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# Properties of Some New Classes of Generalized Exponentially Convex Functions

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**Abstract:** In this paper, we define and introduce some new concepts of the generalized exponentially convex functions. We investigate several properties of the generalized exponentially convex functions and discuss their relations with convex functions. Optimality conditions are characterized by a class of variational inequalities. Several interesting results characterizing the generalized exponentially convex functions are obtained. The results represent a significant improvement of in the previous results.

**Keywords:** Convex functions, monotone operators, strongly convex functions
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## 1 Introduction

Convexity theory describes a broad spectrum of very interesting developments involving a link among various fields of mathematics, physics, economics and engineering sciences. Its development is viewed as the simultaneous pursuit of several research areas. Recently, various extensions and generalizations of convex functions as well as convex sets have been investigated using innovative ideas and techniques. The concept of exponentially convex(concave) functions was considered by Bernstein [1]. Avriel [2,3] introduced the concept of $r$-convex functions. For further properties of the $r$-convex functions in mathematical programming and optimization theory, see Antczak [4] and the references therein. For the applications of exponentially convex functions in information theory, big data analysis, machine learning and statistic, see [1,2,3,4,5,6,7]. Dragomir and Gomm [8] proved that if the function $F$ is a convex function, then $e^{F(x)}$ is exponentially convex functions and derived the Hermite-Hadamard inequalities. For different types of exponentially convex functions, see [9,10,11,12]. Noor and Noor[13,14,15,16,17,18] addressed the fundamental properties of the characterizations of various classes of exponentially convex functions and their variant forms.

Motivated by the ongoing research in this interesting, applicable and dynamic field, we again consider the concept of general exponentially convex functions. We discuss the basic properties of the general exponentially convex functions. It is shown that the general exponentially convex functions include the exponentially convex functions as special case. We have proved that the distinctive properties exponentially convex(concave) functions. Several new concepts have been introduced and investigated. We show that the local minimum of the exponentially convex functions is the global minimum. The difference (sum) of the exponentially convex function and exponentially affine convex function is again a exponentially convex function. The optimal conditions of the differentiable exponentially convex functions can be characterized by a class of variational inequalities, which is itself an interesting outcome of our main results. The ideas and techniques of this paper may inspire the interested reader to explore the applications of the general exponentially convex functions in different branches of pure and applied sciences.

## 2 Formulations and basic facts

Let $K$ be a nonempty closed set in a real Hilbert space $H$. We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the inner product and norm, respectively. Let $F: K \to R$ be a continuous function.

**Definition 1.**[7].*The set $K$ in $H$ is said to be convex set, if*
$u+t(v-u)\in K, \quad \forall u,v\in K, t\in[0,1].$

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**Definition 2.** *A function $F$ is said to be convex, if*

$$F((1-t)u+tv) \leq (1-t)F(u)+tF(v), \forall u,v \in K, t \in [0,1].$$

We now consider a new concept of the generalized exponentially convex function, which is the main motivation of this paper.

**Definition 3.** *A positive function $F$ is said to be generalized exponentially convex function, if there exists $s>1$, such that*

$$s^{F((1-t)u+tv)} \leq (1-t)s^{F(u)}+ts^{F(v)}, \forall u,v \in K, t \in [0,1].$$

One can easily prove that if $F$ is a convex function, then $s^{F(u)}$ is a generalized exponentially convex function. In particular, one can show that $s^{u^2}$ is a generalized exponentially convex function, but it is not a convex function.
We note that if $s=e$, definition 3 reduces to the exponentially convex function, which is mainly due to Noor and Noor [13,14]:

**Definition 4.** *[13,14] A positive function $F$ is said to be exponentially convex function, if*

$$e^{F((1-t)u+tv)} \leq (1-t)e^{F(u)}+te^{F(v)}, \quad \forall u,v \in K, \quad t \in [0,1].$$

We remark that Definition 4 can be rewritten in the following equivalent way, which is due to Avriel [2,3] and Antczak [4].

**Definition 5.** *A function $F$ is said to be exponentially convex function, if*

$$F((1-t)a+tb) \leq \log[(1-t)e^{F(a)}+te^{F(b)}], \quad \forall a,b \in K, \quad t \in [0,1].$$

A function $F$ is called the exponentially concave function, if $-F$ is a exponentially convex function.
It is obvious that two concepts are equivalent. This equivalent has been used to discuss various aspects of the exponentially convex functions. One can also deduce the concept of exponentially convex functions from $r$-convex functions.
For the applications of the exponentially concave function in the communication and information theory, we have the following example.

*Example 1.* [5]The error function

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

becomes an exponentially concave function in the form $erf(\sqrt{x})$, $x \geq 0$, which describes the bit/symbol error probability of communication systems depending on the square root of the underlying signal-to-noise ratio. This reveals that the exponentially concave functions play important part in communication theory and information theory.

**Definition 6.** *The function $F$ on the convex set $K$ is said to be generalized exponentially quasi convex, if, for $s>1$,*

$$s^{F(u+t(v-u))} \leq \max\{s^{F(u)}, s^{F(v)}\}, \quad \forall u,v \in K, t \in [0,1].$$

**Definition 7.** *The function $F$ on the convex set $K$ is said to be generalized exponentially log-convex, if, for $s>1$,*

$$s^{F(u+t(v-u))} \leq (s^{F(u)})^{1-t}(s^{F(v)})^t, \quad \forall u,v \in K, t \in [0,1],$$

*where $F(\cdot)>0$.*

From the aforementioned definitions, we have

$$\begin{aligned} s^{F(u+t(v-u))} &\leq (s^{F(u)})^{1-t}(s^{F(v)})^t \\ &\leq (1-t)s^{F(u)}+ts^{F(v)}) \\ &\leq \max\{s^{F(u)}, s^{F(v)}\}. \end{aligned}$$

This shows that every generalized exponentially log-convex function is a generalized exponentially convex function and generalized exponentially convex function is a generalized exponentially quasi-convex function. However, the converse is untrue.

Let $K=I=[a,b]$ be the interval. We now define the generalized exponentially convex functions on the interval $I=[a,b]$.

**Definition 8.** *Let $I=[a,b]$. Then $F$ is generalized exponentially convex function, if and only if,*

$$\begin{vmatrix} 1 & 1 & 1 \\ a & x & b \\ s^{F(a)} & s^{F(x)} & s^{F(b)} \end{vmatrix} \geq 0; \quad a \leq x \leq b, \quad s>1.$$

One can easily show that the followings are equivalent:

1. $F$ is a generalized exponentially function.
2. $s^{F(x)} \leq s^{F(a)} + \frac{s^{F(b)}-e^{F(a)}}{b-a}(x-a)$.
3. $\frac{s^{F(x)}-s^{F(a)}}{x-a} \leq \frac{s^{F(b)}-s^{F(a)}}{b-a}$.
4. $(b-x)s^{F(a)}+(a-b)s^{F(x)}+(x-a)s^{F(b)}) \geq 0$.
5. $\frac{s^{F(a)}}{(b-a)(a-x)} + \frac{s^{F(x)}}{(x-b)(a-x)} + \frac{s^{F(b}}{(b-a)(x-b)} \leq 0$,

where $x=(1-t)a+tb \in [0,1]$. $\quad s>1$.

**Definition 9.** *A positive function $F$ is said to be generalized exponentially affine convex function, if there exists $s>1$, such that*

$$s^{F((1-t)u+tv)} = (1-t)s^{F(u)}+ts^{F(v)}, \forall u,v \in K, t \in [0,1].$$

One can show that the the product of two generalized exponentially convex functions is again a generalized exponentially convex function.
One can also prove that a function $F$ is a generalized exponentially convex function, if and only if, it satisfies the inequality

$$\begin{aligned} s^{F(\frac{a+b}{2})} &\leq \frac{1}{(b-a)\ln s}\int_a^b s^{F(x)}dx \\ &\leq \frac{1}{2}\{s^{F(a)}+s^{F(b)}\}, \quad a,b \in [a,b]. \end{aligned}$$

which is called the Hermite-Hadamard type inequality.

## 3 Main results

In this section, we consider some basic properties of generalized exponentially convex functions.

**Theorem 1.** *Let $F$ be a strictly generalized exponentially convex function. Then any local minimum of $F$ is a global minimum.*

*Proof.* Let the strictly generalized exponentially convex function $F$ have a local minimum at $u \in K$. Assume the contrary, that is, $s^{F(v)} < s^{F(u)}$ for some $v \in K$. Since $F$ is generalized exponentially convex,

$$s^{F(u+t(v-u))} < ts^{F(v)} + (1-t)s^{F(u)}, \quad \text{for} \quad 0 < t < 1.$$

Thus

$$s^{F(u+t(v-u))} - s^{F(u)} < t[s^{F(v)} - s^{F(u)}] < 0,$$

from which it follows that

$$s^{F(u+t(v-u))} < s^{F(u)},$$

for arbitrary small $t > 0$, contradicting the local minimum.

**Theorem 2.** *If the function $F$ on the convex set $K$ is generalized exponentially convex, the level set $L_\alpha = \{u \in K : s^{F(u)} \leq \alpha, \quad \alpha \in R\}$ is a convex set.*

*Proof.* Let $u, v \in L_\alpha$. Then $s^{F(u)} \leq \alpha$ and $s^{F(v)} \leq \alpha$. Now, $\forall t \in (0,1), \quad w = v + t(u-v) \in K$, since $K$ is a convex set. Thus, by the generalized exponentially convexity of $F$, we have

$$s^{F(v+t(u-v))} \leq (1-t)s^{F(v)} + ts^{F(u)} \leq (1-t)\alpha + t\alpha = \alpha,$$

from which it follows that $v + t(u-v) \in L_\alpha$ Hence $L_\alpha$ is convex set.

**Theorem 3.** *A function $F$ is a generalized exponentially convex on the convex set $K$, if and only if*

$$epi(F) = \{(u,\alpha) : u \in K : s^{F(u)} \leq \alpha, \alpha \in R\}$$

*is a convex set.*

*Proof.* Assume that $F$ is a generalized exponentially convex function. Let

$$(u,\alpha), \quad (v,\beta) \in epi(F).$$

Then it follows that $s^{F(u)} \leq \alpha$ and $s^{F(v)} \leq \beta$. Thus, $\forall t \in [0,1], \quad u, v \in K$, we have

$$s^{F(u+t(v-u))} \leq (1-t)s^{F(u)} + ts^{F(v)} \leq (1-t)\alpha + t\beta,$$

which implies that

$$(u + t(v-u), (1-t)\alpha + t\beta) \in epi(F).$$

Thus $epi(F)$ is a convex set. Conversely, let $epi(F)$ be a convex set. Let $u, v \in K$. Then $(u, s^{F(u)}) \in epi(F)$ and $(v, s^{F(v)}) \in epi(F)$. Since $epi(F)$ is a convex set, we must have

$$(u + t(v-u), (1-t)s^{F(u)} + ts^{F(v)}) \in epi(F),$$

which implies that

$$s^{F(u+t(v-u))} \leq (1-t)s^{F(u)} + ts^{F(u)}.$$

This shows that F is a generalized exponentially convex function.

**Theorem 4.** *The function $F$ is a generalized exponentially quasi convex, if and only if, the level set $L_\alpha = \{u \in K, \alpha \in R : s^{F(u)} \leq \alpha\}$ is a convex set.*

*Proof.* Let $u, v \in L_\alpha$. Then $u, v \in K$ and $\max(s^{F(u)}, s^{F(v)}) \leq \alpha$. Now for $t \in (0,1), w = u + t(v-u) \in K$, We have to prove that $u + t(v-u) \in L_\alpha$. By the exponentially quasi convexity of $F$, we have

$$s^{F(u+t(v-u))} \leq \max\left(s^{F(u)}, s^{F(v)}\right) \leq \alpha,$$

which implies that $u + t(v-u) \in L_\alpha$, indicating that the level set $L_\alpha$ is indeed a convex set.

Conversely, assume that $L_\alpha$ is a convex set. Then for any $u, v \in L_\alpha, t \in [0,1]$, $u + t(v-u) \in L_\alpha$. Let $u, v \in L_\alpha$ for

$$\alpha = max(s^{F(u)}, s^{F(v)}) \quad \text{and} \quad s^{F(v)} \leq s^{F(u)}.$$

Then from the definition of the level set $L_\alpha$, it follows that

$$s^{F(u+t\ (v,u))} \leq \max\left(s^{F(u)}, s^{F(v)}\right) \leq \alpha.$$

Thus $F$ is a generalized exponentially quasi convex function. This completes the proof.

**Theorem 5.** *Let $F$ be a generalized exponentially convex function.. Let $\mu = \inf_{u \in K} F(u)$. Then the set $E = \{u \in K : s^{F(u)} = \mu\}$ is a convex set of $K$. If $F$ is strictly exponentially , then $E$ is a singleton.*

*Proof.* Let $u, v \ \in E$. For $0 < t < 1$, let $w = u + t(v-u)$. Since $F$ is a generalized exponentially convex function, then

$$\begin{aligned} F(w) &= s^{F(u+t(v-u))} \leq (1-t)s^{F(u)} + ts^{F(v)} \\ &= t\mu + (1-t)\mu = \mu, \end{aligned}$$

which implies that $w \in E$. Hence $E$ is a convex set. For the second part, assume that $F(u) = F(v) = \mu$. Since $K$ is a convex set, then for $0 < t < 1, u + t(v-u) \in K$. Furthermore, since $F$ is strictly exponentially convex,

$$\begin{aligned} s^{F(u+t(v-u))} &< (1-t)s^{F(u)} + ts^{F(v)} \\ &= (1-t)\mu + t\mu = \mu. \end{aligned}$$

This contradicts the fact that $\mu = \inf_{u \in K} F(u)$ and hence the result follows.

**Theorem 6.** *If $F$ is a generalized exponentially convex function such that $s^{F(v)} < s^{F(u)}, \forall u, v \in K$, then $F$ is a strictly generalized exponentially quasi convex function.*

*Proof.* By the generalized exponentially convexity of the function $F$, $\forall u, v \in K, t \in [0,1]$, we have

$$s^{F(u+t(v-u))} \leq (1-t)s^{F(u)} + ts^{F(v)} < s^{F}(u),$$

since $s^{F(v)} < s^{F(u)}$, which shows that the function $F$ is strictly exponentially quasi convex.

The nest section addresses some properties of the differentiable exponentially convex functions.

**Theorem 7.** *Let $F$ be a differentiable function on the convex set $K$. Then the function $F$ is generalized exponentially convex function, if and only if,*

$$s^{F(v)} - s^{F(u)}) \geq \langle s^{F(u)} F'(u) ln \ln s, v-u \rangle, \quad \forall v, u \in K, \quad s > 1. \tag{1}$$

*Proof.* Let $F$ be a generalized exponentially convex function. Then

$$s^{F(u+t(v-u))} \leq (1-t)s^{F(u)} + ts^{F(v)}, \qquad \forall u, v \in K,$$

which can be written as

$$s^{F(v)} - s^{F(u)} \geq \{\frac{s^{F(u+t(v-u))} - s^{F(u)}}{t}\}.$$

Taking the limit in the above-mentioned inequality as $t \to 0$ , we have

$$s^{F(v)} - s^{F(u)} \geq \langle s^{F(u)} F'(u) \ln s, v-u) \rangle,$$

which is (1), the required result.

Conversely, let (1) hold. Then $\forall u, v \in K, t \in [0,1]$, $v_t = u + t(v-u) \in K$, we have

$$\begin{aligned} s^{F(v)} - s^{F(v_t)} &\geq \langle s^{F(v_t)} F'(v_t) \ln s, v - v_t) \rangle \\ &= (1-t) \langle s^{F(v_t)} F'(v_t) \ln s, v-u \rangle. \end{aligned} \tag{2}$$

Similarly, we have

$$\begin{aligned} s^{F(u)} - s^{F(v_t)} &\geq \langle e^{F(v_t)} F'(v_t) \ln s, u - v_t) \rangle \\ &= -t \langle e^{F(v_t)} F'(v_t) \ln s, v-u \rangle. \end{aligned} \tag{3}$$

Multiplying (2) by $t$ and (3) by $(1-t)$ and adding the resultant, we have

$$s^{F(u+t(v-u))} \leq (1-t)s^{F(u)} + ts^{F(v)},$$

showing that $F$ is a generalized exponentially convex function.

*Remark.* From (1), we have

$$s^{F(v)-F(u)} - 1 \geq \langle F'(u) \ln s, v-u \rangle, \quad \forall v, u \in K,$$

which can be written as

$$F(v) - F(u) \geq \log\{1 + \langle F'(u) \ln s, v-u \rangle\} \quad \forall v, u \in K, \tag{4}$$

Changing the role of $u$ and $v$ in (4), we also have

$$F(u) - F(v) \geq \log\{1 + \langle F'(v) \ln s, u-v \rangle\} \quad \forall v, u \in K, \tag{5}$$

Adding (4) and (5), we have

$$\begin{aligned} &\langle F'(u) \ln s - F'(v) \ln s, u-v \rangle \\ &\geq (\langle F'(u) \ln s, u-v \rangle)(\langle F'(v) \ln s, u-v \rangle) \end{aligned}$$

which express the monotonicity of the differential $F'(.)$ of the generalized exponentially convex function.

Theorem 7 enables us to introduce the concept of the generalized exponentially monotone operators, which appears to be new ones.

**Definition 10.** *The differential $F'(.)$ is said to be generalized exponentially monotone, if*

$$\langle s^{F(u)} F'(u) \ln s - s^{F(v)} F'(v) \ln s, u-v \rangle \geq 0, \quad \forall u, v \in H.$$

**Definition 11.** *The differential $F'(.)$ is said to be generalized exponentially pseudo-monotone, if*

$$\begin{aligned} &\langle s^{F(u)} F'(u) \ln s, v-u \rangle \geq 0, \\ &\quad \Rightarrow \langle s^{F(v)} F'(v) \ln s, v-u \rangle \geq 0, \quad \forall u, v \in H. \end{aligned}$$

From these definitions, it follows that generalized exponentially monotonicity implies generalized exponentially pseudo-monotonicity, but the converse is untrue.

**Theorem 8.** *Let $F$ be differentiable on the convex set $K$. Then (1) holds, if and only if, $F'$ satisfies*

$$\langle s^{F(u)} F'(u) \ln s - s^{F(v)} F'(v) \ln s, u-v \rangle \geq 0, \quad \forall u, v \in K. \tag{6}$$

*Proof.* Let $F$ be a generalized exponentially convex function on the convex set $K$. Then, from Theorem 3.1, we have

$$s^{F(v)} - s^{F(u)} \geq \langle s^{F(u)} F'(u) \ln s, v-u \rangle, \quad \forall u, v \in K. \tag{7}$$

Changing the role of $u$ and $v$ in (7), we have

$$s^{F(u)} - s^{F(v)} \geq \langle s^{F(v)} F'(v) \ln s, u-v) \rangle, \quad \forall u, v \in K. \tag{8}$$

Adding (7) and (8), we have

$$\langle s^{F(u)} F'(u) \ln s - s^{F(v)} F'(v) \ln s, u-v \rangle \geq 0,$$

which shows that $F'(.)$ is a generalized exponentially monotone.

Conversely, from (6), we have

$$\langle s^{F(v)}F'(v)\ln s, u-v\rangle \leq \langle s^{F(u)}F'(u)\ln s, u-v)\rangle. \tag{9}$$

Since $K$ is an convex set, $\forall u,v\in K,\quad t\in[0,1]$
$v_t = u+t(v-u)\in K$.
Taking $v=v_t$ in (9), we have

$$\begin{aligned}\langle s^{F(v_t)}F'(v_t)\ln s, u-v_t\rangle &\leq \langle s^{F(u)}F'(u)\ln s, u-v_t\rangle\\ &= -t\langle s^{F(u)}F'(u)\ln s, v-u\rangle,\end{aligned}$$

which implies that

$$\langle s^{F(v_t)}F'(v_t)s\ln, v-u\rangle \geq \langle s^{F(u)}F'(u)\ln s, v-u\rangle. \tag{10}$$

Consider the auxiliary function

$$\xi(t) = s^{F(u+t(v-u))},$$

from which, we have

$$\xi(1) = s^{F(v)},\quad \xi(0) = s^{F(u)}.$$

Then, from (10), we have

$$\begin{aligned}\xi'(t) &= \langle s^{F(v_t)}F'(v_t)\ln s, v-u\rangle &(11)\\ &\geq \langle e^{F(u)}F'(u)\ln s, v-u\rangle. &(12)\end{aligned}$$

Integrating (11) between 0 and 1, we have

$$\begin{aligned}\xi(1)-\xi(0) &= \int_0^1 g'(t)dt\\ &\geq \langle s^{F(u)}F'(u)\ln s, v-u\rangle.\end{aligned}$$

Thus it follows that

$$s^{F(v)} - e^{F(u)} \geq \langle s^{F(u)}F'(u)\ln s, v-u\rangle,$$

which is the required (1).

We now give a necessary condition for exponentially pseudo-convex function.

**Theorem 9.** *Let $F'$ be generalized exponentially pseudomonotone. Then $F$ is a generalized exponentially pseudo-convex function.*

*Proof.* Let $F'$ be an exponentially pseudomonotone. Then,

$$\langle s^{F(u)}F'(u)\ln s, v-u\rangle \geq 0, \forall u,v\in K.$$

implies that

$$\langle s^{F(v)}F'(v)\ln s, v-u\rangle \geq 0. \tag{13}$$

Since $K$ is an convex set, $\forall u,v\in K, t\in[0,1]$,
$v_t = u+t(v-u)\in K$.
Taking $v=v_t$ in (13), we have

$$\langle s^{F(v_t)}F'(v_t)\ln s, v-u\rangle \geq 0. \tag{14}$$

Consider the auxiliary function

$$\xi(t) = s^{F(u+t(v-u))} = s^{F(v_t)},\quad \forall u,v\in K, t\in[0,1],$$

which is differentiable, since $F$ is differentiable function. Then, using (14), we have

$$\xi'(t) = \langle s^{F(v_t)}F'(v_t)\ln s, v-u)\rangle \geq 0.$$

Integrating the above-mentioned relation between 0 to 1, we have

$$\xi(1)-\xi(0) = \int_0^1 g'(t)dt \geq 0,$$

that is,

$$s^{F(v)} - s^{F(u)} \geq 0,$$

showing that $F$ is a generalized exponentially pseudo-convex function.

**Definition 12.** *The function $F$ is said to be sharply generalized exponentially pseudo convex, if*

$$\begin{aligned}\langle s^{F(u)}F'(u)\ln s, v-u\rangle &\geq 0\\ &\Rightarrow\\ F(v) &\geq s^{F(v+t(u-v))},\quad \forall u,v\in K, t\in[0,1].\end{aligned}$$

**Theorem 10.** *Let $F$ be an sharply generalized exponentially pseudo convex function on $K$. Then*

$$\langle s^{F(v)}F'(v)\ln s, v-u\rangle \geq 0,\quad \forall u,v\in K.$$

*Proof.* Let $F$ be a sharply generalized exponentially pseudo convex function on $K$. Then

$$s^{F(v)} \geq s^{F(v+t(u-v))},\quad \forall u,v\in K, t\in[0,1].$$

from which we have

$$\begin{aligned}0 &\leq \lim_{t\to 0}\{\frac{s^{F(v+t(u-v))} - s^{F(v)}}{t}\}\\ &= \langle s^{F(v)}F'(v)\ln s, v-u\rangle,\end{aligned}$$

the required result.

**Definition 13.** *A function $F$ is said to be a pseudo convex function, if there exists a strictly positive bifunction $B(.,.)$, such that*

$$\begin{aligned}s^{F(v)} &< s^{F(u)}\\ &\Rightarrow\\ s^{F(u+t(v-u))} &< s^{F(u)} + t(t-1)B(v,u), \forall u,v\in K, t\in[0,1].\end{aligned}$$

**Theorem 11.** *If the function $F$ is exponentially convex function such that $s^{F(v)} < s^{F(u)}$, the function $F$ is exponentially pseudo convex.*

*Proof.* Since $s^{F(v)} < s^{F(u)}$ and $F$ is exponentially convex function, then $\forall u,v\in K,\quad t\in[0,1]$, we have

$$\begin{aligned}s^{F(u+t\ l(v,u))} &\leq s^{F(u)} + t(s^{F(v)} - s^{F(u)})\\ &< s^{F(u)} + t(1-t)(s^{F(v)} - s^{F(u)})\\ &= s^{F(u)} + t(t-1)(s^{F(u)} - s^{F(v)}))\\ &< s^{F(u)} + t(t-1)B(u,v),\end{aligned}$$

where $B(u,v) = s^{F(u)} - s^{F(v)} > 0$, the required result. This shows that the function $F$ is a generalized exponentially convex function.

Now, we show that the difference of exponentially convex function and exponentially affine convex function is again an exponentially convex function.

**Theorem 12.** *Let $f$ be a exponentially affine convex function. Then $F$ is a exponentially convex function, if and only if, $g = F - f$ is a exponentially convex function.*

*Proof.* Let $f$ be exponentially affine convex function. Then

$$s^{f((1-t)u+tv)} = (1-t)s^{f(u)} + ts^{f(v)}, \forall u,v \in K. \quad (15)$$

From the exponentially convexity of $F$, we have

$$s^{F((1-t)u+tv)} \leq (1-t)s^{F(u)} + ts^{F(v)}, \forall u,v \in K. \quad (16)$$

From (15 ) and (16), we have

$$s^{F((1-t)u+tv)} - s^{f((1-t)u+tv)} \quad (17)$$
$$\leq (1-t)(s^{F(u)} - s^{f(u)}) + t(s^{F(v)} - s^{f(v)}), \quad (18)$$

from which it follows that

$$\begin{aligned} s^{H((1-t)u+tv)} &= s^{F((1-t)u+tv)} - s^{f((1-t)u+tv)} \\ &\leq (1-t)(s^{F(u)} - s^{f(u)}) + t(s^{F(v)} - s^{f(v)}), \end{aligned}$$

which show that $H = F - f$ is an exponentially convex function.
The inverse implication is obvious.

We now discuss the optimality condition for the differentiable generalized exponentially convex functions, which is the main motivation of our next result.

**Theorem 13.** *Let $F$ be a differentiable generalized exponentially convex function. Then $u \in K$ is the minimum of the function $F$, if and only if, $u \in K$ satisfies the inequality*

$$\langle s^{F(u)}F'(u)\ln s, v-u\rangle \geq 0, \quad \forall u,v \in K. \quad (19)$$

*Proof.*Let $u \in K$ be a minimum of the function $F$. Then

$$F(u) \leq F(v), \forall v \in K.$$

from which, we have

$$s^{F(u)} \leq s^{F(v)}, \forall v \in K. \quad (20)$$

Since $K$ is a convex set, $\forall u,v \in K$, $t \in [0,1]$, $v_t = (1-t)u + tv \in K$.
Taking $v = v_t$ in (20), we have

$$\begin{aligned} 0 &\leq \lim_{t\to 0}\{\frac{s^{F(u+t(v-u))} - s^{F(u)}}{t}\} \\ &= \langle s^{F(u)}F'(u)\ln s, v-u\rangle. \end{aligned} \quad (21)$$

Since $F$ is differentiable generalized exponentially convex function,

$$s^{F(u+t(v-u))} \leq s^{F(u)} + t(e^{F(v)} - e^{F(u)}, \quad u,v \in K, t \in [0,1],$$

from which, using (21), we have

$$\begin{aligned} s^{F(v)} - s^{F(u)} &\geq \lim_{t\to 0}\{\frac{s^{F(u+t(v-u))} - s^{F(u)}}{t}\} \\ &= \langle s^{F(u)}F'(u)\ln s, v-u\rangle \geq 0, \end{aligned}$$

from which , we have

$$s^{F(v)} - s^{F(u)} \geq 0,$$

which implies that

$$F(u) \leq F(v), \quad \forall v \in K.$$

This shows that $u \in K$ is the minimum of the differentiable exponentially convex function, the required result.

*Remark.*The problem of finding $u \in K$such that

$$\langle s^{F(u)}F'(u)\ln s, v-u\rangle \geq 0, \quad \forall u,v \in K,$$

is called the exponentially variational inequality and appears to be new one. For the applications, formulations, numerical methods and other aspects of variational inequalities, see Noor [19] and the references therein.

## 4 Conclusion

In this paper, we have introduced some new concepts of the generalized exponentially convex functions. We investigated several basic properties of the generalized exponentially convex functions and discussed their relations with convex functions. Optimality conditions are characterized by a class of variational inequalities. Several interesting results characterizing the generalized exponentially convex functions were obtained. The results represent a significant improvement of previously known results We have studied the basic properties of these functions. The interested readers may explore the applications and other properties of the generalized exponentially convex functions in various fields of pure and applied sciences. This is an interesting direction of future research.

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