{D_2D_4} + \frac{1}{D_3D_4} - \beta_1\beta_v\beta_{ah}(\beta_{sh} + \beta_{ah})\frac{N_v^0}{N_h^0}$$

$$a_1 = \frac{1}{D_1D_2} + \frac{1}{D_2D_3} + \frac{1}{D_3D_1} + \frac{1}{D_1D_2D_3D_4} + \beta_1N_h^0(\beta_{ah} - \beta_{sh}) - \beta_2\beta_v(\beta_{sh} + \beta_{ah})\frac{N_v^0}{N_h^0}$$

$$a_0 = \frac{1}{D_1D_2D_3D_4}(1 - R_0) + \frac{\beta_1\beta_{sh}}{D_4} - \frac{\beta_2\beta_{ah}D_4N_h^0}{D_2D_4} - (\frac{1}{D_1} + \frac{1}{D_4})$$

According to the Routh-Hurtwiz criteria, the above characteristics equation will give four negative roots or roots with negative real parts if the following conditions are satisfied:

$$a_3 > 0,\ a_2 > 0,\ a_1 > 0,\ a_0 > 0,\ \begin{vmatrix} a_3 & a_1 \\ 1 & a_2 \end{vmatrix} > 0,\ \begin{vmatrix} a_3 & a_1 & 0 \\ 1 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{vmatrix} > 0,$$

Thus, it follows that the disease-free equilibrium point E_0 is locally asymptotically stable if $R_0 < 1$.

5.2 Global stability of disease-free equilibrium (DFE)

To prove the global stability of disease-free equilibrium, we use the theorem described by Castillo-chavez et al. [20]

Theorem 3.*If the given mathematical model can be written in the form:*

$$\frac{dX}{dt} = F(X,Y),\ and\ \ \frac{dY}{dt} = G(X,Y),\quad G(X,0) = 0 \quad (4)$$

where $X = (N_h, N_v)^T$, $Y = (E_h, I_{sh}, I_{ah})^T$, denoting the number of uninfected and ZIKV infected people,

respectively. Then, the disease-free equilibrium is represented here by

$$E_0 = (X_0, 0) = (\frac{\Lambda_h}{\mu_h}, 0, 0, 0, \frac{\Lambda_v}{\mu_v}, 0)$$

For the global asymptotically stable, the condition (H_1) *and* (H_2) *given below must be satisfied.*

$$H_1 : for \ \frac{dX}{dt} = F(X_0, 0),$$
$$H_2 : G(X,Y) = AY - \widehat{G}(X,Y), \ \ \widehat{G}(X,Y) \geq 0,$$

Here, $A = D_Y G(X_0, 0)$ *is M-matrix (In M-matrix, all the off diagonal element of matrix are non-negative) . If the given system of differential equation in mathematical model satisfies the given condition in (4), then the point* $E_0 = (X_0, 0)$ *is a global asymptotically stable equilibrium of given mathematical model provided* $R_0 < 1$. *And for the given mathematical model, the result is shown in the next theorem, as given below.*

Theorem 4.*The point* $E_0 = (X_0, 0)$ *is global asymptotically stable equilibrium of given mathematical model provided* $R_0 < 1$ *and the condition given in (4) are satisfied under the condition* $\frac{N_v^0}{N_h^0} > \frac{N_v - I_v}{N_h}$.

*Proof.*Using theorem (4.2) to mathematical model system , we consider

$F(X_0, 0) = \Lambda - \mu S, \ \ G(X,Y) = AY - \widehat{G}(X,Y)$

where,

$$A = F - V = \begin{pmatrix} -m_1 & \frac{\beta_1 \Lambda_h}{\mu_v} & \frac{\beta_1 \Lambda_h}{\mu_v} & \beta_2 \\ \beta_{sh} & -m_2 & 0 & 0 \\ \beta_{ah} & 0 & -m_2 & 0 \\ 0 & \frac{\beta_1 \Lambda_v \mu_h}{\Lambda_h \mu_v} & \frac{\beta_1 \Lambda_v \mu_h}{\Lambda_h \mu_v} & -\mu_v \end{pmatrix}$$

then

$$G(X,Y) = AY - \widehat{G}(X,Y) = AY - \begin{pmatrix} \widehat{G}_1(X,Y) \\ \widehat{G}_2(X,Y) \\ \widehat{G}_3(X,Y) \\ \widehat{G}_4(X,Y) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\beta_1 (N_h^0 - S_h)(I_{sh} + I_{ah})}{1 + \alpha (I_{sh} + I_{ah})^2} - \beta_2 (1 - \frac{S_h}{N_h}) I_v \\ 0 \\ 0 \\ \beta_v \left(\frac{N_v^0}{N_h^0} - \frac{N_v - I_v}{N_h} \right) (I_{ah} + I_{sh}) \end{pmatrix}$$

Here, we can easily see $N_h^0 \geq S_h$, so $G(X,Y) \geq 0$ for all (X,Y), provided $\frac{N_v^0}{N_h^0} > \frac{N_v - I_v}{N_h}$. Also, by the definition of M matrix we can say that the matrix A is M matrix. Hence disease-free equilibrium (E_0) is global asymptotically stable under the condition $\frac{N_v^0}{N_h^0} > \frac{N_v - I_v}{N_h}$.

5.3 Local stability of endemic equilibrium (EE)

Theorem 5.*When* $R_0 > 1$, *then endemic equilibrium* E_1 *is locally asymptotically stable under some conditions, otherwise it is unstable.*

The Jacobian matrix of the system (2) at endemic equilibrium point $E_1 = (N_h^*, E_h^*, I_{sh}^*, I_{ah}^*, R_h^*, N_v^*, I_v^*)$ is obtained, as follows:

$$J_1 = \begin{pmatrix} -\mu_h & 0 & -\mu_1 & 0 & 0 & 0 & 0 \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & 0 & p_{27} \\ 0 & \beta_{sh} & -p_{33} & 0 & 0 & 0 & 0 \\ 0 & \beta_{ah} & 0 & -p_{44} & 0 & 0 & 0 \\ 0 & 0 & \gamma_h & \gamma_1 & -\mu_h & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mu_v & 0 \\ p_{71} & 0 & p_{73} & p_{74} & 0 & p_{76} & p_{77} \end{pmatrix}$$

where

$$p_{21} = \frac{\beta_1 (I_{sh}^* + I_{ah}^*)}{1 + \alpha (I_{sh}^* + I_{ah}^*)^2} - \frac{E_h^* + I_{sh}^* + I_{ah}^* + R_h^*}{(N_h^*)^2}$$

$$p_{22} = -\frac{\beta_1 (I_{sh}^* + I_{ah}^*)}{1 + \alpha (I_{sh}^* + I_{ah}^*)^2} - \frac{\beta_2 I_v^*}{N_h^*} - (\beta_{sh} + \beta_{ah} + \mu_h)$$

$$p_{23} = \frac{\beta_1 (N_h^* - E_h^* - 2I_{sh}^* - 2I_{ah}^* - R_h^*)}{1 + (I_{sh}^* + I_{ah}^*)^2} - \frac{\beta_2 I_v^*}{N_h^*} - \frac{2\alpha\beta_2 (N_h^* - E_h^* - I_{sh}^* - I_{ah}^* - R_h^*)(I_{sh}^* + I_{ah}^*)^2)}{(1 + (I_{sh}^* + I_{ah}^*)^2)^2}$$

$$p_{24} = \frac{\beta_1 (N_h^* - E_h^* - 2I_{sh}^* - 2I_{ah}^* - R_h^*)}{1 + (I_{sh}^* + I_{ah}^*)^2} - \frac{\beta_2 I_v^*}{N_h^*} - \frac{2\alpha\beta_2 (N_h^* - E_h^* - I_{sh}^* - I_{ah}^* - R_h^*)(I_{sh}^* + I_{ah}^*)^2)}{(1 + (I_{sh}^* + I_{ah}^*)^2)^2}$$

$$p_{25} = \frac{-\beta_1 (I_{sh}^* + I_{ah}^*)}{1 + (I_{sh}^* + I_{ah}^*)^2} - \frac{\beta_2 I_v^*}{N_h^*}$$

$$p_{27} = \frac{\beta_2 (N_h^* - E_h^* - 2I_{sh}^* - 2I_{ah}^* - R_h^*)}{N_h^*}$$

$$p_{33} = p_{44} = (\gamma_h + \mu_h + \mu_1)$$

$$p_{71} = \frac{\beta_v (I_{sh}^* + I_{ah}^*)(N_v^* - I_v^*)}{(N_h^*)^*}$$

$$p_{73} = \frac{\beta_v (N_v^* - I_v^*)}{N_h^*}$$

$$p_{76} = \frac{\beta_v (I_{sh}^* + I_{ah}^*)}{N_h^*}$$

$$p_{77} = -\frac{\beta_v (I_{sh}^* + I_{ah}^*)}{N_h^*} - \mu_v$$

Clearly, one eigenvalue of the matrix J_1 is $-\mu_h$ and remaining eigenvalues are the roots of the following Polynomial equation:

$$\lambda^6 + c_5\lambda^5 + c_4\lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0$$

where,

$$\begin{aligned}
c_5 &= k_1 + k_2 + 2\mu_h - (p_{22} + p_{77}) \\
c_4 &= \mu_h^2 + 2\mu_h(p_{77} - k_1)(p_{22} - k_2) - p_{73} - \beta_{ah}p_{74} \\
c_3 &= 2\mu_h(p_{77} - k_1)(p_{22} - k_2) - 2\mu_h(k_2p_{22} + k_1p_{77}) \\
&\quad + \gamma_1(\beta_{ah}p_{27} + p_{25}) + \beta_{ah}p_{74}(k_1 + p_{22}) \\
&\quad + p_{74}(p_{77} + 2\mu_h - k_2) + \beta_{sh}(p23 - \mu_1) - \gamma_h p_{27} \\
c_2 &= k_1p_{77} + k_2p_{22} + 2\mu_h k_2 p_{22}(p_{77} - k_1) \\
&\quad + 2\mu_h k_1 p_{22}(p_{77} - k_2) - \gamma_h p_{27}(k_2 + 2\mu_h) \\
&\quad - \gamma_1\beta_{ah}p_{27}(k_2 + 2\mu_h) + \beta_{ah}p_{74}(\mu_h p_{77} - k_1\mu_h) \\
&\quad + (p_{22} - \mu_h)(k_1 + \mu_h) - p_{73}k_1\mu_h + p_{73}\mu_h^2 \\
&\quad + 2p_{73}\mu_h(p_{22} - k_2) + \beta_{sh}(p_{23} - \mu_1) - \gamma_1 p_{23} \\
c_1 &= 2\mu_h k_1 k_2 p_{22} p_{77} + \mu_h^2(p_{77} - k_1)k_2 p_{22} \\
&\quad + \mu_h^2(p_{22} - k_2)k_1 p_{77} + \mu_1\beta_{sh} \\
&\quad + p_{77}(k_1 + k_2) - \gamma_h p_{27}(\mu_h^2 + 2\mu_h k_2) \\
&\quad + \gamma_1\beta_{ah}p_{27}(\mu_h^2 + 2\mu_h k_1) + \mu_h\beta_{ah}p_{74}p_{77}(k_1 + \mu_h) \\
&\quad + \mu_h\beta_{ah}p_{74}k_1(p_{22} - \mu_h) + 2p_{73}p_{77}\mu_h k_2 \\
&\quad + \mu_h^2 p_{73}(p_{77} - k_1)\beta_{sh}\mu_h p_{23}p_{77} + \beta_{sh}\mu_h p_{23}(\mu_h - p_{77}) \\
&\quad + \beta_{sh}\gamma_h p_{77}(\mu_h - p_{77}) + \gamma_1 p_{25}p_{77}(k_1 + \mu_h) \\
&\quad + \beta_{sh}\gamma_1 p_{77}(k_2 + \mu_h) - (\gamma_1 p_{25}k_1 + \mu_1\beta_{sh}k_2)\mu_h \\
c_0 &= k_1k_2p_{22}p_{77} - \mu_h^2\gamma_h k_2 p_{27} - \beta_{ah}\mu_h^2\gamma_1 p27 \\
&\quad + \beta_{ah}\mu_h^2 k_1 p_{74}p_{77} + \mu_h^2 k_1 p_{73}p_{77} - \beta_{ah}\mu_h^2 p_{23}p_{77} \\
&\quad - k_1\mu_h + \mu_h\gamma_1 k_1 p_{25}p_{77} - \beta_{sh}\mu_h\gamma_h p_{25}p_{77} \\
&\quad + \mu_h\beta_{sh}\mu_h k_2 p_{77}p_{71}p_{27}(k_2 + \mu_h)
\end{aligned}$$

According to the Routh-Hurwitz criterion, the above equation will give negative roots or negative real parts if the following condition is satisfied:

$$c_5 > 0, \quad \begin{vmatrix} c_5 & c_3 \\ 1 & c_4 \end{vmatrix} > 0, \quad \begin{vmatrix} c_5 & c_3 & c_1 \\ 1 & c_4 & c_2 \\ 0 & c_5 & c_3 \end{vmatrix} > 0,$$

$$\begin{vmatrix} c_5 & c_3 & c_1 & 0 \\ 1 & c_4 & c_2 & c_0 \\ 0 & c_5 & c_3 & c_1 \\ 0 & 1 & c_4 & c_2 \end{vmatrix} > 0, \quad \begin{vmatrix} c_5 & c_3 & c_1 & 0 & 0 \\ 1 & c_4 & c_2 & c_0 & 0 \\ 0 & c_5 & c_3 & c_1 & 0 \\ 0 & 1 & c_4 & c_2 & c_0 \\ 0 & 0 & c_5 & c_3 & c_1 \end{vmatrix} > 0$$

Hence, the endemic equilibrium point E_1 of the system is locally asymptotically stable when $R_0 > 1$.

6 Sensitivity Analysis

In this section, we present the impact of the change in values of the parameters on the functional value of the basic reproduction number R_0. Sensitivity index of R_0 that depends differentiably on any of its parameter P as described in [23,24]

$$Y_P^{R_0} = \frac{P}{R_0}\frac{\partial R_0}{\partial P}$$

Here, the parameter $\beta_1, \beta_2, \beta_v, \beta_{sh}, \beta_{ah}$ are the leading parameters, which control the basic reproduction number R_0. Sensitivity of R_0 is given below:

$$\begin{aligned}
Y_{\beta_1}^{R_0} &= \frac{\beta_1}{R_0}\frac{\partial R_0}{\partial \beta_1} \\
&= \frac{\beta_1}{2R_0}\left[N_h^0 D_1(\beta_{sh}D_2 + \beta_{ah}D_3)\left(1 + \frac{1}{\sqrt{\bar{X}}}\right)\right] \\
Y_{\beta_2}^{R_0} &= \frac{\beta_2}{R_0}\frac{\partial R_0}{\partial \beta_2} \\
&= \frac{\beta_2}{R_0}\left[\frac{\beta_v N_v^0(\beta_{sh}D_2 + \beta_{ah}D_3)D_1D_4}{N_h^0\sqrt{\bar{X}}}\right] \\
Y_{\beta_{sh}}^{R_0} &= \frac{\beta_{sh}}{R_0}\frac{\partial R_0}{\partial \beta_{sh}} \\
&= \frac{\beta_{sh}}{2R_0}(\beta_1 N_h^0 D_1 D_2) \\
&\quad + \frac{\beta_{sh}}{2R_0\sqrt{\bar{X}}}\left(P_4 + \frac{2\beta_2\beta_v N_v^0 D_1 D_2 D_4}{N_h^0}\right) \\
Y_{\beta_{ah}}^{R_0} &= \frac{\beta_{ah}}{R_0}\frac{\partial R_0}{\partial \beta_{ah}} \\
&= \frac{\beta_{ah}}{2R_0}\left[\beta_1 N_h^0 D_1 D_3 + \frac{\beta_1 N_h^0 D_1(\beta_{sh}D_2 + \beta_{ah}D_3)D_3}{\sqrt{\bar{X}}}\right] \\
Y_{\beta_v}^{R_0} &= \frac{\beta_v}{R_0}\frac{\partial R_0}{\partial \beta_v} \\
&= \frac{\beta_v}{R_0}\left[\frac{\beta_2 N_v^0(\beta_{sh}D_2 + \beta_{ah}D_3)D_1D_4}{N_h^0\sqrt{\bar{X}}}\right] \\
Y_{\beta_2}^{R_0} &= Y_{\beta_v}^{R_0}
\end{aligned}$$

where,

$$\begin{aligned}
&\bar{X} = (\beta_1 N_h^0 D_1(\beta_{sh}D_2 + \beta_{ah}D_3))^2 \\
&+ \frac{4\beta_2\beta_v N_v^0(\beta_{sh}D_2 + \beta_{ah}D_3)D_1D_4}{N_h^0}, \\
&P_4 = \beta_1 N_h^0 D_1(\beta_{sh}D_2 + \beta_{ah}D_3)D_2
\end{aligned}$$

As the above partial derivatives are positive, we conclude that the basic reproduction number R_0 increases based on the increase in the control parameters. It is noticeable that $Y_{\beta_2}^{R_0} = Y_{\beta_v}^{R_0}$, so we can conclude minor changes in β_1, β_2, β_v, β_{sh}, β_{ah}. We will have the same outcome on R_0. In Figures 6, 7, 8 , 9 and 10, we have demonstrated the effect of the parameters β_1, β_2, β_v, β_{sh}, β_{ah} on R_0

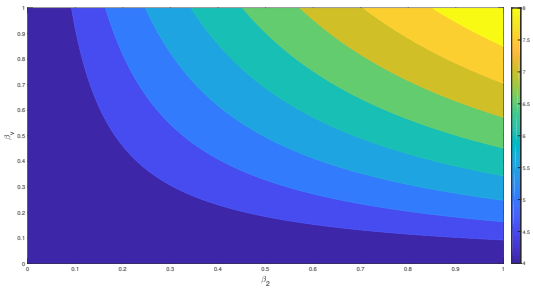

Fig. 6: Influnce of β_2 *and* β_v on R_0

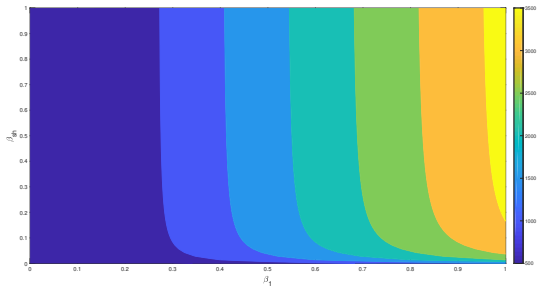

Fig. 9: Influnce of β_1 *and* β_{sh} on R_0

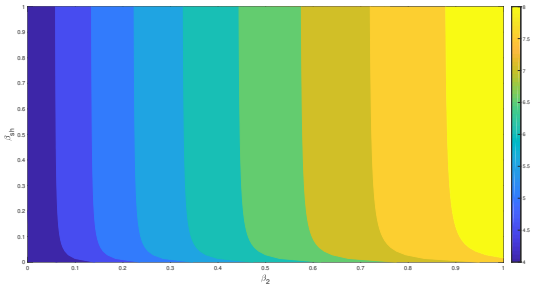

Fig. 7: Influnce of β_2 *and* β_{sh} on R_0

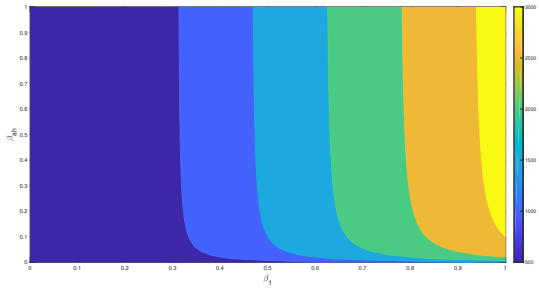

Fig. 10: Influnce of β_1 *and* β_{ah} on R_0

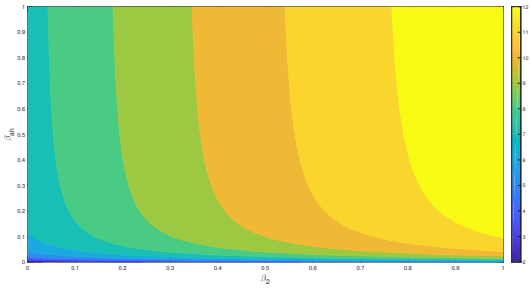

Fig. 8: Influnce of β_2 *and* β_{ah} on R_0

7 Numerical Simulation

Here, we illustrate the mathematical findings using MATLAB program for the Numerical simulation of the model.

$\Lambda_h = 2;\ \Lambda_v = 50;\ \beta_1 = 0.0012;\ \beta_2 = 0.0045;\ \beta_{sh} = 0.0069;$

$\beta_{ah} = 0.0075; \beta_v = 0.0056; \alpha = 0.02;\ \mu_h = 0.0072;$

$\mu_1 = 0.096;\ \gamma_h = 2.019;\ \gamma_1 = 2.009;\ \mu_v = 0.15$

The above set of parameter values is considered for disease-free equilibrium, and we get $R_0 = 0.1088 < 1$. Stability of the the disease-free equilibrium point $E_0(379.25, 0, 0, 0, 0, 276.08, 0)$ is manifested in Figure 11. Later, we change our parameter value $\gamma_h,\ \gamma_1,\ \beta_v,\ \mu_h,\ \mu_v,$ *and* μ_1, then we get $R_0 = 3.6724 > 1$. Stability of the endemic equilibrium point $E_1(300.05, 48.32, 123.69, 71.56, 149.02, 166.65, 169.76)$, which shown in Figure 12, is also stable. In Figure 13, we have shown the simulation of S_h and R_h for the stability of endemic equilibrium. The effects of different values of the parameters γ_h and γ_1 which correspond to infective human are demonstrated in Figures 14 and 15, respectively. Figures 14 and 15, exhibit that with the increase in these parameters, the infected population decreases.

8 Optimal Control Model

In this section, we have extended the mathematical model (2) to optimal control problem by incorporating three time-dependent optimal control parameters, namely u_1, u_2 and u_3. If u_1, u_2 and u_3 equal zero, no effort is placed in these controls at time t, and if they equal one, maximum effort is applied. Thus, optimal control variables are given, as follows:

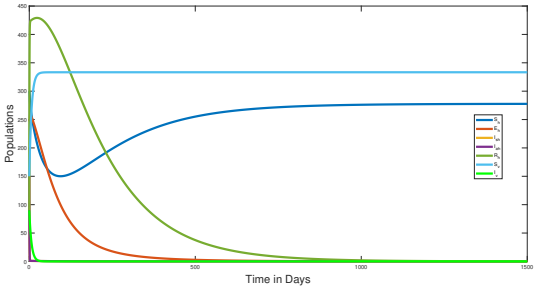

Fig. 11: Variation of S_h E_h ,I_{sh},I_{ah},R_h S_v, I_v showing the stability of disease-free equilibrium point with $R_0 = 0.1088 < 0$.

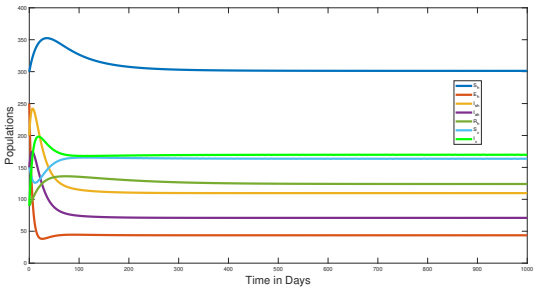

Fig. 12: Variation of S_h E_h ,I_{sh},I_{ah},R_h S_v, I_v showing the stability of endemic equilibrium point with $R_0 = 3.6724 > 0$.

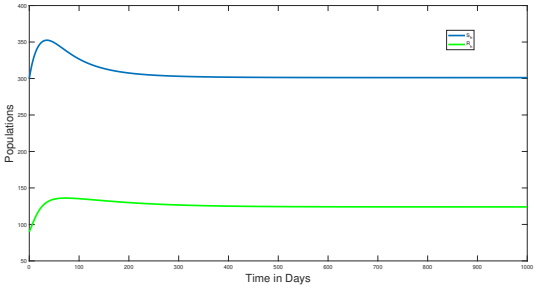

Fig. 13: Variation of (S_h, R_h) showing the stability of endemic equilibrium point

(i)The control variable u_1 represents the reduction in the transmission between human-to-human sexual transmission through the use of condoms.

(ii)The control variable u_2 represents the use of insecticide-treated bed nets, mosquito repulsive lotions and electronic devices to reduce mosquito biting rate.

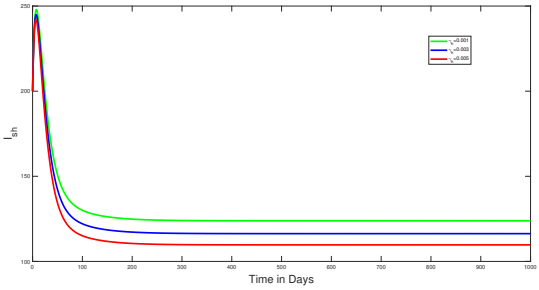

Fig. 14: Variation of of I_{sh} with time for different values of γ_h

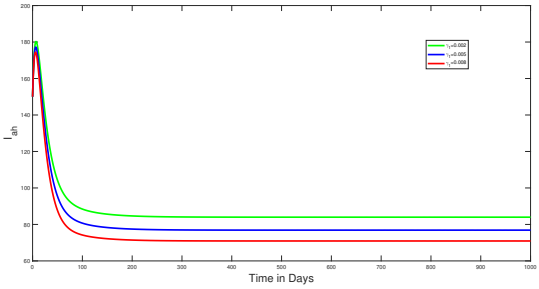

Fig. 15: Variation of of I_{ah} with time for different values of γ_1

(iii)The control variable u_3 corresponds to additional death rate of mosquitoes due to control efforts. The control strategies use an indoor insect fogger or indoor insect spray to kill mosquitoes.

Based on the above-mentioned assumptions, the optimal control model is given, as follows:

$$\begin{aligned}
\frac{dS_h}{dt} &= \Lambda_h - \beta_1(1-u_1)\frac{S_h(I_{sh}+I_{ah})}{1+\alpha(I_{sh}+I_{ah})^2} \\
&\quad -(1-u_2)\beta_2\frac{S_hI_v}{N_h} - \mu_hS_h \\
\frac{dE_h}{dt} &= \beta_1(1-u_1)\frac{S_h(I_{sh}+I_{ah})}{1+\alpha(I_{sh}+I_{ah})^2} + (1-u_2)\beta_2\frac{S_hI_v}{N_h} \\
&\quad -(\beta_{sh}+\beta_{ah}+\mu_h)E_h \\
\frac{dI_{sh}}{dt} &= \beta_{sh}E_h - (\gamma_h+\mu_h+\mu_1)I_{sh} \\
\frac{dI_{ah}}{dt} &= \beta_{ah}E_h - (\gamma_1+\mu_h+\mu_1)I_{ah} \\
\frac{dR_h}{dt} &= \gamma_hI_{sh} + \gamma_1I_{ah} - \mu_hR_h \\
\frac{dS_v}{dt} &= \Lambda_v - (1-u_2)\beta_v\frac{S_v(I_{sh}+I_{ah})}{N_h} - (\mu_v+u_3)S_v \\
\frac{dI_v}{dt} &= (1-u_2)\beta_v\frac{S_v(I_{sh}+I_{ah})}{N_h} - (\mu_v+u_3)I_v.
\end{aligned} \tag{5}$$

8.1 Optimal Control Analysis

In this section, we analyze the optimal control model (5) using optimal control theory. The objective functional for fixed time t_f is given below:

$$J = \int_0^{t_f} (K_1(E_h + I_{sh} + I_{ah}) + K_2(S_v + I_v) + (\frac{1}{2}K_3{u_1}^2 + \frac{1}{2}K_4u_2^2 + + \frac{1}{2}K_5u_3^2))dt$$

Here, the parameter $K_1 \geq 0$, $K_2 \geq 0$, $K_3 \geq 0$, $K_4 \geq 0$, $K_5 \geq 0$ and they represent the weight constants. The term ${K_3u_1}^2$ denotes the cost associated with human-to-human sexual contact protection efforts, the term ${K_4u_2}^2$ signifies the cost associated with mosquito-to-human contact protection efforts and the term ${K_5u_3}^3$ represents the cost associated with mosquito-reduction efforts. The quadratic control functions have been used in line with the related literature on optimal control problems ([16],[17],[23],[25],[26] [27],[28]).

Our objective is to find the control parameters ${u_1}^*$,${u_2}^*$, ${u_3}^*$ such that

$$J(u^*) = \min_{u\in\Omega} J(u_1,u_2,u_3), \tag{6}$$

where Ω is the control set and is defined as

$\Omega = \{u_1,u_2,u_3 : \text{measurable and } 0 \leq u_1 \leq 1\}$, $0 \leq u_2 \leq 1\}$, $0 \leq u_3 \leq 1\}$ and $t \in [0,t_f]$.

The Lagrangian of this problem is defined as : $L(E_h,I_{sh},I_{ah},S_v,I_v,u_1,u_2,u_3) = K_1(E_h + I_{sh} + I_{ah}) + K_2(S_v + I_v) + \frac{1}{2}K_3{u_1}^2 + \frac{1}{2}K_4u_2^2 + \frac{1}{2}K_5u_3^2$

For our problem, we formed Hamiltonian $\mathscr{H}$: $\mathscr{H} = L(E_h,I_{sh},I_{ah},S_v,I_v,u_1,u_2,u_3) + \lambda_1\frac{dS_h}{dt} + \lambda_2\frac{dE_h}{dt} + \lambda_3\frac{dI_{sh}}{dt} + \lambda_4\frac{dI_{ah}}{dt} + \lambda_5\frac{dR_h}{dt} + \lambda_6\frac{dS_v}{dt} + \lambda_7\frac{dI_v}{dt}$,

where $\lambda_i, (i = 1,2,....7)$ are the adjoint variables. Now, the differential equation corresponding to adjoint variables can be written as

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= \mu_h\lambda_1 + (1-u_1)\beta_1\frac{(I_{sh}+I_{ah})}{1+\alpha(I_{sh}+I_{ah})^2}(\lambda_1-\lambda_2) \\
&\quad +(1-u_2)\frac{\beta_2(N_h-S_h)I_v}{N_h^2}(\lambda_2-\lambda_1) \\
&\quad +(1-u_2)\frac{\beta_v S_v I_h}{N_h^2}(\lambda_7-\lambda_6) \\
\frac{d\lambda_2}{dt} &= -K_1 + \mu_h\lambda_2 + (1-u_2)\frac{\beta_2 S_h I_v}{N_h^2}(\lambda_1-\lambda_2) \\
&\quad +\beta_{sh}(\lambda_2-\lambda_3) + \beta_{ah}(\lambda_2-\lambda_4) \\
&\quad +(1-u_1)\frac{\beta_v(I_{sh}+I_{ah})}{N_h^2}(\lambda_7-\lambda_6) \\
\frac{d\lambda_3}{dt} &= -K_1 + (1-u_1)\frac{\beta_1 S_h}{1+\alpha(I_{sh}+I_{ah})^2}(\lambda_1-\lambda_2) \\
&\quad +(1-u_1)\frac{2\alpha S_h(I_{sh}+I_{ah})^2}{[1+\alpha(I_{sh}+I_{ah})^2]^2}(\lambda_2-\lambda_1) \\
&\quad +(1-u_2)\frac{\beta_2 S_h I_v}{N_h^2}(\lambda_2-\lambda_1) + \gamma_h(\lambda_3-\lambda_5) + \mu_h\lambda_3 \\
&\quad \mu_1\lambda_3 + (1-u_2)\frac{\beta_v S_v\,[N_h-(I_{sh}+I_{ah})]}{N_h^2}(\lambda_6-\lambda_7) \\
\frac{d\lambda_4}{dt} &= -K_1 + (1-u_1)\frac{\beta_1 S_h}{1+\alpha(I_{sh}+I_{ah})^2}(\lambda_1-\lambda_2) \\
&\quad +(1-u_1)\frac{2\alpha S_h(I_{sh}+I_{ah})^2}{[1+\alpha(I_{sh}+I_{ah})^2]^2}(\lambda_2-\lambda_1) \\
&\quad +(1-u_2)\frac{\beta_2 S_h I_v}{N_h^2}(\lambda_2-\lambda_1) + \gamma_1(\lambda_4-\lambda_5) + \mu_h\lambda_4 \\
&\quad +(1-u_2)\frac{\beta_v S_v\,[N_h-(I_{sh}+I_{ah})]}{N_h^2}(\lambda_6-\lambda_7) \\
\frac{d\lambda_5}{dt} &= \mu_h\lambda_5 + (1-u_2)\frac{\beta_2 S_h I_v}{N_h^2}(\lambda_2-\lambda_1) \\
&\quad +(1-u_2)\frac{\beta_v S_h(I_{sh}+I_{ah})}{N_h^2}(\lambda_7-\lambda_6) \\
\frac{d\lambda_6}{dt} &= -K_2 + (1-u_2)\frac{\beta_v(Ish+Iah)}{N_h}(\lambda_7-\lambda_6) \\
&\quad +(\mu_v+u_3)\lambda_6 \qquad (7) \\
\frac{d\lambda_7}{dt} &= -K_2 + (\mu_v+u_3)\lambda_7 + (1-u_2)\frac{\beta_2 S_g}{N_h}(\lambda_1-\lambda_2)
\end{aligned}$$

Let $\widetilde{S_h}$, $\widetilde{E_h}$, $\widetilde{I_{sh}}$, $\widetilde{I_{ah}}$, $\widetilde{R_h}$, $\widetilde{S_v}$, $\widetilde{I_v}$ be the optimum values of S_h, E_h, I_{sh}, I_{ah}, R_h, S_v, I_v respectively, and $\widetilde{\lambda_1}$, $\widetilde{\lambda_2}$, $\widetilde{\lambda_3}$, $\widetilde{\lambda_4}$, $\widetilde{\lambda_5}$, $\widetilde{\lambda_6}$, $\widetilde{\lambda_7}$ be the solution of the system (8).

Using [29,30], we state and prove the following theorem:

Theorem 6. *There exist optimal controls* $({u_1}^*,{u_2}^*,{u_3}^*) \in \Omega$ *such that* $J({u_1}^*,{u_2}^*,{u_3}^*) = \min J(u_1,u_2,u_3)$ *subject to system* (5).

Proof. To prove this theorem, we use [29]. Here, the state variables and the controls are positive. For this minimizing problem, the necessary convexity of the objective functional in (u_1,u_2,u_3) is satisfied. The control variable set $u_1,u_2,u_3 \in \Omega$ is also convex and closed by the definition. The integrand of the functional $K_1(E_h + I_{sh} + I_{ah}) + K_2(S_v + I_v) + \frac{1}{2}K_3{u_1}^2 + \frac{1}{2}K_4u_2^2 + \frac{1}{2}K_5u_3^2$ is convex on the control set Ω and the state variables are bounded.

Since there exist optimal controls for minimizing the functional subject to equations (5) and (7), we use Pontryagin's maximum principle to derive the necessary conditions to find the optimal solutions, as follows:
If (x,u) is an optimal solution of an optimal control problem, there exists a non-trivial vector function $\lambda = \lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ satisfying the following equalities.

$$\frac{dx}{dt} = \frac{\partial H(t,x,u,\lambda)}{\partial \lambda}$$

$$0 = \frac{\partial H(t,x,u,\lambda)}{\partial u}$$

$$\frac{d\lambda}{dt} = -\frac{\partial H(t,x,u,\lambda)}{\partial x}$$

Employing Pontryagin's maximum principle [30] and Theorem (8.1), we prove the following theorem:

Theorem 7. *The optimal controls $(u_1{}^*, u_2{}^*, u_3{}^*)$, which minimize J over the region Ω, given by*

$$u_1{}^* = min\{1, max(0, \widetilde{u_1})\}$$

$$u_2{}^* = min\{1, max(0, \widetilde{u_2})\}$$

$$u_3{}^* = min\{1, max(0, \widetilde{u_3})\}$$

where

$$\widetilde{u_1} = \frac{\beta_1 S_h^*(I_{sh}^* + I_{ah}^*)}{K_3\left[1+\alpha(I_{sh}^* + I_{ah}^*)^2\right]}(\lambda_2 - \lambda_1),$$
$$\widetilde{u_2} = \frac{\beta_2 S_h^* I_v^*(\lambda_2 - \lambda_1) + \beta_v S_v^*(Ish^* + I_{ah}^*)(\lambda_7 - \lambda_6)}{K_4 N_h^*},$$
$$\widetilde{u_3} = \frac{S_v^*\lambda_6 + I_v^*\lambda_7}{K_5}$$

Proof. Using optimality condition :

$$\frac{\partial \mathscr{H}}{\partial u_1} = 0, \ \frac{\partial \mathscr{H}}{\partial u_2} = 0, \ \frac{\partial \mathscr{H}}{\partial u_3} = 0,$$

we get,

$$\frac{\partial \mathscr{H}}{\partial u_1} = u_1 K_3 + \beta_1 \frac{S_h^*(I_{sh}^* + I_{ah}^*)}{1+\alpha(I_{sh}^* + I_{ah}^*)^2}(\lambda_1 - \lambda_2) = 0$$
$$\implies u_1 = \frac{\beta_1 S_h^*(I_{sh}^* + I_{ah}^*)}{K_3\left[1+\alpha(I_{sh}^* + I_{ah}^*)^2\right]}(\lambda_2 - \lambda_1) = \widetilde{u_1}$$
$$\frac{\partial \mathscr{H}}{\partial u_2} = u_2 K_4 + \frac{\beta_2 S_h^* I_v^*}{N_h^*}(\lambda_1 - \lambda_2) + \frac{\beta_v S_v^*(Ish^* + I_{ah}^*)}{N_h^*}(\lambda_6 - \lambda_7)$$
$$\implies u_2 = \frac{\beta_2 S_h^* I_v^*(\lambda_2 - \lambda_1) + \beta_v S_v^*(Ish^* + I_{ah}^*)(\lambda_7 - \lambda_6)}{N_h^* K_4}$$
$$= \widetilde{u_2}$$
$$\frac{\partial \mathscr{H}}{\partial u_3} = u_3 K_5 - (S_v^*\lambda_6 + I_v^*\lambda_7)$$
$$\implies u_3 = \frac{S_v^*\lambda_6 + I_v^*\lambda_7}{K_5} = \widetilde{u_3}$$

Again, lower and upper bounds for these controls are 0 and 1, respectively. i.e. $u_1 = u_2 = u_3 = 0$ if $\widetilde{u_1} < 0$, $\widetilde{u_2} < 0$, $\widetilde{u_3} < 0$ and $u_1 = u_2 = u_3 = 1$ if $\widetilde{u_1} > 1$, $\widetilde{u_2} > 1$ and $\widetilde{u_3} > 1$, otherwise $u_1 = \widetilde{u_1}$, $u_2 = \widetilde{u_2}$ and $u_3 = \widetilde{u_3}$. Hence, for these controls $u_1{}^*$,$u_2{}^*$ and $u_3{}^*$, we get optimum value of the function J.

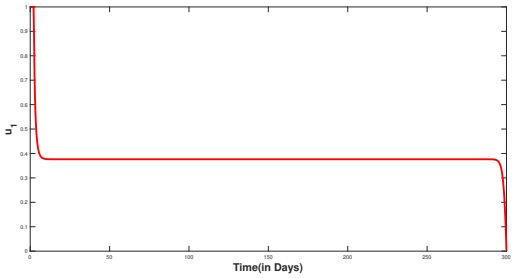

Fig. 16: Control profile of u_1

9 Numerical Simulation of Optimal Control

We simulate our optimal control model by keeping the parameters corresponding to stability of endemic equilibrium point E_1 of the model (1). With the help of MATLAB, the optimal control model is simulated. The weight constants for the optimal control problem are taken as

$$K_1 = 1, K_2 = 1, K_3 = 45, K_4 = 65, K_5 = 85.$$

We solve the optimality system (5) by iterative method with the help of forward and backward difference approximations [30]. We consider the time interval as [0, 300]. Profiles for optimal control u_1,u_2 and u_3 are shown, respectively , in Figures 16, 17 and 18. Finally, to see the effects of optimal controls, the infected human and infected vectors are plotted against time with and without optimal control in Figugrs 19 and 20 respectively. It is easy to notice that optimal control is more effective in reducing the number of infectives. The three optimal control application is the best control strategy to minimize the number of infectives, and will definitely reduce the spread of ZIKV.

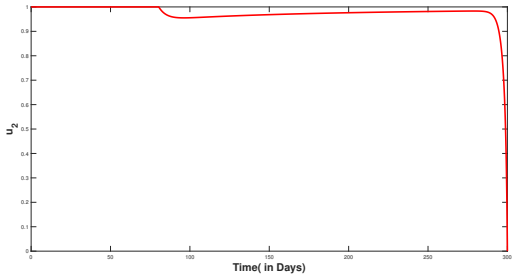

Fig. 17: Control profile of u_2

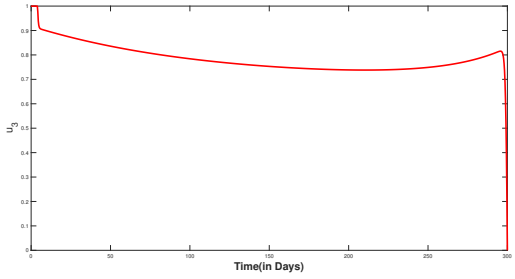

Fig. 18: Control profile of u_3

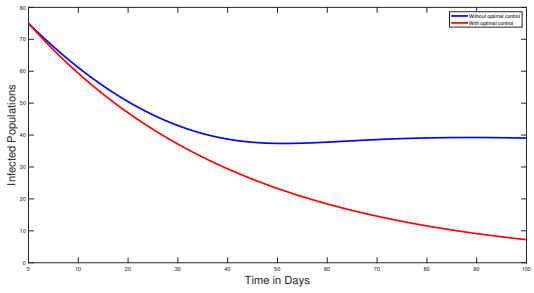

Fig. 19: Variation of I_h with and without optimal control

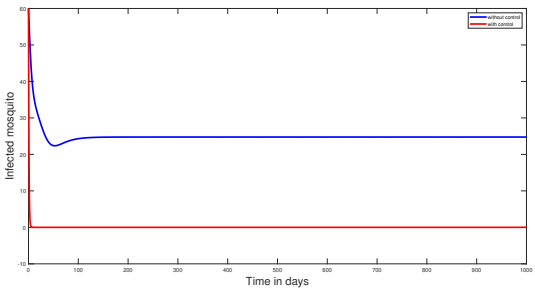

Fig. 20: Variation of I_v with and without optimal control

10 Conclusion

In this paper, we have proposed and analyzed transmission dynamics of the ZIKV epidemic model. We investigated the disease-free and endemic equilibria and computed the basic reproduction number of the disease in detail. The disease-free equilibrium (DFE) is locally asymptotically stable whenever the basic reproduction number $R_0 < 1$. Existence of backward bifurcation analysis was presented in detail. The endemic equilibrium (EE) is also locally asymptotically stable whenever the basic reproduction number $R_0 > 1$.

Numerical simulation was performed and it was shown that an increase in the parameter γ_h (which is the rate of Recovery in a human) decreased in the equilibrium level of the infective human. Also, the non-monotonic incident rate gives a positive result on the equilibrium level of infected humans as an increase in the parameter α corresponds to a decrease in the equilibrium level of infected humans.

Finally, we extended our model to optimal control problem and different control strategies were presented. To observe the effect of optimal controls, we extended the numerical simulation to this model, too. It has been observed that the optimal control model gives better results by reducing the infective levels during the stipulated period.

Acknowledgement

The authors would like to thank the editor and anonymous referees for their valuable comments and suggestions which improved the paper.

The authors are also grateful to Dr. Mini Ghosh and Akhil Kumar Srivastav of Division of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Chennai for their useful comments that helped prepare the paper.

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