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Separation Axioms Modulo a Tolerance Relation

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Abstract: In this paper we present separation axioms T_0, T_1, T_2, R_0, module a tolerance relation. That is, the separation axioms are generalized, fixing previously closeness or tolerable level of error. We also investigate their properties and relations between these axioms and some types of classic separation axioms.

Keywords: Separation axioms,Topological space, tolerance relation

Dedicated to the memory of Professor Jorge Medina Sancho

1 Introduction and preliminaries

In a topological space, it is insufficient to have different elements. It is also important to know when they are topologically distinguishable. The separation axioms allow us to define when they have this characteristic.

Definition 1.([1]) *Let X be a nonempty set. A collection $\mathbf{g}$ of subsets of X is called a generalized topology on X if $\emptyset \in \mathbf{g}$ and it is closed under arbitrary unions.*

In this article $\mathbf{g}$ will always denote a generalized topology and the members of $\mathbf{g}$ are called $\mathbf{g}-$open while the complement of a $\mathbf{g}-$open set are called $\mathbf{g}-$closed.

Given $(X,\mathbf{g})$ a generalized topological space, two points are topologically distinguishable if they do not exactly have the same neighborhoods.

Definition 2(See [5]). *A binary relation R (i.e. a subset $R \subseteq X \times X$) which is reflexive and symmetric relation on X, but not necessarily transitive, is called tolerance relation.*

Sossinsky comments in [5] At first glance it seems dubious that a definition as simple and general as that of tolerance can give rise to a meaningful theory. But it does and I, for one, must admit that I haven't stopped being surprised by this strange circumstance.

Tolerance relations are useful to describe the brain [4] and applications to linguistics [6]. Applications also involve almost-fixed points theorems and almost-solutions existence theorems [5], rough set defined by tolerances [7].

A topological space X is Hausdorff if and only its diagonal is closed. In [3] the authors, in the context of a generalized topology on a set X, give a characterizations of some separation axioms between T_0 and T_2 in terms of properties of the diagonal in $X \times X$, where the diagonal of $X \times X$ is the subset

$$\Delta = \{(x,x) \in X \times X : x \in X\}.$$

Now, if we define a tolerance relation ρ on X, since ρ is reflexive, it is true that $\Delta \subseteq \rho$.

At this point is valid to ask, can separation axioms be established in terms of a tolerance relation defined on X?. To give an affirmative answer to this question will be the major objective of this paper.

Let us start by presenting some basic definitions and notations.

First we present the fundamental separation axioms. Numbering from 0 to 2 refers to an increasing degree of separation.

0.-A space X is said to be a T_0 space, or it satisfies the T_0 axiom, if for any two distinct points $x,y \in X$, there exists an open set $U \subseteq X$ that contains only one.

1.-A space X is said to be a T_1 space, or that it satisfies the T_1 axiom, if for any two distinct points $x,y \in X$ there exists two open sets U and V such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$.

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Every T_1 space is also a T_0 space and a T_1 spaces preserve a topological property.

2.-A space X is said to be a T_2 space, or that it satisfies the T_2 axiom if for any two distinct points $x,y \in X$ there exists two open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.
A T_2 spaces is also called a Hausdorff space.

A T_2 space preserve a topological property. Every T_2 space is a T_1 space.

The three previous axioms allow to distinguish two points topologically.

Definition 3(See [2,3,1]). *Let $A \subseteq X$, the closure of A, denoted by $k_{\mathbf{g}}(A)$, is the smallest $\mathbf{g}-$closed set such that $A \subseteq k_{\mathbf{g}}(A)$.*

If $A \subseteq X$, then $a \in k_{\mathbf{g}}(A)$ if and only if $U \cap A \neq \emptyset$ for every open neighborhood U of a.

After this, we present two more axioms that topologically characterize different points.

0.-A space X is said to be a R_0 space, or that it satisfies the R_0 axiom, if for all $x,y \in X$ and $k_{\mathbf{g}}(x) \neq k_{\mathbf{g}}(y)$, then $k_{\mathbf{g}}(x) \cap k_{\mathbf{g}}(y) = \emptyset$.

1.-A space X is said to be a R_1 space, or that it satisfies the R_1 axiom, if for all $x,y \in X$, and $k_{\mathbf{g}}(x) \neq k_{\mathbf{g}}(y)$, then there exist disjoint open sets K and K' such that $k_{\mathbf{g}}(x) \subseteq K$ and $k_{\mathbf{g}}(y) \subseteq K'$.

Every R_1 space is also a R_0 space. Both R_0 and R_1 spaces preserve topological properties.

In the study of separation axioms, set operations similar to the closure operator are frequently used. These operations are naturally extended to the context of a generalized topology $\mathbf{g}$.

Definition 4.[2] *Let $A \subseteq X$, the intersection of all $\mathbf{g}-$open subset of X containing A is called the kernel of A, denoted by $\chi_{\mathbf{g}}(A)$; this means*

$$\chi_{\mathbf{g}}(A) = \bigcap \{H \in \mathbf{g} : A \subseteq H\}.$$

Definition 5(See [2,3,1]). *Let $A \subseteq X$, the following set is defined $sat_{\mathbf{g}}(A) = \bigcup_{x\in A} k_{\mathbf{g}}(\{x\})$.*

It is easy to show that $\chi_{\mathbf{g}}(A) = A$ for all $A \in \mathbf{g}$. Moreover, $x \in \chi_{\mathbf{g}}(y)$ if and only if $y \in k_{\mathbf{g}}(x)$ for any $x,y \in X$.

$\mathbf{g}-$open sets in the product topology over $X \times X$ have the form $A \times B$, where A and B are open sets in X.

Proposition 1(See [3]). *For any $(x,y) \in X \times X$ the following holds:*

1.$k_{\mathbf{g}}(x,y) = k_{\mathbf{g}}(x) \times k_{\mathbf{g}}(y)$, and
2.$\chi_{\mathbf{g}}(x,y) = \chi_{\mathbf{g}}(x) \times \chi_{\mathbf{g}}(y)$.

In [3], the following sets are defined:

Definition 6. *Let $(X,\mathbf{g})$ be a topological space and ρ be a tolerance relation over X.*

1.$(x,y) \in L_{\mathbf{g}}$ if and only if $\forall A \in \mathbf{g}[x \in A \Rightarrow y \in A]$,
2.$(x,y) \in E_{\mathbf{g}}$ if and only if $\forall A \in \mathbf{g}[x \in A \Leftrightarrow y \in A]$,
3.$L^{\rho}_{\mathbf{g}} = L_{\mathbf{g}} \cup \rho$,
4.$E^{\rho}_{\mathbf{g}} = E_{\mathbf{g}} \cup \rho$,
5.$Q_{\rho}(x) = \{y \in X : (x,y) \in \rho\}$.

Notice that $\Delta \subseteq E_{\mathbf{g}} \subseteq L^{\rho}_{\mathbf{g}}$. Moreover, $L^{\rho}_{\mathbf{g}}$ a is transitive relation and $E_{\mathbf{g}}$ is an equivalence relation on X.

In [3], the results formulated are denoted by Lemma 1 and Theorem 1

Lemma 1.

1.$(x,y) \in L_{\mathbf{g}}$ if and only if $y \in \chi_{\mathbf{g}}(x)$ and if and only if $x \in k_{\mathbf{g}}(y)$.
2.$(x,y) \in E_{\mathbf{g}}$ if and only if $k_{\mathbf{g}}(x) = k_{\mathbf{g}}(y)$ and if and only if $\chi_{\mathbf{g}}(x) = \chi_{\mathbf{g}}(y)$.

Theorem 1.

1.$\mathbf{g}$ satisfies the T_1 separation axiom if and only if $\chi_{\mathbf{g}}(\Delta) = \Delta$, if and only if $L_{\mathbf{g}} = \Delta$.
2.$\mathbf{g}$ satisfies the T_0 separation axiom if and only if $E_{\mathbf{g}} = \Delta$.

2 Main Results

Now, some results that are useful in the development of the paper are presented:

Lemma 2. *$(x,y) \notin sat_{\mathbf{g}}(\rho)$ if and only if $\chi_{\mathbf{g}}(x,y) \cap \rho = \emptyset$.*

Proof. Let $(x,y) \notin sat_{\mathbf{g}}(\rho)$ and $(x_0,y_0) \in \chi_{\mathbf{g}}(x,y) \cap \rho$, then $(x,y) \in k_{\mathbf{g}}((x_0,y_0))$ and $(x_0,y_0) \in \rho$. Thus, $(x,y) \in sat_{\mathbf{g}}(\rho)$. This signifies is a contradiction, so $\chi_{\mathbf{g}}(x,y) \cap \rho = \emptyset$.

Conversely, suppose that $\chi_{\mathbf{g}}(x,y) \cap \rho = \emptyset$ and let $(x,y) \in sat_{\mathbf{g}}(\rho)$, then $(x,y) \in k_{\mathbf{g}}((x_0,y_0))$ for some $(x_0,y_0) \in \rho$. Hence, $(x_0,y_0) \in \chi_{\mathbf{g}}(x,y) \cap \rho$. From this contradiction, we obtain the desired result.

Lemma 3. *$(x,y) \notin \chi_{\mathbf{g}}(\rho)$ if and only if $k_{\mathbf{g}}(x,y) \cap \rho = \emptyset$.*

Proof. Let $(x,y) \notin \chi_{\mathbf{g}}(\rho)$ and $(x_0,y_0) \in k_{\mathbf{g}}(x,y) \cap \rho$, then $(x,y) \notin \chi_{\mathbf{g}}((x_0,y_0))$. Therefore, $(x_0,y_0) \notin k_{\mathbf{g}}((x,y))$, and so $k_{\mathbf{g}}(x,y) \cap \rho = \emptyset$.

Conversely, if $(x_0,y_0) \in \rho$, then $(x_0,y_0) \notin k_{\mathbf{g}}(x,y)$, which means $(x,y) \notin \chi_{\mathbf{g}}((x_0,y_0))$. Consequently $(x,y) \notin \chi_{\mathbf{g}}(\rho)$.

Theorem 2. *Let $(X,\mathbf{g})$ be a topological space and ρ be a tolerance relation over X. Then,*

(1)$\chi_{\mathbf{g}}(\rho)$,
(2)$k_{\mathbf{g}}(\rho)$, and
(3)$sat(\rho)$

are tolerance relations.

Proof. (1)Since $\Delta \subseteq \rho \subseteq \chi_{\mathbf{g}}(\rho)$, then $\chi_{\mathbf{g}}(\rho)$ is reflexive. If $(x,y) \in \chi_{\mathbf{g}}(\rho)$, there exists $(a,b) \in \rho$ such that $(x,y) \in \chi_{\mathbf{g}}(a,b)$. Therefore $(y,x) \in \chi_{\mathbf{g}}(b,a)$, and since $(b,a) \in \rho$, then $(y,x) \in \chi_{\mathbf{g}}(\rho)$ and so $\chi_{\mathbf{g}}(\rho)$ is symmetric.

(2)To demonstrate the second part, since $\Delta \subseteq \rho \subseteq k_{\mathbf{g}}(\rho)$ then $k_{\mathbf{g}}(\rho)$ is reflexive.
Suppose that $(x,y) \in k_{\mathbf{g}}(\rho)$ and let C a $\mathbf{g}-$open neighbourhood of (y,x), then there exists $O_y, O_x \in \mathbf{g}$ such that $(y,x) \in O_y \times O_x \subseteq C$. Also $(x,y) \in O_x \times O_y$, then there exists $(c,d) \in O_x, O_y$ and $(c,d) \in \rho$. Because $(d,c) \in \rho$ and $(d,c) \in C$ then $(y,x) \in k_{\mathbf{g}}(\rho)$.

(3)Finally $\Delta \subseteq \rho \subseteq sat(\rho)$, then $sat(\rho)$ is reflexive.
If $(x,y) \in sat(\rho)$, for some $(a,b) \in \rho$ $(x,y) \in k_{\mathbf{g}}(a,b)$. And, since $(b,a) \in \rho$ and $(y,x) \in k_{\mathbf{g}}(b,a)$, then $(y,x) \in sat(\rho)$, and so $sat(\rho)$ is symmetric.

Now we state the fundamental separation axioms in terms of a topology $(X,\mathbf{g})$ modulo a tolerance relation ρ.

$T_2(mod\ \rho)$ space

Definition 7. *X is said to be a $T_2(mod\ \rho)$ space if for every pair of point $x,y \in X$ with $(x,y) \notin \rho$ there exist open neighbourhoods U of x and V of y such that $(U \times V) \cap \rho = \emptyset$.*

Let's look at the two examples below.

Example 1. 1.Let (X,τ) be a topological space such that x_0, y_0 are isolated points in X. Let $\rho = X \times X \setminus \{(x_0,y_0),(y_0,x_0)\}$ be a tolerance relation, then (X,τ) is a $T_2(mod\ \rho)$ space.
2.Let $\mathbb{R}$ be the real line with its usual topology and $\rho = \mathbb{R} \times \mathbb{R} \setminus \{(r_0,q_0),(q_0,r_0)\}$ with $r_0 \neq q_0$ $r_0, q_0 \in \mathbb{R}$. Then, $\mathbb{R}$ is not a $T_2(mod\ \rho)$ space.

Theorem 3. *X is a $T_2(mod\ \rho)$ space if and only if ρ is a closed set of $X \times X$.*

Proof. Suppose that X is a $T_2(mod\ \rho)$ space. If $(x,y) \in k_{\mathbf{g}}(\rho) \setminus \rho$, then there exist open neighbourhoods U of x and V of y such that $(U \times V) \cap \rho = \emptyset$.

On the other hand, since $(x,y) \in k_{\mathbf{g}}(\rho)$ for all open set A in $X \times X$, that contains (x,y), we have $A \cap \rho \neq \emptyset$, in particular $(U \times V) \cap \rho \neq \emptyset$.

From this contradiction we get the needed result.

Conversely, suppose that $k_{\mathbf{g}}(\rho) = \rho$ and let $x,y \in X$ with $(x,y) \notin \rho$. Then there exists an open set A in $X \times X$ such that $(x,y) \in A$ and $A \cap \rho = \emptyset$. It means that exist open neighbourhoods U of x and V of y such that $(U \times V) \cap \rho = \emptyset$.

Theorem 4. *Let $\{\rho_i\}_{i \in I}$ be a the family of tolerance relations. If X is a $T_2(mod\ \rho_i)$ space for each $i \in I$, then X is a $T_2(mod \bigcap_{i \in I} \rho_i)$ space.*

Proof. By Theorem 3, if the set $\{\rho_i : i \in I\}$ is an arbitrary collection of closed subsets, then $\bigcap_{i \in I} \rho_i$ is closed. Hence

$$k_{\mathbf{g}}\left(\bigcap_{i \in I} \rho_i\right) = \bigcap_{i \in I} \rho_i.$$

It is worth asking whether the reciprocal of this theorem is fulfilled. To answer this, it is sufficient to consider the real line $\mathbb{R}$ with its usual topology and P the set of all tolerance relations over $\mathbb{R}$. The intersection of all tolerance relations over $\mathbb{R}$ is $\Delta = \bigcap_{\rho_i \in P} \rho_i$ the identity relation. Accordingly, $\mathbb{R}$ is $T_2(mod\,\Delta)$, i.e. the classic T_2 axiom. However it was a given example of a tolerance relation where $\mathbb{R}$ is not $T_2(mod\,\rho_i)$.

Continuity of functions is one of the core concepts of topology. We establish a relation between the axioms of separation and a continuous function,

Theorem 5. *Let $(X,\mathbf{g}_X)$ and $(Y,\mathbf{g}_Y)$ be topological spaces and let $f,g : X \to Y$ be continuous, being Y a $T_2(mod\ \rho)$ space, then $\{x \in X : (f(x),g(x)) \in \rho\}$ is closed.*

Proof. Let $A = \{x \in X : (f(x),g(x)) \notin \rho\}$ and suppose $a \in A$. Since $(f(a),g(a)) \notin \rho$ there exist open sets $U,V \subseteq Y$, such that $f(a) \in U$, $g(a) \in V$ and $(U \times V) \cap \rho = \emptyset$.

Let $W = f^{-1}(U) \cap f^{-1}(V)$, then W is open in X and $a \in W$. Moreover, $W \subseteq A$. Thus A is open, so $\{x \in X : (f(x),g(x)) \in \rho\}$ is closed.

Corollary 1. *Let $(X,\mathbf{g}_X)$ be a topological space and let $f : X \to Y$ be a continuous map being Y a $T_2(mod\ \rho)$ space, then $\{x \in X : (f(x),x) \in \rho\}$ is closed.*

Note that the reciprocal is untrue. Let f,g be the identity functions in $\mathbb{R}$ and $\rho = \mathbb{R} \times \mathbb{R} \setminus \{(r_0,q_0),(q_0,r_0)\}$, then $\{x : (f(x),g(x)) \in \rho\} = \{x : (x,x) \in \rho\} = \mathbb{R}$ is a closed set but $\mathbb{R}$ is not a $T_2(mod\ \rho)$ space.

$T_1(mod\ \rho)$ space

Definition 8. *X is said to be a $T_1(mod\ \rho)$ space if for every pair of points $x,y \in X$ and $(x,y) \notin \rho$, there exists an open neighbourhood U of x such that $Q_\rho(y) \cap U = \emptyset$.*

Theorem 6. *X is a $T_1(mod\ \rho)$ space if and only if $\forall x \in X$: $Q_\rho(x)$ is closed.*

Proof. Suppose X is a $T_1(mod\ \rho)$ space. Let $x \in X$ and $y \in X \setminus Q_\rho(x)$, that is $(x,y) \notin \rho$. Thus, there exists an open set U containing y such that $U \subseteq X \setminus Q_\rho(x)$, therefore $X \setminus Q_\rho(x)$ is open.

Conversely, if for all $x \in X$, $Q_\rho(x)$ is closed then if $y \in XQ_\rho(x)$, there exist an open set U containing y such that $U \subseteq X \setminus Q_\rho(x)$. Hence, U and $Q_\rho(x)$ are disjoint sets.

Lemma 4. *Let $\{\rho_i\}_{i \in I}$ be an arbitrary family of partial order relations. Then, $Q_{\bigcap_{i \in I} \rho_i}(x) = \bigcap_{i \in I} Q_{\rho_i}(x)$.*

*Proof.*Let $x \in X$.
$w \in Q_{\bigcap_{i\in I}\rho_i}(x) \iff (x,w) \in \bigcap_{i\in I}\rho_i \iff w \in Q_{\rho_i}(x), \forall i \in I \iff w \in \bigcap_{i\in I} Q_{\rho_i}(x)$.

Theorem 7.*Let $\{\rho_i\}_{i\in I}$ be an arbitrary family of partial order relations. If X is a $T_1(mod\rho_i)$ space, for each $i \in I$, then X is a $T_1(mod \bigcap_{i\in I}\rho_i)$ space.*

*Proof.*This follows from Theorem 6 and Lemma 4.

Let $\mathbb{R}$ be the real line with its usual topology and $\rho = \mathbb{R}\times\mathbb{R}\setminus\{(r_0,q_0),(q_0,r_0)\}$ with $r_0 \neq q_0$ $r_0,q_0 \in \mathbb{R}$. Let P be the set of all tolerance relations over $\mathbb{R}$. The intersection of all elements of P is $\Delta = \bigcap_{\rho_i\in P}\rho_i$ the identity relation. So, $\mathbb{R}$ is $T_1(mod\ \Delta)$ because $Q_\Delta(x) = \{x\}$ is a closed set, but $\mathbb{R}$ is not a $T_1(mod\ \rho)$ space, because $Q_\rho(r_0) = \mathbb{R}\setminus\{q_0\}$ is not a closed set. This means that the reciprocal of the preceding theorem is untrue.

Lemma 5.*Define $Q^*_\rho(x) = \bigcap_{z\in Q_\rho(x)} Q_\rho(z)$, if $(x,y) \in \rho$ then $Q^*_\rho(x) \times Q^*_\rho(y) \subseteq \rho$.*

*Proof.*Let $(a,b) \in Q^*_\rho(x) \times Q^*_\rho(y)$. So $(a,w) \in \rho$ $\forall w$ with $(x,w) \in \rho$, in particular $(a,y) \in \rho$. By the same argument $(b,z) \in \rho$ $\forall z$ with (z,y), in particular (a,b) because of the symmetry property of ρ.

Theorem 8.*If X is $T_1(mod\ \rho)$, then $sat(\rho) = \rho$.*

*Proof.*Let $x \in X$, and $Q^*_\rho(x) = \bigcap_{z\in Q_\rho(x)}(Q_\rho(z))$. By Theorem 6 $Q^*_\rho(x)$ is a closed set containing x because it is the intersection of closed sets, so $k_{\mathbf{g}}(x) \subseteq Q^*_\rho(x)$. If $(x,y) \in \rho$ then $k_{\mathbf{g}}(x,y) \in Q^*_\rho(x) \times Q^*_\rho(y)$. Therefore, by Lemma 5, $k_{\mathbf{g}}(x,y) \subseteq \rho$.

Note that the real line $\mathbb{R}$ with its usual topology and $\rho = \mathbb{R}\times\mathbb{R}\setminus\{(r_0,q_0),(q_0,r_0)\}$ with $r_0 \neq q_0$ $r_0,q_0 \in \mathbb{R}$. Then $sat(\rho) = \rho$ $\mathbb{R}$ is not a $T_1(mod\ \rho)$ space because $Q_\rho(r_0) = \mathbb{R}\setminus\{q_0\}$ is not a closed set.

Theorem 9.*$sat(\rho) = \rho$ if and only if for all $x \in X$, $k_{\mathbf{g}}(x) \subseteq Q^*_\rho(x)$.*

*Proof.*Let $x \in X$, and $a \in k_{\mathbf{g}}(x)$. We have $k_{\mathbf{g}}(a) \in k_{\mathbf{g}}(x)$ and $k_{\mathbf{g}}(x)\times k_{\mathbf{g}}(x) \subseteq \rho = sat(\rho)$. Then for all z such that $(x,z) \in \rho$ implies $k_{\mathbf{g}}(a)\times k_{\mathbf{g}}(z) \subseteq k_{\mathbf{g}}(x)\times k_{\mathbf{g}}(z) \subseteq \rho$, so $(a,z) \in \rho$. Finally $a \in Q^*_\rho(x)$.

To demonstrate the reciprocal, suppose $k_{\mathbf{g}}(x) \subseteq Q^*_\rho(x)$ for every $x \in X$. Let $(a,b) \in sat(\rho)$, then there exists $(x,y) \in \rho$ such that $(a,b) \in k_g(x,y)$. But $k_{\mathbf{g}}(x,y) = k_{\mathbf{g}}(x)\times k_{\mathbf{g}}(y) \subseteq Q^*_\rho(x)\times Q^*_\rho(y)$ and by Lemma 5 $(a,b) \in k_{\mathbf{g}}(x,y) \subseteq \rho$. Finally $sat(\rho) \subseteq \rho$.

$T_0(mod\ \rho)$ spaces

Definition 9.*X is said to be a $T_0(mod\ \rho)$ space if for every pair of points $x,y \in X$, $(x,y) \notin \rho$ there exists an open neighbourhood U such that $x \in U$ and $Q_\rho(y)\cap U = \emptyset$ or $y \in U$ and $Q_\rho(x)\cap U = \emptyset$.*

Note that all the above $T_2(mod\ \rho)$ spaces are also $T_1(mod\ \rho)$ spaces.

Furthermore, consider $(\mathbb{R},\mathbf{g})$ with $\mathbf{g}$ the generalized topology generated by the elements of its basis $B = \{\mathbb{R} - [k,k+1] : k \in \mathbb{Z}\}$. If $\rho = \bigcup_{k\in\mathbb{Z}}[k,k+1]\times[k,k+1]$, then this space is a $T_1(mod\ \rho)$ space, but it is not a T_1 space.

Theorem 10.*X is a $T_0(mod\ \rho)$ space, then $E^\rho_{\mathbf{g}} = \rho$.*

*Proof.*Let X be a $T_0(mod\ \rho)$ space and $(x,y) \notin \rho$, then without loss of generality we can assume that there exists a open neighbourhood U such that $x \in U$ and $y \notin U$. This implies that, $\chi_{\mathbf{g}}(x) \neq \chi_{\mathbf{g}}(y)$, so it follows that $(x,y) \notin E_{\mathbf{g}}$ suggesting that we have proven that $E^\rho_{\mathbf{g}} \subseteq \rho$.

$R_0(mod\ \rho)$ spaces

Definition 10.*X is said to be a $R_0(mod\ \rho)$ space if for every pair of points $x,y \in X$ with $(x,y) \notin \rho$ and $k_{\mathbf{g}}(x) \neq k_{\mathbf{g}}(y)$, then $(k_{\mathbf{g}}(x)\times k_{\mathbf{g}}(y))\cap\rho = \emptyset$.*

Theorem 11.*X is a $R_0(mod\ \rho)$ space if and only if $\chi_{\mathbf{g}}(\rho) \subseteq E^\rho_{\mathbf{g}}$.*

*Proof.*Suppose that X is a $R_0(mod\ \rho)$ space and $(x,y) \in \chi_{\mathbf{g}}(\rho)$ with $(x,y) \notin \rho$.

Two possibilities are considered: First, $k_{\mathbf{g}}(x) = k_{\mathbf{g}}(y)$. When applying Lemma 1, we obtain $(x,y) \in E^\rho$, so $(x,y) \in E^\rho_{\mathbf{g}}$.

Secondly let suppose that it is not possible that $k_{\mathbf{g}}(x) \neq k_{\mathbf{g}}(y)$, indeed if $(c,d) \in \rho$ then, given that X is a $R_0(mod\ \rho)$ space, $(c,d) \notin k_{\mathbf{g}}(x)\times k_{\mathbf{g}}(y)$. So $(c,d) \notin k_{\mathbf{g}}((x,y))$, then $(x,y) \notin \chi_{\mathbf{g}}((x,y))\cap\chi_{\mathbf{g}}(\rho)$, which contradicts the choice of (x,y).

Reciprocally, let $\chi_{\mathbf{g}}(\rho) \subseteq E^\rho_{\mathbf{g}}$ and $x,y \in X$ with $k_{\mathbf{g}}(x) \neq k_{\mathbf{g}}(y)$ and $(x,y) \notin \rho$. Under these conditions, if $(a,b) \in (k_{\mathbf{g}}(x)\times k_{\mathbf{g}}(y))\cap\rho$, then $(a,b) \in k_{\mathbf{g}}(x,y)$, Consequently, $(x,y) \in \chi_{\mathbf{g}}(a,b) \subseteq \chi_{\mathbf{g}}(\rho) \subseteq E^\rho_{\mathbf{g}}$, which means $(x,y) \in E_{\mathbf{g}}$ and it contradicts the choice of (x,y).

$\rho = \Delta$, $(X,\mathbf{g})$ is $T_1(mod\ \rho)$ if only if $(X,\mathbf{g})$ is $T_0(mod\ \rho)$ and is $R_0(mod\ \rho)$. Is it true for any tolerance relation $\rho \neq \Delta$?

$R_1(mod\ \rho)$ spaces

Definition 11.*X is said to be a $R_1(mod\ \rho)$ space if for every pair of points $x,y \in X$ with $(x,y) \notin \rho$ and $k_{\mathbf{g}}(x) \neq k_{\mathbf{g}}(y)$, there exist sets $A,B \in \mathbf{g}$ with $k_{\mathbf{g}}(x) \subseteq A$ and $k_{\mathbf{g}}(y) \subseteq B$ such that $(A\times B)\cap\rho = \emptyset$.*

Let $\mathbb{R}$ be the real line with its usual topology and (a_1,b_1) and (a_2,b_2) disjoint subsets of the real line. Define $\rho = \mathbb{R}\times\mathbb{R}\setminus((a_1,b_2)\times(a_2,b_2)\cup(a_2,b_2)\times(a_1,b_1))$, then $\mathbb{R}$ is a $R_1(mod\ \rho)$ space.

Theorem 12. *X is a $R_1(mod\ \rho)$ space if and only if $k_{\mathbf{g}}(\rho) \subseteq E^{\rho}_{\mathbf{g}}$.*

Proof. Suppose X is a $R_1(mod\ \rho)$ space, and $(x,y) \in k_{\mathbf{g}}(\rho)$. We consider three cases: First, if $(x,y) \in \rho$, obviously $(x,y) \in E^{\rho}_{\mathbf{g}}$. Second, we consider that $(x,y) \notin \rho$ and $k_{\mathbf{g}}(x) = k_{\mathbf{g}}(y)$, and in such a case, Lemma 1 indicates that $(x,y) \in E_{\mathbf{g}}$; therefore, $(x,y) \in E^{\rho}_{\mathbf{g}}$. Third we consider that $(x,y) \notin \rho$ and $k_{\mathbf{g}}(x) \neq k_{\mathbf{g}}(y)$. Since X is a $R_1(mod\ \rho)$ space, there exist sets $A,B \in \mathbf{g}$ with $k_{\mathbf{g}}(x) \subseteq A$ and $k_{\mathbf{g}}(y) \subseteq B$ such that $(A\times B)\cap\rho = \emptyset$, which implies that $(x,y) \notin k_{\mathbf{g}}(\rho)$, from this contradiction with the choice of (x,y) follows the desired result.

Theorem 13. *Let $\{\rho_i\}_{i\in I}$ be an arbitrary family of order relations. If X is a $R_1(mod\ \rho_i)$ space for each $i \in I$, then X is a $R_1(mod\ \bigcap_{i\in I}\rho_i)$ space.*

Proof. For each ρ_i, we have $k_{\mathbf{g}}(\rho_i) \subseteq E^{\rho_i}_{\mathbf{g}}$. Then

$$k_{\mathbf{g}}(\bigcap_{i\in I}\rho_i) \subseteq \bigcap_{i\in I} k_{\mathbf{g}}(\rho_i) \subseteq \bigcap_{i\in I} E_{\mathbf{g}}\cup\rho_i = E_{\mathbf{g}}\cup\bigcap_{i\in I}\rho_i$$

Accordingly, X is a $R_1(mod\ \bigcap_{i\in I}\rho_i)$ space.

Consider the real line $\mathbb{R}$ with the usual topology and the tolerance relation $\rho = \mathbb{R}\times\mathbb{R}\setminus([a_1,b_2]\times[a_2,b_2]\cup[a_2,b_2]\times[a_1,b_1])$. If P is the set of all tolerance relations, then $\Delta = \bigcap_{\rho_i\in P}\rho_i$, so $\mathbb{R}$ is a $R_1(mod\ \Delta)$, but it is not a $R_1(mod\ \rho)$ space.

3 Conclusion

A tolerance on a set is a mathematical structure formalizing resemblance or the idea of being the same up to a small error. In this paper, we presented separation axioms T_0, T_1, T_2, R_0, module a tolerance relation. That is, the separation axioms are generalized, specifying in advance closeness or tolerable level of error. We also explored their properties and, relations between these axioms and some types of classic separation axioms.

4 Applications

While working on a problem involving topological spaces, it is often necessary to add an extra condition called separation axiom. Separation axioms give us the opportunity to topologically identify different sets, concepts used in both topology and functional analysis, such as when we want to obtain the uniqueness of the limit. However, tolerance relationships provide an opportunity for research to define the level of difference or tolerable level of error. Thus, since tolerance levels naturally appear in different branches of mathematics, the authors consider that the conjunction between separation axioms and tolerance relationships will allow the use of other powerful mathematical tools to solve problems, both in topology as well as functional analysis, and mathematical applications.

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