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Performance Evaluation of Some Confidence Intervals for Estimating the Shape Parameter of the Two-Parameter Lomax Distribution

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Abstract: The Lomax distribution has had wide application in a variety of fields. It has been used in the analysis of income data, business failure data, and atmospheric data. In this article, we propose new approaches for the confidence intervals of the shape parameter of the Lomax distribution based on the maximum likelihood (ML) approach, bootstrap (BS) approach, Bayesian approach using Jeffreyss Prior (BJ) and Conjugate Prior (BC), and generalized probability weighted moment (GPWM). The performance of each method is assessed by simulation in terms of the coverage probabilities and average widths by the Monte Carlo simulation. An extensive simulation study indicates that the ML approach performs better than other approaches because it provides coverage probability close to the nominal confidence level, and the average widths are narrow. Moreover, the GPWM approach is regarded as a recommended method when the parameters α and β are high. We also illustrate our confidence intervals using a real world example in the area of meteorology.

Keywords: Confidence interval, interval estimation, shape parameter, Lomax distribution

1 Introduction

The Lomax distribution (i.e. also known as the Pareto of the second type) was first introduced by K. S. Lomax in 1954 [1]. It is one of the well-known distributions that benefit in various fields, such as actuarial science, economics, engineering, reliability, as well as medical and biological sciences. It provides a very good alternative to common lifetime distributions when the population may have a heavy-tailed distribution [2]. The applications of the Lomax distribution in data modeling of income and wealth were investigated by Harris [3] and Atkinson and Harrison [4]. The model of size spectra data in aquatic ecology was investigated by Vidondo et al. [5]. Hassan and Al-Ghamdi [6] used the Lomax distribution in reliability and life-testing problems to define the optimal times of changing stress level. Moreover, the Lomax distribution is considered an important model of lifetime models, because it belongs to the family of decreasing failure rate [7]. Some details about the Lomax distribution are given by Arnold [8] and Johnson et al. [9] and some properties for the Lomax distribution have been discussed by Ahsanullah [10], Amin [14], Balakrishnan [12] and Lee et al. [13]. Let $\mathbf{X} = (X_1, X_2, \dots, X_n)$ be a random sample from a Lomax distribution, denoted as $\mathscr{LOM}(\alpha, \beta)$. The distribution function of this distribution is given by

$$F(x;\alpha,\beta) = 1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}, \quad x > 0, \ (\alpha, \beta > 0) \quad (1)$$

where α is a shape parameter and β is a scale parameter. Therefore, the probability density function (p.d.f.) is given by

$$f(x;\alpha,\beta) = \frac{\alpha}{\beta}\left(1 + \frac{x}{\beta}\right)^{-(\alpha+1)}, \quad x > 0, \ (\alpha, \beta > 0). \quad (2)$$

The survival function or reliability function of the Lomax distribution is given by

$$R(x;\alpha,\beta) = 1 - F(x) = \left(1 + \frac{x}{\beta}\right)^{-\alpha} \quad (3)$$

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and the hazard function is given by

$$h(x;\alpha,\beta)=\frac{f(x)}{R(x)}=\frac{\alpha}{\beta+x}. \tag{4}$$

Then the mean and variance of the delta-lognormal distribution are

$$E(X)=\begin{cases}\frac{\beta}{\alpha-1} & ;\quad \text{for } \alpha\geq 1\\ \text{undefined} & ;\quad \text{Otherwise}\end{cases} \tag{5}$$

and

$$V(X)=\begin{cases}\frac{\alpha\beta^2}{(\alpha-1)^2(\alpha-2)} & ;\quad \text{for } \alpha\geq 1\\ \infty & ;\quad \text{for } 1<\alpha\leq 2.\end{cases} \tag{6}$$

The probability density function of the Lomax distribution is a heavy-tailed distribution to the right. Nevertheless, the asymmetry of the distribution decreases with β. The Lomax density for some values α with $\beta=1$ is shown in Fig. 1.

Many authors have investigated inferential issues for the parameters of the Lomax distribution. In the literature, some properties and moments for the Lomax distribution are available. The record values were discussed by Lee and Lim in 2009 [13] and Amin in 2011 [14], respectively. In 2012, Moghadam et al. [15] derived the moments and inference for the order statistics and generalized order statistics. The estimation of the parameters of the Lomax distribution based on a generalized probability weighted moment was derived by Abdullah and Abdullah [16]. Several authors studied parameter estimation of the Lomax distribution. For example, Abd-Ellah [17] and Nasiri and Hosseini [18] compared the performance of Bayesian and non-Bayesian estimation from the Lomax distribution based on record values. Furthermore, Ahmad et al. [19] estimated the parameters of the Lomax distribution using Jefferys and an extension of Jefferys prior under different loss functions. Ahmad et al. [20] derived Bayes estimators using extension of Jeffreys prior, Gamma prior under the Entropy loss function and Precautionary loss function. More recently, Venegas et al. [21] derived various structural properties including expressions for the moments, skewness and kurtosis coefficients of the Lomax-Rayleigh model and estimated model parameters using the moment and maximum likelihood methods. Under the topic of interval estimation for the future observation, Al-Hussaini et al. [22] proposed interval estimations based on the Bayesian analysis for future observations based on a type I censored sample from a non-homogeneous population. Abd-Ellah [23] studied the problem of Bayesian prediction bounds for certain order statistics for samples from the Lomax distribution. No work has been done on confidence intervals for the parameters of the Lomax distribution that have been wildly reported in many applications.

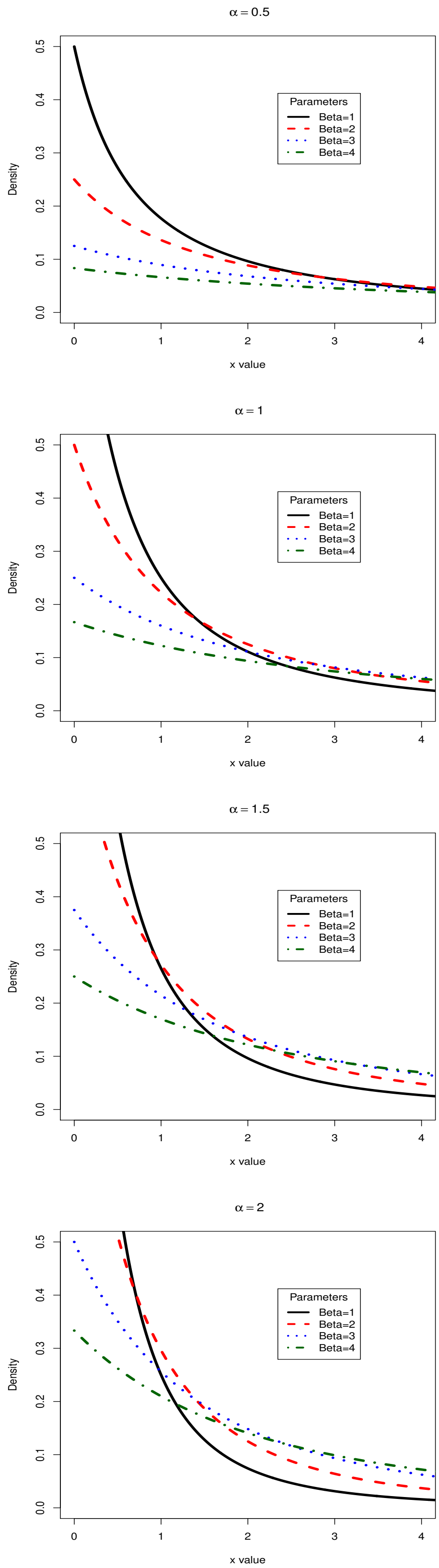

Fig. 1: Comparison of the Lomax density with different values of α and β.

Therefore, we are interested in estimating the confidence interval for the shape parameter for the Lomax distribution. In this article, we propose five methods of interval estimation for the shape parameter of the Lomax distribution. The first method is a confidence interval construction based on the maximum likelihood (ML) approach. The second method is a confidence interval construction based on the bootstrap (BS) approach. The third and fourth methods are confidence interval constructions based on the Bayesian approach using Jeffreyss Prior and Conjugate Prior. The last method is a confidence interval construction based on the generalized probability weighted moment (GPWM). The performance of all these methods are examined using Monte Carlo simulations by evaluating coverage probabilities and average lengths of each confidence interval.

The rest of this paper is organized as follows. In the first Section, we briefly describe the properties of the Lomax distribution. The details of the interval estimation for the shape parameter of the Lomax distribution are presented in Section 2. A simulation study and results discussion are provided in Section 3.1 and Section 3.2. In Section 3.3, we illustrate these methods with several examples. Finally, summary and concluding Remarks are given in Section 3.4.

2 Materials and Methods

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a random sample from a Lomax distribution, denoted as $\mathscr{LOM}(\alpha,\beta)$. The concepts of the following methods are used to obtain confidence intervals for α. Now, we describe all five methods in more detail.

2.1 Confidence Intervals Based on Maximum Likelihood (ML) Approach

The log-likelihood function is given by

$$lnL(\alpha,\beta) = n\ln\alpha - n\ln\beta - (\alpha+1)\sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\beta}\right).$$

The maximum likelihood estimator (MLE) of θ, denoted as $\hat{\theta}_{ML}$, is obtained by solving the equation

$$\frac{\partial lnL(\theta)}{\partial\theta} = 0.$$

Then

$$\frac{n}{\alpha} - \sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\beta}\right) = 0$$

and

$$-\frac{n}{\beta} + \frac{(\alpha+1)\sum_{i=1}^{n}x_i}{\beta^2\left(1+\frac{\sum_{i=1}^{n}x_i}{n\beta}\right)} = 0.$$

The estimators of θ can then be obtained as

$$\hat{\theta} = \begin{bmatrix}\hat{\alpha}\\ \hat{\beta}\end{bmatrix} = \begin{bmatrix}\frac{n}{\sum_{i=1}^{n}\ln\left(1+\frac{x}{\beta}\right)}\\ \frac{\alpha\sum_{i=1}^{n}x_i}{n}\end{bmatrix}. \quad (7)$$

Therefore, the Fisher information matrix can be written, as follows

$$I(\theta) = -E\left[\frac{\partial^2}{\partial^2\theta}\ln L(\theta|data)\right]$$

$$= \begin{bmatrix}-\frac{\partial^2\ln L}{\partial\alpha^2} & -\frac{\partial^2\ln L}{\partial\alpha\partial\beta}\\ -\frac{\partial^2\ln L}{\partial\alpha\partial\beta} & -\frac{\partial^2\ln L}{\partial\beta^2}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{\alpha^2} & -\frac{1}{\beta(\alpha+1)}\\ -\frac{1}{\beta(\alpha+1)} & \frac{\alpha}{\beta^2(\alpha+2)}\end{bmatrix}. \quad (8)$$

The MLE of α can then be obtained as

$$\hat{\alpha}_{ML} = \frac{n}{\sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\beta}\right)}. \quad (9)$$

The asymptotic Fisher information matrix is obtained by taking the expected value of the second and mixed partial derivatives of $lnL(\alpha,\beta)$ with respect to α and β. Furthermore, the inverse of the Fisher information matrix is an estimator of the asymptotic variance-covariance matrix:

$$Var(\hat{\theta}_{ML}) = \left[I(\hat{\theta}_{ML})\right]^{-1}.$$

Thus, the variance of the ML estimator for α can be denoted as

$$Var(\hat{\alpha}_{ML}) = \frac{\alpha^2}{n}. \quad (10)$$

The standard errors are the square roots of the diagonal elements of the variance-covariance matrix. For the asymptotic distribution of a maximum likelihood estimate of α, we write

$$\hat{\alpha}_{ML} \sim \mathscr{N}\left(\alpha, \frac{\alpha^2}{n}\right). \quad (11)$$

The estimated standard error of the maximum likelihood estimates is given by:

$$SE(\hat{\alpha}_{ML}) = \frac{\alpha}{\sqrt{n}}. \quad (12)$$

The $(1-\nu)100\%$ two-sided confidence intervals for α based on the maximum likelihood approach are, as follows:

$$CI(\alpha)_{ML} = [\alpha_L, \alpha_U] = \left[\hat{\alpha}_{ML} - Z_{(\nu/2)}\frac{\alpha}{\sqrt{n}}, \hat{\alpha}_{ML} + Z_{(\nu/2)}\frac{\alpha}{\sqrt{n}}\right] \quad (13)$$

where ν is the significance level, and $Z_{\nu/2}$ is the upper $100(1-\nu/2)th$ percentile of $\mathscr{N}(0,1)$.

2.2 Confidence Interval Based on the Bootstrap (BS) Approach

In the proposed approach, we construct the bootstrap confidence interval based on bootstrap percentiles, which performs well in terms of coverage probability and average length for various sample sizes [24]. We have considered bias-correcting the estimators using a parametric bootstrap. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be a random sample of size n, where each element is a random drawn from the random variable X from a Lomax distribution, as well as $\mathscr{LOM}(\alpha, \beta)$ and $\breve{\alpha}$ be an estimator of α based on x; this can be written as $\breve{\alpha} = s(x)$. Afterwards, let $\mathbf{B}$ be bootstrap samples $(\mathbf{x}^{*1}, \mathbf{x}^{*2}, \ldots, \mathbf{x}^{*B})$ that are generated independently from the original sample x. These bootstrap-bias-corrected estimators are calculated by [25]

$$\breve{\alpha} = 2\hat{\alpha} - 1/B \sum_{j=1}^{B} \hat{\alpha}_{(j)}. \tag{14}$$

where $\hat{\alpha}_{(j)}$ is the MLE of α obtained from the j^{th} of the B bootstrap samples and $j = 1, 2, \ldots, B$. Suppose we want to use the B bootstrap samples to form a 95% confidence interval. We need to calculate the $\breve{\alpha}$ number of B values to an ordered value in the list of B standardized bootstrap estimates of α. The respective bootstrap replications are denoted as $(\breve{\alpha}_1, \breve{\alpha}_2, \ldots, \breve{\alpha}_B)$, where $\breve{\alpha}_j = s(\mathbf{x}_j)$. Therefore, the percentile $(1-\nu)100\%$ bootstrap confidence interval for α is given by

$$CI(\alpha)_{BS} = [\alpha_L, \alpha_U] = \left[\breve{\alpha}_{(\nu/2)}, \breve{\alpha}_{(1-\nu/2)}\right],$$

where $\breve{\alpha}_{(\nu/2)}$ is the $B(\nu/2)$th ordered value in the list of B standardized bootstrap estimates of α.

2.3 Bayesian Confidence Intervals

Here, we introduce our Bayesian confidence intervals for α of the Lomax distribution. The likelihood function of the Lomax distribution can be presented as follows

$$L(\theta|data) \propto \left(\frac{\alpha}{\beta}\right)^n \prod_{i=1}^{n} \left(1 + \frac{x}{\beta}\right)^{-(\alpha+1)}, \tag{15}$$

where $\theta = [\alpha, \beta]'$.

In statistical inference, any Bayesian Confidence interval is based on the posterior pdf. The posterior distribution of the parameter α is obtained using different non-informative and informative prior distributions of the parameter α and the likelihood function of the above density. In this research, the Bayesian procedures for different choices of prior distributions for θ were explored and used to compare performance of each method by conducting a simulation study.

2.3.1 Jeffreys's Prior

Since θ is unknown, the prior distribution of α is given by

$$g(\theta) \propto \sqrt{I(\theta)}.$$

Therefore, the Jeffreys' prior distribution of α can be written as [19]

$$g(\alpha) \propto \frac{1}{\alpha}. \tag{16}$$

The likelihood used to form the posterior density is also considered a function of parameter α. The posterior distribution of α is given by

$$\pi(\alpha|x) \propto L(x|\alpha)g(\alpha).$$

By combining the likelihood function (15) and the prior density function (16), the joint posterior density function can be written as

$$\pi(\alpha|x)_1 \propto \left(\frac{\alpha}{\beta}\right)^n \prod_{i=1}^{n} \left(1 + \frac{x_i}{\beta}\right)^{-(\alpha+1)} \frac{1}{\alpha}$$

or

$$\pi(\alpha|x)_1 \propto \frac{\alpha^{n-1}}{\beta^n} \exp\left(-(\alpha+1)\sum_{i=1}^{n} \ln\left(1 + \frac{x_i}{\beta}\right)\right).$$

The posterior distribution of α was shown by Ahmad et al. [19] as follows

$$\pi(\alpha|x)_1 = \frac{\left[\sum_{i=1}^{n} \ln\left(1 + \frac{x_i}{\beta}\right)\right]^n}{\Gamma(n)} \alpha^{n-1} \exp(-\alpha \sum_{i=1}^{n} \ln(1 + \frac{x_i}{\beta})) \tag{17}$$

where $\frac{\left[\sum_{i=1}^{n} \ln\left(1+\frac{x_i}{\beta}\right)\right]^n}{\Gamma(n)}$ is independence of α and $\Gamma(n) = \int_0^\infty \alpha^{n-1} \exp(-\alpha) d\alpha$. The posterior distribution of α is obtained as

$$\pi(\alpha|x)_1 = \alpha^{n-1} \left[\frac{(\sum_{i=1}^{n} \ln(1 + \frac{x_i}{\beta}))^n}{\Gamma(n)}\right] \exp(-\alpha \sum_{i=1}^{n} \ln(1 + \frac{x_i}{\beta})) \tag{18}$$

or

$$\pi(\alpha|x)_1 = \alpha^{n-1} AB \tag{19}$$

where

$$A = \frac{\left(\sum_{i=1}^{n} \ln\left(1 + \frac{x_i}{\beta}\right)\right)^n}{\Gamma(n)} \tag{20}$$

and

$$B = \exp\left(-\alpha \sum_{i=1}^{n} \ln\left(1 + \frac{x_i}{\beta}\right)\right). \tag{21}$$

Therefore, the $(1-\nu)100\%$ Bayesian confidence interval under Jeffreys's prior for α is given by

$$CI(\alpha)_{BJ} = \left[\alpha^*_{BJ(L)}, \alpha^*_{BJ(U)}\right] \tag{22}$$

where $\alpha^*_{BJ(L)}$ is the $\alpha^*_{BJ}(\nu/2)^{th}$ ordered value in the list of estimate of α and $\alpha^*_{BJ(U)}$ is the $\alpha^*(1-\nu/2)^{th}$ ordered value in the list of α^*_{BJ} estimates of α. Thus the coverage probability of the Bayesian confidence interval under Jeffreys's prior can be computed using the Algorithm 1.

Algorithm 1. For a given α,β, and n, the Bayesian confidence interval under Jeffreys's prior can be computed by the following steps.

1. Generate set $x_1,x_2,\dots,x_n$ from $\mathscr{LOM}(\alpha,\beta)$.
2. Compute $\Gamma(n)$.
3. Compute A and B following equation (20) and (21).
4. Compute α_{BJ} from their posterior distribution by substituting A and B in equation (19).
5. Repeat Steps 2-4 a total of 10,000 times and create an ordered array of α_{BJ}.
6. Compute the posterior density intervals for α from equation (22). If $\alpha^*_{BJ(L)} \le \alpha \le \alpha^*_{BJ(U)}$, then set $cp=1$; otherwise, set $cp=0$.
7. Repeat Steps 1-6 a total of 10,000 times and compute the coverage probability and the average width.

2.3.2 Conjugate Prior

The conjugate prior of α is determined based on a gamma distribution. This prior was first used by Papadopoulos [26]. Ahmad et al. [20] presented the conjugate prior distribution of α as

$$g(\alpha) \propto \frac{a^b}{\Gamma(b)}\exp(-a\alpha)\,\alpha^{b-1}, a>0, b>0 \text{ and } \alpha>0, \tag{23}$$

where a and b are hyper parameters. Combining the likelihood function (15) and the prior density function (23), the posterior density of α is given by

$$\pi(\alpha|x)_2 \propto \left(\frac{\alpha}{\beta}\right)^n \prod_{i=1}^{n}\left(1+\frac{x_i}{\beta}\right)^{-(\alpha+1)} \frac{a^b}{\Gamma(b)}\exp(-a\alpha)\,\alpha^{b-1}$$

or

$$\pi(\alpha|x)_2 \propto \frac{\alpha^n}{\beta^n}\frac{a^b}{\Gamma(b)}\exp(-a\alpha)\,\alpha^{b-1} \cdot\exp(-(\alpha+1)\sum_{i=1}^{n}\ln(1+\frac{x_i}{\beta})).$$

The posterior distribution of α is given by

$$\pi(\alpha|x)_2 = \frac{\left[\left(a+\Sigma_{i=1}^n \ln\left(1+\frac{x_i}{\beta}\right)\right)\right]^{n+b}}{\Gamma(n+b)}\alpha^{n+b-1} \cdot\exp(-\alpha(a+\sum_{i=1}^{n}(1+\frac{x_i}{\beta}))) \tag{24}$$

where $\frac{\left[\left(a+\Sigma_{i=1}^n \ln\left(1+\frac{x_i}{\beta}\right)\right)\right]^{n+b}}{\Gamma(n+b)}$ is independent of α and $\Gamma(n+b) = \int_0^\infty \alpha^{(n+b-1)}\exp(-\alpha)d\alpha$. Therefore, the posterior distribution of α can be written as

$$\pi(\alpha|x)_2 = \alpha^{(n+b-1)}\frac{\left(a+\Sigma_{i=1}^n \ln\left(1+\frac{x_i}{\beta}\right)\right)^{n+b}}{\Gamma(n+b)} \cdot\exp(-\alpha(a+\sum_{i=1}^{n}\ln(1+\frac{x}{\beta}))). \tag{25}$$

Subsequently, the posterior distribution is given by

$$\pi(\alpha|x)_2 = \alpha^{(n+b-1)}CD \tag{26}$$

where

$$C = \frac{\left[\left(a+\Sigma_{i=1}^n \ln\left(1+\frac{x_i}{\beta}\right)\right)\right]^{n+b}}{\Gamma(n+b)} \tag{27}$$

and

$$D = \exp\left(-\alpha\left(a+\sum_{i=1}^{n}\ln\left(1+\frac{x}{\beta}\right)\right)\right). \tag{28}$$

Therefore, the $(1-\nu)100\%$ Bayesian confidence interval under conjugate prior for α is given by

$$CI(\alpha)_{BC} = \left[\alpha^*_{BC(L)}, \alpha^*_{BC(U)}\right] \tag{29}$$

where $\alpha^*_{BC(L)}$ is the $\alpha^*_{BC}(\nu/2)^{th}$ ordered value in the list of estimate of α^* and α^*_U is the $\alpha^*_{BC}(1-\nu/2)^{th}$ ordered value in the list of α^*_{BC} estimate of α. Thus, the coverage probability of the Bayesian confidence interval under the Conjugate prior can be computed using Algorithm 2.

Algorithm 2. For a given α,β, and n, the Bayesian confidence interval under the conjugate prior can be computed by the following steps.

1. Generate set $x_1,x_2,\dots,x_n$ from $\mathscr{LOM}(\alpha,\beta)$.
2. Compute hyper parameters where $a=0.5$ and $b=0.1$.
3. Compute $\Gamma(n+b)$.
4. Compute C and D following equation (27) and (28).
5. Compute α_{BC} from their posterior distribution by substituting C and D in equation (26).
6. Repeat Steps 2-5 a total of 10,000 times and create an ordered array of α_{BC}.
7. Compute the posterior density intervals for α from equation (29). If $\alpha^*_{BC(L)} \le \alpha \le \alpha^*_{BC(U)}$, then set $cp=1$; otherwise, set $cp=0$.
8. Repeat Steps 1-7 a total of 10,000 times and compute the coverage probability and the average width.

2.4 Confidence Interval Based on the Generalized Probability Weighted Moment (GPWM)

Finally, we consider the generalized probability weighted moment (GPWM) approach for constructing a confidence interval for the shape parameter of the Lomax distribution. The generalized probability weighted moments (GPWM) method was introduced by Rasmussen [27]. It is an alternate method to the classical moments, and is more efficient at producing robust parameter estimates, especially in the cases of small samples. The idea of the GPWM for order $p=1$ and $v=0$ is given by

$$\begin{aligned} M_{1,u,0} &= E\left(X\left(F\left(X\right)\right)^{u}\right) \\ &= \int_0^\infty xF\left(X\right)^{u} f(x)dx \\ &= \int_0^1 x(F)F^{u}dF, \end{aligned} \tag{30}$$

where X is a continuous variable, F is the cumulative distribution function of X, and u is neither a small nor nonnegative integer. Abd-Elfattah and Alharbey [28] applied the GPWM method to estimate the parameters of the Lomax distribution. They used the GPWM of the form $M_{1,u,0}$ because it has a simpler analytical structure than $M_{1,u,v}$. The procedure of the GPWM estimation for the construction of confidence intervals for the shape parameter of the Lomax distribution is summarized in the following steps.

Step 1. Obtain the inverse cumulative distribution function $x(F)$ of the Lomax distribution which is given by

$$xF\left(X\right) = \beta\left(\left(1-F\right)^{\frac{-1}{\alpha}}-1\right), \quad x>0, \ (\alpha,\beta>0). \tag{31}$$

Step 2. Calculate the theoretical GPWM from $M_{1,u,0}$ where u takes values $1,2,\ldots$ depending on the number of the unknown parameters. Since the Lomax distribution has two parameters, the formula $M_{1,u,0}$ takes the following forms

$$\begin{aligned} M_{1,u,0} &= \int_0^1 \beta\left(\left(1-F\right)^{\frac{-1}{\alpha}}-1\right)F^{u}dF \\ &= \beta\left[B(u+1,1-\frac{1}{\alpha})-\frac{1}{u+1}\right], \end{aligned} \tag{32}$$

where $B(m,n)$ is the beta function. Let $u=u_1,u_2$, the formula of $M_{1,u_1,0}$ and $M_{1,u_2,0}$ be written as

$$M_{1,u_1,0} = \beta\left[B(u_1+1,1-\frac{1}{\alpha})-\frac{1}{u_1+1}\right], \tag{33}$$

and

$$M_{1,u_2,0} = \beta\left[B(u_2+1,1-\frac{1}{\alpha})-\frac{1}{u_2+1}\right]. \tag{34}$$

Step 3. Obtain the parameter to construct the confidence interval for the shape parameter of the Lomax distribution. We focus on the parameter α and obtain the parameter α in the term of $M_{1,u_1,0}$ and $M_{1,u_2,0}$. From equation (33) and (34), we have

$$\beta = \frac{M_{1,u_1,0}}{\left[B(u_1+1,1-\frac{1}{\alpha})-\frac{1}{u_1+1}\right]} \tag{35}$$

and

$$\beta = \frac{M_{1,u_2,0}}{\left[B(u_2+1,1-\frac{1}{\alpha})-\frac{1}{u_2+1}\right]}. \tag{36}$$

Substituting equation (35) into equation (36) yields

$$\frac{M_{1,u_1,0}}{\left[B(u_1+1,1-\frac{1}{\alpha})-\frac{1}{u_1+1}\right]} = \frac{M_{1,u_2,0}}{\left[B(u_2+1,1-\frac{1}{\alpha})-\frac{1}{u_2+1}\right]}$$

or

$$\frac{M_{1,u_1,0}}{[B(u_1+1,1-\frac{1}{\alpha})-\frac{1}{u_1+1}]} - \frac{M_{1,u_2,0}}{[B(u_2+1,1-\frac{1}{\alpha})-\frac{1}{u_2+1}]} = 0. \tag{37}$$

Step 4. Calculate the sample estimators of $M_{1,u_1,0}$ and $M_{1,u_2,0}$ using the idea of Hosking [29]. The formula $\hat{M}_{1,u,v}$ is based on the order of the complete sample $x_1<x_2<\cdots<x_n$ of size n. The sample estimators of $M_{1,u,v}$ are given by

$$\hat{M}_{1,u,v} = \sum_{i=1}^{n} x_{(i)} p_i^{u} (1-p_i)^{v} \tag{38}$$

where $p_i = \frac{i-0.35}{n}$ and $x_{(i)}$ is the ith observation in the ordered sample. Since $v=0$, the sample estimators of $M_{1,u,0}$ can be written as

$$\hat{M}_{1,u_1,0} = \frac{1}{n}\sum_{i=1}^{n} x_i \left(\frac{i-0.35}{n}\right)^{u_1} \tag{39}$$

and

$$\hat{M}_{1,u_2,0} = \frac{1}{n}\sum_{i=1}^{n} x_i \left(\frac{i-0.35}{n}\right)^{u_2}. \tag{40}$$

Step 5. Solve for the shape parameter estimator α of GPWM by substituting equation (39) and (40) into equation (37) as

$$\frac{\hat{M}_{1,u_1,0}}{[B(u_1+1,1-\frac{1}{\alpha})-\frac{1}{u_1+1}]} - \frac{\hat{M}_{1,u_2,0}}{[B(u_2+1,1-\frac{1}{\alpha})-\frac{1}{u_2+1}]} = 0. \tag{41}$$

The shape parameter estimator of GPWM or $\hat{\alpha}_{GPWM}$ will be solved from equation (41). Since no closed forms of the solution exist, an iterative numerical search can be used to obtain the shape parameter estimator of GPWM or $\hat{\alpha}_{GPWM}$.

Step 6. Calculate the variance of the shape parameter estimator of GPWM or $Var(\hat{\alpha}_{GPWM})$. The asymptotic variance of the GPWM estimator was reported by Hosking [29] using the asymptotic variance covariance of the PWM estimators $\hat{M}_{1,u_1,0}$ and $\hat{M}_{1,u_2,0}$ to approximate the asymptotic variance of α. We have

$$\hat{\alpha}_{GPWM} = \Upsilon(\hat{M}_{1,u_1,0},\hat{M}_{1,u_2,0}) \tag{42}$$

and

$$\hat{\beta}_{GPWM} = \Phi(\hat{M}_{1,u_1,0},\hat{M}_{1,u_2,0}) \tag{43}$$

then the vector $(\hat{M}_{1,u_1,0}, \hat{M}_{1,u_2,0})$ has a Gaussian limiting distribution with mean vector $(M_{1,u_1,0}, M_{1,u_2,0})$ and covariance matrix

$$Cov = \frac{1}{n}(A_{u_1,u_2}) \tag{44}$$

where $A_{u_1,u_2} = I_{u_1,u_2} + I_{u_2,u_1}$ and I_{u_1,u_2} is the variance of PWM which is

$$I_{u_1,u_2} = \int_0^\infty \int_0^y [F(x)]^{u_1+1}[F(y)]^{u_2}[1-F(y)]dxdy \tag{45}$$

and

$$I_{u_2,u_1} = \int_0^\infty \int_0^y [F(x)]^{u_2+1}[F(y)]^{u_1}[1-F(y)]dxdy. \tag{46}$$

The first-order approximations of the variances and covariance of $\hat{\alpha}_{GPWM}$ and $\hat{\beta}_{GPWM}$ can be computed using the variances and covariance of the GPWM estimators of $M_{1,u_1,0}$ and $M_{1,u_2,0}$ by calculating these functions,

$$W_1 = \begin{bmatrix} Var(\hat{\alpha}_{GPWM}) \\ Var\left(\hat{\beta}_{GPWM}\right) \\ Cov\left(\hat{\alpha}_{GPWM}, \hat{\beta}_{GPWM}\right) \end{bmatrix}$$

$$W_2 = \begin{bmatrix} M_{11}^2 & M_{12}^2 & 2M_{11}M_{12} \\ M_{21}^2 & M_{22}^2 & 2M_{21}M_{22} \\ M_{11}M_{21} & M_{12}M_{22} & M_{11}M_{22}+M_{21}M_{12} \end{bmatrix}^{-1}$$

$$W_3 = \begin{bmatrix} Var\left(\hat{M}_{1,u_1,0}\right) \\ Var\left(\hat{M}_{1,u_2,0}\right) \\ Cov\left(\hat{M}_{1,u_1,0}, \hat{M}_{1,u_2,0}\right) \end{bmatrix}$$

and

$$W_1 = W_2 \cdot W_3 \tag{47}$$

where $Var\left(\hat{M}_{1,u_1,0}\right), Var\left(\hat{M}_{1,u_2,0}\right)$, and $Cov\left(\hat{M}_{1,u_1,0}, \hat{M}_{1,u_2,0}\right)$ are given in equation (44). The values of M_{ij} can be explicitly obtained from equation (33) and (34), as follows:

$$\begin{aligned} M_{11} &= \frac{\partial M_{1,u_1,0}}{\partial \alpha} \\ &= \frac{\beta}{\alpha^2 \Gamma(u_2+2+\frac{1}{\alpha})} \\ &\quad \cdot \left[\Gamma'(1-\alpha^{-1})[\Gamma(u_1+1) - \Gamma'(u_1+1)\Gamma(1-\alpha^{-1})\right] \end{aligned} \tag{48}$$

$$M_{12} = \frac{\partial M_{1,u_1,0}}{\partial \beta} = B(u_2+1, 1-\frac{1}{\alpha}) - \frac{1}{u_2+1} \tag{49}$$

where $\Gamma'(z) = \Gamma(z)\Psi_0(z)$ is the derivatives of the gamma function, which is described in terms of the polygamma function. Furthermore, M_{11} and M_{22} are obtained from $M_{1,u_1,0}$ and $M_{1,u_2,0}$ by replacing u_1 by u_1.

Step 7. Calculate the $(1-\nu)100\%$ two-sided confidence intervals for α based on the generalized probability weighted moment (GPWM) as

$$CI(\alpha)_{GPWM} = \left[\alpha_{GPWM(L)}, \alpha_{GPWM(U)}\right] \tag{50}$$

where $\alpha_{GPWM(L)} = \hat{\alpha}_{GPWM} - Z_{(\nu/2)}\sqrt{Var(\hat{\alpha}_{GPWM})}$ and $\alpha_{GPWM(U)} = \hat{\alpha}_{GPWM} + Z_{(\nu/2)}\sqrt{Var(\hat{\alpha}_{GPWM})}$.

3 Results and Discussion

3.1 Simulation technique

A Monte Carlo simulation was conducted using the **R** statistical software [30] to evaluate the performance of the proposed confidence intervals for the shape parameter of the Lomax distribution. The simulation studies were carried out to evaluate the coverage probabilities (CP) and average widths (AW) of each confidence interval. The important factor in judging the effectiveness of confidence intervals is the performance of coverage probabilities. Normally, a preferable confidence interval should have a coverage probability close to the nominal confidence level. Moreover, the average width has also been a short length interval. The nominal values of 0.90 and 0.95 are calculated based on 10,000 replications and for the bootstrap computations, 10,000 bootstrapped data sets are used. The estimated coverage probability and the average width are given by

$$CP = \frac{\sum_{i=1}^{10,000} I_i(L \le \alpha \le U)}{10,000}$$

and

$$AW = \frac{\sum_{i=1}^{10,000}(U_i - L_i)}{10,000}$$

where $\sum_{i=1}^{10,000} I_i(L \le \alpha \le U)$ denotes the number of simulation runs for which the shape parameter (α) lies within the confidence interval.

All data sets were generated from a Lomax distribution, $\mathscr{LOM}(\alpha, \beta)$. For these simulations, we used 80 unique sets of parameter values. These include situations of varying $n, \alpha,$ and β, using $n = 25, 50, 100, 500, 1000; \alpha = 0.5, 1.0, 1.5, 2.0$ and $\beta = 1.0, 2.0, 4.0, 6.0$. A simulation study in each design is used to calculate the coverage probabilities and the average length of the confidence interval. The results of 95% and 99% confidence intervals for α are shown in Tables 1-10. In the following, the notation "a" is used when the coverage probability of the confidence interval is reported and "b" is used for the average width of the confidence interval.

Table 1: Coverage probabilities and average widths of α for $n = 25$ at the 0.95 nominal level

α	β	ML		BS		BJ		BC		GPWM	
		CP[a]	AW[b]	CP[a]	AW[b]	CP[a]	AW[b]	CP[a]	AW[b]	CP[a]	AW[b]
0.5	1.0	0.9466	0.7167	0.9484	0.8452	0.9337	3.6568	0.9406	3.6594	0.9361	1.2027
	2.0	0.9346	0.7166	0.9469	0.8470	0.9340	3.6565	0.9334	3.6509	0.9543	1.2026
	4.0	0.9474	0.7170	0.9511	0.8482	0.9349	3.6569	0.9461	3.6511	0.9481	1.2030
	6.0	0.9485	0.7169	0.9500	0.8455	0.9423	3.6569	0.9536	3.6510	0.9516	1.2029
1.0	1.0	0.9482	1.4341	0.9764	1.6922	0.9392	1.8285	0.9447	1.8930	0.9282	1.5281
	2.0	0.9455	1.4335	0.9434	1.6939	0.9343	1.8283	0.9311	1.8312	0.9475	1.5275
	4.0	0.9531	1.4344	0.9516	1.6922	0.9333	1.8283	0.9320	1.8193	0.9446	1.5284
	6.0	0.9588	1.4333	0.9494	1.6929	0.9442	1.8283	0.9399	1.8144	0.9482	1.5273
1.5	1.0	0.9582	2.1506	0.9526	2.5415	0.9238	1.2190	0.9276	1.2103	0.9534	1.8526
	2.0	0.9666	2.1498	0.9496	2.5412	0.9228	1.2189	0.9238	1.2041	0.9662	1.8519
	4.0	0.9561	2.1493	0.9501	2.5354	0.9242	1.2189	0.9236	1.2007	0.9612	1.8513
	6.0	0.9580	2.1503	0.9564	2.5433	0.9251	1.2189	0.9202	1.2002	0.9441	1.8524
2.0	1.0	0.9490	2.8671	0.9465	3.3860	0.9190	0.9142	0.9285	0.9069	0.9168	2.1771
	2.0	0.9544	2.8680	0.9510	3.3778	0.9185	0.9141	0.9210	0.9070	0.9424	2.1780
	4.0	0.9629	2.8670	0.9525	3.3834	0.9171	0.9142	0.9248	0.9048	0.9698	2.1770
	6.0	0.9599	2.8685	0.9468	3.3771	0.9194	0.9142	0.9206	0.9027	0.9506	2.1785

Table 2: Coverage probabilities and average widths of α for $n = 50$ at the 0.95 nominal level

α	β	ML		BS		BJ		BC		GPWM	
		CP[a]	AW[b]	CP[a]	AW[b]	CP[a]	AW[b]	CP[a]	AW[b]	CP[a]	AW[b]
0.5	1.0	0.9540	0.6487	0.9543	0.5753	0.9372	5.1762	0.9401	3.9717	0.9476	1.2495
	2.0	0.9475	0.6489	0.9481	0.5769	0.9371	5.1760	0.9439	3.9477	0.9502	1.2497
	4.0	0.9493	0.6487	0.9476	0.5768	0.9496	5.1764	0.9428	3.9468	0.9491	1.2495
	6.0	0.9501	0.6487	0.9493	0.5762	0.9464	5.1761	0.9446	3.9742	0.9508	1.2496
1.0	1.0	0.9594	1.2979	0.9589	1.1540	0.9354	2.5882	0.9421	1.9084	0.9365	1.6215
	2.0	0.9434	1.2974	0.9522	1.1514	0.9408	2.5881	0.9416	1.9085	0.9538	1.6210
	4.0	0.9524	1.2979	0.9475	1.1532	0.9431	2.5881	0.9407	1.9082	0.9525	1.6215
	6.0	0.9519	1.2974	0.9498	1.1515	0.9439	2.5878	0.9393	1.9086	0.9490	1.6210
1.5	1.0	0.9405	1.9461	0.9472	1.7312	0.9419	1.7253	0.9403	1.3754	0.9397	1.9926
	2.0	0.9450	1.9467	0.9527	1.7290	0.9439	1.7255	0.9426	1.3732	0.9369	1.9931
	4.0	0.9474	1.9466	0.9463	1.7291	0.9418	1.7254	0.9404	1.3472	0.9568	1.9931
	6.0	0.9471	1.9475	0.9507	1.7287	0.9432	1.7255	0.9403	1.3247	0.9541	1.9940
2.0	1.0	0.9746	2.5953	0.9602	2.3021	0.9383	1.2941	0.9365	0.9935	0.9411	2.3646
	2.0	0.9675	2.5944	0.9532	2.3072	0.9366	1.2940	0.9459	0.9935	0.9311	2.3637
	4.0	0.9610	2.5952	0.9517	2.3033	0.9368	1.2941	0.9420	0.9543	0.9496	2.3644
	6.0	0.9542	2.5954	0.9497	2.3036	0.9311	1.2939	0.9421	0.9542	0.9528	2.3647

Table 3: Coverage probabilities and average widths of α for $n = 100$ at the 0.95 nominal level

α	β	ML		BS		BJ		BC		GPWM	
		CP[a]	AW[b]	CP[a]	AW[b]	CP[a]	AW[b]	CP[a]	AW[b]	CP[a]	AW[b]
0.5	1.0	0.9485	0.6029	0.9526	0.3995	0.9455	7.3235	0.9410	5.9714	0.9464	1.2849
	2.0	0.9487	0.6031	0.9499	0.3998	0.9419	7.3241	0.9366	5.6971	0.9496	1.2851
	4.0	0.9509	0.6029	0.9499	0.4010	0.9438	7.3238	0.9449	5.6274	0.9504	1.2849
	6.0	0.9513	0.6030	0.9498	0.3989	0.9419	7.3238	0.9447	5.6291	0.9502	1.2850
1.0	1.0	0.9657	1.2062	0.9516	0.6677	0.9371	3.6620	0.9308	2.8149	0.9499	1.6923
	2.0	0.9459	1.2062	0.9493	0.6721	0.9408	3.6618	0.9333	2.8149	0.9510	1.6922
	4.0	0.9488	1.2061	0.9490	0.6686	0.9495	3.6619	0.9456	2.8144	0.9511	1.6921
	6.0	0.9497	1.2060	0.9508	0.6710	0.9420	3.6615	0.9422	2.8149	0.9512	1.6920
1.5	1.0	0.9514	1.8095	0.9428	0.9399	0.9387	2.4412	0.9418	1.8966	0.9488	2.0995
	2.0	0.9447	1.8092	0.9480	0.9419	0.9363	2.4411	0.9393	1.8766	0.9402	2.0992
	4.0	0.9459	1.8095	0.9501	0.9359	0.9334	2.4414	0.9301	1.8657	0.9488	2.0995
	6.0	0.9518	1.8086	0.9489	0.9442	0.9364	2.4412	0.9349	1.8568	0.9496	2.0986
2.0	1.0	0.9567	2.4121	0.9507	1.2135	0.9391	1.8310	0.9366	1.4743	0.9534	2.5061
	2.0	0.9482	2.4127	0.9441	1.2002	0.9407	1.8309	0.9348	1.4427	0.9511	2.5067
	4.0	0.9505	2.4116	0.9511	1.2097	0.9262	1.8310	0.9200	1.4074	0.9533	2.5056
	6.0	0.9523	2.4124	0.9510	1.2031	0.9219	1.8308	0.9296	1.4043	0.9516	2.5064

3.2 Results discussion

Several interesting features appear in Tables 1-10. First, the bootstrap intervals have smaller limits than the normal-based intervals, especially in small sample size cases. The comparison of the two bootstrap confidence intervals shows that the coverage probabilities of the BJ and BC approaches are almost similar, and the average widths of both methods get narrow when the parameter α increases. The average width of the BC method is slightly better than that of the BJ approach. The results in terms of the coverage probabilities of all five approaches show that the ML approach outperforms other methods. We conclude that the average widths of the ML approach is the shortest if n is small. Performance of the ML and BS approaches is almost identical for large sample sizes. However, the BS approach takes much longer to execute than the ML approach. For increasing the parameters α

Table 4: Coverage probabilities and average widths of α for $n = 500$ at the 0.95 nominal level

α	β	ML		BS		BJ		BC		GPWM	
		CP[a]	AW[b]	CP[a]	AW[b]	CP[a]	AW[b]	CP[a]	AW[b]	CP[a]	AW[b]
0.5	1.0	0.9506	0.5449	0.9494	0.2699	0.9445	8.2245	0.9431	7.9826	0.9500	1.3352
	2.0	0.9495	0.5448	0.9501	0.2702	0.9462	8.2240	0.9442	7.9683	0.9502	1.3352
	4.0	0.9504	0.5448	0.9499	0.2714	0.9448	8.2238	0.9445	7.8251	0.9502	1.3351
	6.0	0.9500	0.5449	0.9500	0.2693	0.9453	4.5622	0.9448	7.6826	0.9500	1.3353
1.0	1.0	0.9484	1.0896	0.9493	0.5381	0.9377	4.5619	0.9408	3.9841	0.9500	1.7923
	2.0	0.9514	1.0897	0.9505	0.5425	0.9475	4.5621	0.9460	3.9425	0.9505	1.7924
	4.0	0.9509	1.0896	0.9500	0.5390	0.9451	4.5615	0.9463	3.8413	0.9499	1.7923
	6.0	0.9497	1.0897	0.9498	0.5414	0.9456	3.3412	0.9446	3.4125	0.9502	1.7924
1.5	1.0	0.9522	1.6346	0.9492	0.8103	0.9451	3.3411	0.9407	2.6609	0.9500	2.2496
	2.0	0.9501	1.6345	0.9499	0.8123	0.9493	3.3415	0.9481	2.6584	0.9498	2.2496
	4.0	0.9493	1.6348	0.9498	0.8063	0.9431	3.3413	0.9443	2.6561	0.9505	2.2499
	6.0	0.9494	1.6342	0.9497	0.8146	0.9436	2.7312	0.9448	2.6087	0.9503	2.2493
2.0	1.0	0.9514	2.1794	0.9494	1.0839	0.9428	2.7310	0.9276	1.9206	0.9465	2.7068
	2.0	0.9467	2.1796	0.9500	1.0706	0.9423	2.7311	0.9484	1.9064	0.9486	2.7070
	4.0	0.9485	2.1792	0.9502	1.0801	0.9453	2.7309	0.9462	1.8921	0.9501	2.7066
	6.0	0.9510	2.1791	0.9502	1.0735	0.9442	2.7276	0.9448	1.8206	0.9500	2.7065

Table 5: Coverage probabilities and average widths of α for $n = 1000$ at the 0.95 nominal level

α	β	ML		BS		BJ		BC		GPWM	
		CP[a]	AW[b]	CP[a]	AW[b]	CP[a]	AW[b]	CP[a]	AW[b]	CP[a]	AW[b]
0.5	1.0	0.9491	0.5315	0.9504	0.1761	0.9428	8.9772	0.9438	8.6447	0.9495	1.3475
	2.0	0.9501	0.5315	0.9498	0.1763	0.9448	8.9762	0.9460	8.6442	0.9502	1.3475
	4.0	0.9510	0.5315	0.9453	0.1761	0.9450	8.9766	0.9454	8.6405	0.9469	1.3475
	6.0	0.9501	0.5314	0.9470	0.1763	0.9450	8.9762	0.9453	8.6375	0.9485	1.3475
1.0	1.0	0.9503	1.0631	0.9502	0.3528	0.9456	6.3885	0.9465	6.3253	0.9495	1.8172
	2.0	0.9499	1.0630	0.9497	0.3527	0.9434	6.3882	0.9438	6.3223	0.9498	1.8170
	4.0	0.9503	1.0630	0.9500	0.3522	0.9448	6.3884	0.9441	6.3204	0.9501	1.8170
	6.0	0.9499	1.0630	0.9492	0.3542	0.9447	6.3878	0.9448	6.3167	0.9479	1.8170
1.5	1.0	0.9524	1.5943	0.9497	0.6650	0.9476	5.5787	0.9465	5.2902	0.9503	2.2864
	2.0	0.9505	1.5945	0.9502	0.6608	0.9440	5.5255	0.9442	5.2497	0.9502	2.2865
	4.0	0.9499	1.5944	0.9498	0.6669	0.9439	5.5552	0.9451	5.2147	0.9501	2.2864
	6.0	0.9499	1.5943	0.9499	0.6582	0.9456	5.5295	0.9457	5.2145	0.9498	2.2863
2.0	1.0	0.9553	2.1261	0.9496	0.9622	0.9522	5.0941	0.9423	4.1665	0.9502	2.7562
	2.0	0.9514	2.1260	0.9496	0.9807	0.9476	5.0448	0.9418	4.1612	0.9499	2.7561
	4.0	0.9498	2.1258	0.9501	0.9673	0.9456	5.0418	0.9449	4.1611	0.9499	2.7559
	6.0	0.9497	2.1262	0.9500	0.9743	0.9452	5.0080	0.9450	4.1513	0.9500	2.7563

Table 6: Coverage probabilities and average widths of α for $n = 25$ at the 0.99 nominal level

α	β	ML		BS		BJ		BC		GPWM	
		CP[a]	AW[b]	CP[a]	AW[b]	CP[a]	AW[b]	CP[a]	AW[b]	CP[a]	AW[b]
0.5	1.0	0.9894	0.7214	0.9614	0.8786	0.9702	3.6938	0.9726	3.6809	0.9699	1.2216
	2.0	0.9851	0.7234	0.9682	0.8711	0.9643	3.6692	0.9764	3.6595	0.9801	1.2140
	4.0	0.9916	0.7254	0.9853	0.8525	0.9695	3.6622	0.9611	3.6554	0.9824	1.2046
	6.0	0.9868	0.7236	0.9839	0.8514	0.9808	3.6611	0.9869	3.6591	0.9866	1.2071
1.0	1.0	0.9959	1.4753	0.9763	1.7167	0.9692	1.8686	0.9646	1.9810	0.9630	1.5420
	2.0	0.9903	1.4837	0.9746	1.7626	0.9444	1.8613	0.9665	1.8816	0.9738	1.5614
	4.0	0.9851	1.4667	0.9709	1.7129	0.9785	1.8415	0.9709	1.8251	0.9827	1.5399
	6.0	0.9870	1.4357	0.9869	1.6959	0.9832	1.8385	0.9751	1.8343	0.9801	1.5348
1.5	1.0	0.9945	2.1978	0.9772	2.6062	0.9720	1.2464	0.9732	1.3162	0.9781	1.9894
	2.0	0.9917	2.1667	0.9755	2.5423	0.9700	1.2609	0.9711	1.2071	0.9833	1.8927
	4.0	0.9887	2.1827	0.9875	2.5381	0.9608	1.2364	0.9637	1.2029	0.9813	1.8640
	6.0	0.9849	2.1527	0.9896	2.5602	0.9608	1.2368	0.9465	1.2153	0.9988	1.8615
2.0	1.0	0.9873	2.8771	0.9706	3.4569	0.9613	1.1505	0.9140	1.0640	0.9855	2.4016
	2.0	0.9879	2.9450	0.9816	3.4474	0.9777	0.9216	0.9735	0.9573	0.9835	2.1875
	4.0	0.9910	2.8794	0.9750	3.4124	0.9366	0.9658	0.9696	0.9307	0.9814	2.1956
	6.0	0.9917	2.8999	0.9725	3.3983	0.9531	0.9317	0.9626	0.9083	0.9799	2.1884

and β, the coverage probabilities for GPWM approach get closer to the nominal confidence level in all cases. The GPWM approach performs quite well even when the sample sizes are small.

3.3 Illustrative Examples

To illustrate the computation of confidence intervals proposed in this article, we use the data of observations for precipitation in Florida meteorological study by Simpson [31]. We consider a sample of computer file sizes (in bytes) for all 269 files with the *.ini extension on a Windows-based personal computer. The website of http://web.uvic.ca/~dgiles/downloads/data can be used to download these data. The data showed appropriateness for the Lomax distribution by Ferreira et al. [32]. The 95% confidence intervals for α are listed in Table 3 which

Table 7: Coverage probabilities and average widths of α for $n = 50$ at the 0.99 nominal level

α	β	ML		BS		BJ		BC		GPWM	
		CP^a	AW^b	CP^a	AW^b	CP^a	AW^b	CP^a	AW^b	CP^a	AW^b
0.5	1.0	0.9954	0.6509	0.9883	0.5793	0.9653	5.1786	0.9707	3.9771	0.9862	1.2497
	2.0	0.9835	0.6492	0.9818	0.5831	0.9703	5.1835	0.9783	3.9571	0.9846	1.2523
	4.0	0.9867	0.6492	0.9864	0.5782	0.9846	5.1824	0.9801	3.9489	0.9847	1.2505
	6.0	0.9834	0.6495	0.9864	0.5767	0.9843	5.1773	0.9823	3.9756	0.9869	1.2522
1.0	1.0	0.9908	1.3408	0.9820	1.1867	0.9844	2.6200	0.9853	1.9204	0.9788	1.6698
	2.0	0.9962	1.3158	0.9806	1.1697	0.9802	2.5971	0.9804	1.9088	0.9854	1.6245
	4.0	0.9838	1.3064	0.9880	1.1620	0.9757	2.5928	0.9757	1.9088	0.9843	1.6298
	6.0	0.9841	1.3002	0.9810	1.1533	0.9786	2.5880	0.9718	1.9118	0.9863	1.6278
1.5	1.0	0.9908	1.9640	0.9702	1.7396	0.9805	1.7352	0.9503	1.3870	0.9693	1.9957
	2.0	0.9976	1.9472	0.9874	1.7468	0.9735	1.7317	0.9716	1.3998	0.9696	2.0009
	4.0	0.9870	1.9487	0.9849	1.7301	0.9758	1.7310	0.9823	1.3564	0.9810	1.9995
	6.0	0.9911	1.9496	0.9827	1.7296	0.9779	1.7276	0.9736	1.3278	0.9881	2.0102
2.0	1.0	0.9901	2.6753	0.9854	2.3441	0.9549	1.2987	0.9722	0.9988	0.9835	2.4069
	2.0	0.9852	2.5961	0.9887	2.3158	0.9427	1.3007	0.9814	1.0091	0.9788	2.3689
	4.0	0.9847	2.6075	0.9885	2.3069	0.9754	1.2944	0.9758	0.9688	0.9830	2.3737
	6.0	0.9841	2.6011	0.9804	2.3060	0.9635	1.3073	0.9847	0.9550	0.9816	2.3673

Table 8: Coverage probabilities and average widths of α for $n = 100$ at the 0.99 nominal level

α	β	ML		BS		BJ		BC		GPWM	
		CP^a	AW^b	CP^a	AW^b	CP^a	AW^b	CP^a	AW^b	CP^a	AW^b
0.5	1.0	0.9847	0.6046	0.9867	0.4010	0.9788	7.3237	0.9777	5.9770	0.9814	1.2971
	2.0	0.9840	0.6076	0.9838	0.4016	0.9777	7.3262	0.9714	5.6972	0.9847	1.2866
	4.0	0.9866	0.6038	0.9858	0.4038	0.9783	7.3277	0.9813	5.6293	0.9869	1.2866
	6.0	0.9866	0.6038	0.9868	0.3990	0.9782	7.3241	0.9803	5.6292	0.9871	1.2853
1.0	1.0	0.9875	1.2147	0.9889	0.6711	0.9752	3.6722	0.9710	2.8268	0.9964	1.6959
	2.0	0.9855	1.2130	0.9820	0.6757	0.9757	3.6635	0.9705	2.8194	0.9843	1.7004
	4.0	0.9880	1.2075	0.9821	0.6700	0.9834	3.6638	0.9820	2.8182	0.9863	1.6941
	6.0	0.9850	1.2069	0.9863	0.6728	0.9779	3.6617	0.9782	2.8150	0.9870	1.6966
1.5	1.0	0.9927	1.8306	0.9817	0.9482	0.9769	2.4629	0.9722	1.9145	0.9857	2.1357
	2.0	0.9847	1.8120	0.9788	0.9462	0.9691	2.4441	0.9744	1.8970	0.9727	2.1086
	4.0	0.9863	1.8139	0.9838	0.9397	0.9665	2.4452	0.9669	1.8699	0.9821	2.1024
	6.0	0.9881	1.8162	0.9846	0.9451	0.9737	2.4413	0.9696	1.8614	0.9869	2.0992
2.0	1.0	0.9843	2.4301	0.9825	1.2244	0.9658	1.8487	0.9741	1.4837	0.9825	2.5251
	2.0	0.9878	2.4202	0.9832	1.2048	0.9794	1.8428	0.9629	1.4553	0.9859	2.5153
	4.0	0.9874	2.4139	0.9823	1.2115	0.9643	1.8335	0.9546	1.4153	0.9896	2.5093
	6.0	0.9892	2.4164	0.9876	1.2077	0.9583	1.8322	0.9670	1.4043	0.9892	2.5113

Table 9: Coverage probabilities and average widths of α for $n = 500$ at the 0.99 nominal level

α	β	ML		BS		BJ		BC		GPWM	
		CP^a	AW^b	CP^a	AW^b	CP^a	AW^b	CP^a	AW^b	CP^a	AW^b
0.5	1.0	0.9865	0.5451	0.9845	0.2700	0.9797	8.2247	0.9792	7.9834	0.9863	1.3363
	2.0	0.9860	0.5453	0.9857	0.2712	0.9824	8.2250	0.9800	7.9698	0.9859	1.3356
	4.0	0.9864	0.5454	0.9857	0.2726	0.9807	8.2250	0.9807	7.8257	0.9865	1.3364
	6.0	0.9861	0.5457	0.9860	0.2702	0.9814	4.5634	0.9807	7.6840	0.9860	1.3364
1.0	1.0	0.9871	1.0914	0.9828	0.5421	0.9741	4.5654	0.9784	3.9855	0.9849	1.7938
	2.0	0.9880	1.0913	0.9867	0.5446	0.9830	4.5627	0.9827	3.9436	0.9874	1.7937
	4.0	0.9869	1.0914	0.9860	0.5408	0.9810	4.5615	0.9828	3.8429	0.9865	1.7930
	6.0	0.9858	1.0904	0.9828	0.5426	0.9814	3.3426	0.9808	3.4132	0.9860	1.7929
1.5	1.0	0.9882	1.6365	0.9872	0.8135	0.9830	3.3412	0.9792	2.6624	0.9875	2.2575
	2.0	0.9870	1.6376	0.9849	0.8133	0.9844	3.3422	0.9838	2.6604	0.9858	2.2513
	4.0	0.9854	1.6353	0.9849	0.8082	0.9794	3.3417	0.9802	2.6564	0.9860	2.2511
	6.0	0.9863	1.6351	0.9851	0.8164	0.9795	2.7314	0.9809	2.6095	0.9859	2.2501
2.0	1.0	0.9913	2.1900	0.9881	1.0855	0.9802	2.7373	0.9623	1.9242	0.9834	2.7125
	2.0	0.9870	2.1829	0.9844	1.0733	0.9773	2.7368	0.9858	1.9097	0.9824	2.7088
	4.0	0.9874	2.1811	0.9846	1.0839	0.9807	2.7322	0.9831	1.8944	0.9851	2.7082
	6.0	0.9868	2.1800	0.9866	1.0743	0.9796	2.7289	0.9807	1.8224	0.9867	2.7089

indicates that the width of the BS approach provides the shortest width among all methods.

3.4 Summary and Concluding Remarks

In this paper, we proposed five methods of interval estimation for the shape parameter of the Lomax distribution. These methods are the ML, BS, BJ, BC, and GPWM approaches. Moreover, the Monte Carlo simulation numerically evaluated the performances of all methods that are considered as coverage probability and average width. The findings revealed that the ML method performed well for all cases because the coverage probabilities were consistently greater than, or close to, the nominal confidence level, while the average widths were mostly shorter than other methods. Furthermore, the GPWM method is regarded as a recommended method when large α and β exist. Moreover, underestimation

Table 10: Coverage probabilities and average widths of α for $n = 1000$ at the 0.99 nominal level

α	β	ML		BS		BJ		BC		GPWM	
		CP^a	AW^b	CP^a	AW^b	CP^a	AW^b	CP^a	AW^b	CP^a	AW^b
0.5	1.0	0.9898	0.5317	0.9864	0.1770	0.9787	8.9783	0.9796	8.6464	0.9849	1.3491
	2.0	0.9861	0.5331	0.9858	0.1767	0.9809	8.9772	0.9819	8.6457	0.9863	1.3488
	4.0	0.9860	0.5328	0.9859	0.1769	0.9809	8.9774	0.9815	8.6415	0.9860	1.3485
	6.0	0.9861	0.5326	0.9859	0.1774	0.9811	8.9770	0.9813	8.6385	0.9860	1.3485
1.0	1.0	0.9868	1.0651	0.9866	0.3536	0.9826	6.3910	0.9826	6.3268	0.9857	1.8173
	2.0	0.9860	1.0632	0.9857	0.3546	0.9794	6.3891	0.9799	6.3228	0.9857	1.8171
	4.0	0.9863	1.0634	0.9856	0.3531	0.9805	6.3897	0.9803	6.3215	0.9864	1.8178
	6.0	0.9858	1.0638	0.9860	0.3548	0.9808	6.3887	0.9808	6.3176	0.9859	1.8180
1.5	1.0	0.9890	1.5963	0.9868	0.6669	0.9859	5.5822	0.9809	5.2904	0.9879	2.2897
	2.0	0.9862	1.5949	0.9855	0.6635	0.9795	5.5268	0.9803	5.2511	0.9865	2.2876
	4.0	0.9858	1.5954	0.9858	0.6677	0.9801	5.5564	0.9811	5.2165	0.9861	2.2873
	6.0	0.9861	1.5958	0.9859	0.6593	0.9816	5.5306	0.9819	5.2160	0.9858	2.2875
2.0	1.0	0.9929	2.1269	0.9869	0.9679	0.9878	5.1000	0.9768	4.1681	0.9897	2.7585
	2.0	0.9879	2.1271	0.9867	0.9827	0.9839	5.0459	0.9765	4.1615	0.9868	2.7564
	4.0	0.9860	2.1268	0.9861	0.9682	0.9815	5.0432	0.9814	4.1629	0.9858	2.7566
	6.0	0.9857	2.1272	0.9857	0.9746	0.9812	5.0089	0.9809	4.1521	0.9858	2.7572

occurred for many cases when applying the Bayesian methods.

Table 11: Interval estimations of α the 0.95 nominal level

Methods	Interval estimations	
	(Lower bound, Upper bound)	widths
ML	(0.4173, 0.9302)	0.5129
BS	(0.2215, 0.6873)	0.4658
BJ	(0.4197, 0.5902)	1.1705
BC	(0.1570, 1.1925)	1.0355
GPWM	(0.3270, 1.1732)	0.8462

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References

[1] K. S. Lomax, Business Failures; Another example of the analysis of failure data, *Journal of the American Statistical Association*, **49**, 847-852 (1954).
[2] M.C. Bryson, Heavy-tailed distributions: properties and tests, *Technometrics*, **16**(1.1), 61-68 (1974).
[3] C. M. Harris, The Pareto distribution as a queue service discipline, *Operations Research*, **16**(2), 307-313 (1968).
[4] A. B. Atkinson, and A. J. Harrison, *Distribution of Personal Wealth in Britain*, Cambridge University Press, Cambridge, UK, (1978).
[5] B. Vidondo, Y.T. Prairie, J.M. Blanco, and C.M. Duarte, Some aspects of the analysis of size spectra in aquatic ecology, *Limnology and Oceanography*, **42**, 184-192 (1997).
[6] A. S. Hassan, and A. S. Al-Ghamdi, Optimum step-stress accelerated life testing for lomax distribution, *Journal of Applied Sciences Research*, **5**(2), 2153-2164 (2009).
[7] M. Chahkandi, and M. Ganjali, On some lifetime distributions with decreasing failure rate, *Computational Statistics and Data Analysis*, **53**(12), 4433-4440 (2009).
[8] B. C. Arnold, *Pareto Distributions*, International Cooperative Publishing House, Maryland, USA, (1983).
[9] N. L. Johnson, S. Kotz, and N. Balakrishnan, *Contineous Univariate Distributions: Vol. 1*, Wiley, New York, USA, (1994).
[10] M. Ahsanullah, Recoed values of the Lomax distribution, *Statistica Neerlandica*, **45**, 21-29 (1991).
[11] E. A. Amin, Kth upper record values and their moments, *International Mathematical Forum*, **6**, 3013-3021 (2011).
[12] N. Balakrishnan, and M. Ahsanullah, *Relations for single and product moments of record values from Lomax distribution*, Sankhya Series B, **17**, 140-146 (2009).
[13] M-Y. Lee, and E-K. Lim, Characterization of the Lomax, exponential and Pareto distributions by conditional expectations of recoed values, *Journal of the Chungcheong Mathematical Society*, **22**, 149-153 (2009).
[14] E.A. Amin, Kth upper record values and their moments, *International Mathematical Forum*, **6**, 3013-3021 (2011).
[15] M.S. Moghadam, F. Yahmaei, and M. Babanezhad, Inference for Lomax distribution under generalized order statistics, *Applied Mathematical Sciences*, **6**, 5241-5251 (2012).
[16] M.A. Abdullah, and H.A. Abdullah, Estimation of Lomax parameters based on Generalized Probability Weighted Moment, *Journal of King Abdulaziz University-Science*, **22**(2), 171-184 (2010).
[17] A.H. Abd-Ellah, Comparision of estimates using record statistics from Lomax model: Bayesian and non-Bayesian approaches, *Journal of Statistical Research of Iran*, **3**, 139-158 (2006).
[18] P. Nasiri, and S. Hosseini, Statistical inferences for Lomax distribution based on record values (Bayesian and classical), *Journal of Modern Applied Statistical Methods*, **11**, 179-189 (2012).
[19] A. Ahmad, S.P. Ahmad, and A. Ahmed, Bayesian Analysis of shape parameter of Lomax distribution under different loss functions, *International Journal of Statistics and Mathematics*, **2**(1), 055-065 (2015).
[20] A. Ahmad, S. P. Ahmad, and A. Ahmed, Bayesian analysis of lomax distribution under asymmetric loss functions, *Journal of Statistics Applications and Probability Letters*, **3**(1), 35-44 (2016).
[21] O. Venegas, Y. A. Iriarte, J. M. Astorga, and H. W. Gomez, Lomax-Rayleigh distribution with an application,

Applied Mathematics and Information Sciences, **13**(5), 741-748 (2019).

[22] E.K. Al-Hussaini, Z.F. Jaheen, and A.M. Nigm, Bayesian prediction based on finite mixture of Lomax component model and type I censoring, *Statistics*, **35**, 259-268 (2001).

[23] A.H. Abd-Ellah, Bayesian one sample prediction bounds for the Lomax distribution, *Indian Journal of Pure and Applied Mathematics*, **30**, 101-109 (2003).

[24] N. Jantakoon and A. Volodin, Interval estimation for the shape and scale parameters of the Birnbaum Saunders distribution, *Lobachevskii Journal of Mathematics*, **40**(8), 1164-1177 (2019).

[25] D. E. Giles, H. Feng, and R. T. Godwin, On the bias of the maximum likelihood estimator for the two-parameter lomax distribution, *Communications in Statistics Theory and Methods*, **42**, 1934-1950 (2013).

[26] A.S. Papadopoulos, The Burr distribution as a failure model from a Bayesian approach, *IEEE Transactions on Reliability*, **R-27**(5), 369-371 (1978).

[27] P. Rasmussen, Generalized probability weighted moments: application to the generalized probability weighted moments: pareto distribution, *Water Resource Research*, **37**(6), 1745-1751 (2001).

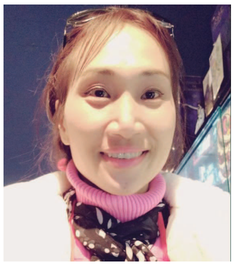
[28] A. M. Abd-Elfattah, and A. H. Alharbey, Estimation of Lomax Parameters Based on Generalized Probability Weighted Moment, *Journal of King Abdulaziz University-Science*, **22**(2), 171-184 (2010).

[29] J. R. M. Hosking, *The theory of probability weighted moments, Research Report RC 12210*, IBM Research Division, Yorktown Heights, New York, USA, (1986).

[30] R Development Core Team, *R: a language and environment for statistical computing*, R Foundation for Statistical Computing, (2005).

[31] J. Simpson, Use of the gamma distribution in single-cloud rainfall analysis, *Monthly Weather Review*, **100**, 309-312 (1972).

[32] . P. Ferreira, J. Gonzales, V. Tomazella, R. Ehlers, F. Louzada, and E. Silva, *Objective Bayesian analysis for the lomax distribution*, ArXiv 1602.08450v1, 1-19 (2016).



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