Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/140405

Adaptation Of Residual Power Series Approach For Solving Time-Fractional Nonlinear Kline-Gordon Equations With Conformable Derivative

Rasha Amryeen[1], *Fatimah Noor Harun*[1], *Mohammed Al-Smadi*[2,*] *and Azwani Alias*[1]

[1]Pusat Pengajian Informatik & Matematik Gunaan, Universiti Malaysia Terengganu, 21030 Kuala Nerus, Terengganu, Malaysia
[2]Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun 26816, Jordan

Received: 12 Feb. 2019, Revised: 2 Apr. 2020, Accepted: 18 Apr. 2020
Published online: 1 Jul. 2020

Abstract: In this paper, the time-fractional nonlinear Kline-Gordon equations are considered and solved using the adaptive of residual power series method. The fractional derivative is considered in a conformable sense. Analytical solutions are obtained based on conformable Taylor series expansion by substituting the truncated conformable series solutions to residual error functions. This adaptation can be implemented as a novel alternative technique to handle many nonlinear issues occurring in physics and engineering. Effectiveness, validity, and feasibility of the proposed method are demonstrated by testing some numerical applications. Tabular and graphic results indicate that the method is superior, accurate and appropriate for solving these fractional partial differential models with compatible derivatives.

Keywords: Fractional partial differential equations, Klein-Gordon equations, Conformable derivative, Residual errors, Approximate solutions

1 Introduction

The theory of fractional calculus has gained much attention in various fields of science and engineering due to vast array of applications and the critical role it plays to describe the complex dynamic behavior of real-world problems such as fluid flow, traffic, biological populations, and diffusion system [[1]-[6]]. The fractional state has many advantages over the classical order, which helps simplify control over the modeling of nature without any lack of genetic characteristics and memory effort. The fractional operator is a powerful mathematical tool that plays an important role in simulating many nonlinear problems, including electrical circuits, electromagnetic waves, damping laws, signal processing, and rheology [[7]-[11]].

The Klein-Gordon equation (KGE) is considered one of the most popular nonlinear partial differential equations that gained much attention in describing relativistic electrons, solitons, quantum, fluid dynamics, and mechanics [[12]-[15]]. It also plays an important role in many other applications, including optics, plasma ions, and solid-state problems [[16]-[18]]. On the other hand, several effective numeric-analytic methods have recently been used to obtain approximate solutions to nonlinear fractional Klein-Gordon equations. For instance, the homotopy perturbation method has been applied for solving a class of nonlinear FKGEs [15]. In [16], the homotopy analysis method has been implemented to approximate solutions of the nonlinear FKGEs. The Riccati expansion method has been employed for solving nonlinear space-time FKGEs [17]. In [18], the modified reduced differential transform method has been introduced for providing numeric solutions for nonlinear space-time FKGEs. However, other categories of advanced numerical methods for different topics can be found in [[19]-[28]].

In this work, we extend the scope of application of the residual power series method in the sense of conformable derivative to construct multiple time-fractional power series solutions to time-fractional nonlinear Klein–Gordon equations with conformable derivative in

* Corresponding author e-mail: mhm.smadi@bau.edu.jo

the following form:

$$T_t^{\alpha}\omega(x,t)=\omega_{xx}(x,t)+\lambda\omega(x,t)+\mu\omega^2(x,t)+\gamma\omega^3(x,t), \tag{1}$$

subject to the following initial condition

$$\omega(x,0)=\omega_0(x), \tag{2}$$

where $0<\alpha\leq 1$, λ,μ and $\gamma\in\mathbb{R}$, $\omega_0(x)$ is given analytical function of x, $\omega(x,t)$ is an unknown analytical function to be defined, and T_t^{α} denotes the time-conformable derivative of order α. Here, we assume that FKGEs (1) and (2) have smooth unique solution in the interval of interest.

The fractional power series method (FPSM) is an effective analytical technique for identifying and defining FPS solutions for many types of ordinary differential equations, partial differential equations, integrodifferential equations, fuzzy differential equations, and integral equations that include different categories of fractional operators [[29]-[33]]. This method is characterized as a systematic and easy-to-use alternative technique for creating FPS solutions for both linear and nonlinear problems without being linearized, discretized or exposed to perturbation. Unlike the traditional technique of the power series, FPSM does not require a comparison of corresponding coefficients or finding a recursion relationship, whereas the series coefficients are calculated by determining the residual error functions associated with the compatible fractional derivatives and then producing a system of algebraic equations for one or more variables.

The remainder of the present work is organized as follows: In Section 2, the definitions and characteristics of the conformable derivative and the fractional power series are presented. The proposed approach is described in Section 3 to provide a representation of the FPS solution for both linear and nonlinear FKGEs. In Section 4, some numerical examples are implemented to show the versatility, capabilities, and applicability of the FPSM. Section 5 is devoted to conclusion.

2 Preliminaries

The essentials resulting from conformable fractional calculus theory are presented briefly, and the most important definitions and theories of the fractional power series method are also presented in conformable sense.

Definition 2.1: [34] Let f be n-differentiable at $t>s$, the conformable fractional derivative starting from s of a function $f:[s,\infty)\to\mathbb{R}$ of order $\alpha\in(n-1,n]$, $t>s$, is defined by

$$\frac{d^{\alpha}f}{dt^{\alpha}}=\lim_{\varepsilon\to 0}\frac{f^{(\lceil\alpha\rceil-1)}\left(t+\varepsilon(t-s)^{\lceil\alpha\rceil-\alpha}\right)-f^{(\lceil\alpha\rceil-1)}(t)}{\varepsilon} \tag{3}$$

and $T^{\alpha}f(s)=\lim_{t\to s^+}T^{\alpha}f(t)$ provided $f(t)$ is α-differintiable in some $(0,s)$, $s>0$, and $\lim_{t\to s^+}T^{\alpha}f(t)$ exists, where $\lceil\alpha\rceil$ is the smallest integer greater than or equal α.

It is worth noting here that f is called α-differentiable at a point t whenever f has a conformable fractional derivative of order α at a point t. Some features of the α-differentiable are provided in [[35]-[38]]. In the next theorem, we mention some of these features.

Theorem 2.1: [35] Let $\alpha\in(0,\ 1]$ and assume f, g be α-differentiable at a point $t>s$. Then

1. $\frac{d^{\alpha}}{dt^{\alpha}}(kf+hg)=k\,f^{(\alpha)}+h\,g^{(\alpha)},\forall k,\ h\in\mathbb{R}$.
2. $\frac{d^{\alpha}}{dt^{\alpha}}(\lambda)=0$, λ is a constant.
3. $\frac{d^{\alpha}}{dt^{\alpha}}(\lambda f)=\lambda\frac{d^{\alpha}}{dt^{\alpha}}(f)$.
4. $\frac{d^{\alpha}}{dt^{\alpha}}((t-a)^p)=p(t-a)^{p-\alpha},\forall p\in\mathbb{R}$.
5. If f is differentiable, then

$$\frac{d^{\alpha}}{dt^{\alpha}}f(t)=(t-a)^{1-\alpha}\frac{d}{dt}f(t).$$

Corollary 2.1: For $\alpha\in(n-1,n]$, if $f:[0,\infty)\to\mathbb{R}$ is α-differentiable at $t>s$, then f is continuous at s.

Definition 2.2: [35] The conformable fractional integral starting from s of order $\alpha\in(n-1,n]$ of $f(t)$ is defined as

$$I_s^{\alpha}f(t)=\frac{1}{(n-1)!}\int_s^t\frac{(t-\tau)^{n-1}f(\tau)}{(\tau-s)^{n-\alpha}}d\tau,\quad t>\tau\geq s\geq 0. \tag{4}$$

Theorem 2.2: Let $\alpha\in(n-1,n]$ and assume f be n-times differentiable function. Then

1. $\frac{d^{\alpha}}{dt^{\alpha}}(I_s^{\alpha}f(t))=f(t)$,
2. $I_s^{\alpha}\left(\frac{d^{\alpha}}{dt^{\alpha}}f(t)\right)=f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(s)(t-s)^k}{k!}$.

Definition 2.4: [36] Let $\partial^k u/\partial t^k$ and $\partial^k u/\partial x^k$, $k=1,2,\ldots,n-1$, be defined on $I\times[s,\infty)$, then the conformable time-fractional differential operator of order α of a function $u(x,t):I\times[s,\infty)\to\mathbb{R}$ is defined by

$$\begin{aligned}T_t^{\alpha}u(x,t)&=\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}\\&=\lim_{\varepsilon\to 0}\frac{u_t^{(n-1)}\left(x,t+\varepsilon(t-s)^{n-\alpha}\right)-u_t^{(n-1)}(x,t)}{\varepsilon},\\&\alpha\in(n-1,n],\ t>s\geq 0,\end{aligned} \tag{5}$$

and the space-fractional differential operator of order β of $u(x,t):I\times[s,\infty)\to\mathbb{R}$ is defined by

$$\begin{aligned}T_x^{\beta}(x,t)&=\frac{\partial^{\beta}u(x,t)}{\partial x^{\beta}}\\&=\lim_{\varepsilon\to 0}\frac{u_x^{(n-1)}\left(x+\varepsilon(x-s)^{n-\beta},t\right)-u_x^{(n-1)}(x,t)}{\varepsilon},\\&\beta\in(n-1,n],\ x>s\geq 0.\end{aligned} \tag{6}$$

Definition 2.5: [36] The conformable fractional integral starting from s of order $\alpha \in (n-1,n]$ of a function $u(x,t) : I \times [s,\infty) \to \mathbb{R}$ is defined by

$$I_s^{\alpha} u(x,t) = \begin{cases} \frac{1}{(n-1)!} \int_s^t \frac{(t-\tau)^{n-1} u(x,\tau)}{(\tau-s)^{n-\alpha}} d\tau, & x \in I,\ t > \tau \geq s \geq 0, \\ u(x,t), & \alpha = 0. \end{cases} \tag{7}$$

Definition 2.6: [38] For $0 \leq n-1 < \alpha \leq n$, a power series (PS) of the form

$$\sum_{k=0}^{\infty} f_k(x)(t-t_0)^{k\alpha} = f_0(x) + f_1(x)(t-t_0)^{\alpha} + f_2(x)(t-t_0)^{2\alpha} + \dots, \quad x \in I,\ t_0 \leq t < t_0 + R^{1/\alpha},\ R > 0, \tag{8}$$

is called a multiple fractional PS at $t = t_0$, where t is a variable and the function $f_k(x)$ is called the coefficients of the PS.

As the classical power series, it is clear that all terms of the multiple fractional PS (8) vanish as soon as $t = t_0$ except the first term, which means the multiple fractional PS is convergent when $t = t_0$. Furthermore, for $t \geq t_0$ this multiple fractional series is definitely convergent for $|t-t_0| < R^{1/\alpha}$, $(R > 0)$, where $R^{1/\alpha}$ is the radius of convergence of the series.

Theorem 2.3: Let $0 \leq n-1 < \alpha \leq n$ and assume $u(x,t) : I \times [t_0, t_0 + R^{1/\alpha}) \to \mathbb{R}$ can be expressed as the following multiple fractional PS about $t = t_0$:

$$u(x,t) = \sum_{k=0}^{\infty} f_k(x)(t-t_0)^{k\alpha}, \quad x \in I,\ t_0 \leq t < t_0 + R^{1/\alpha},\ R > 0. \tag{9}$$

Let $u(x,t)$ be continuous on $I \times [t_0, t_0 + R^{1/\alpha})$ and $\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t) = T_t^{k\alpha} u(x,t) \in C(t_0, t_0+R^{1/\alpha})$, for $k = 1,2,\dots$, then the coefficients $f_k(x)$ are given by $f_k(x) = \frac{T_t^{k\alpha} u(x,t_0)}{\alpha^k (k)!}$, where $T_t^{k\alpha}$ stands for sequential conformable time-fractional derivative of order k that is defined by $T^{k\alpha} u(x,t) = \underbrace{T^{\alpha} \cdot T^{\alpha} \cdots T^{\alpha} u(x,t)}_{k-times}$.

***Proof*:** Let $u(x,t)$ be a function of two variables that can be expressed as the multiple fractional PS of Eq. (9), i.e. $u(x,t) = f_0(x) + f_1(x)(t-t_0)^{\alpha} + f_2(x)(t-t_0)^{2\alpha} + \dots$, $t_0 \leq t < t_0 + R^{1/\alpha}, R > 0$. Then, if we put $t = t_0$, one can obtain $f_0(x) = u(x,t_0)$.

Applying the operator T_t^{α} once to $u(x,t)$ leads to

$$T_t^{\alpha} u(x,t) = \alpha f_1(x) + 2\alpha f_2(x)(t-t_0)^{\alpha} + \dots,$$

and evaluating the result at $t = t_0$ leads to $T_t^{k\alpha} u(x,t_0) = \alpha f_1(x)$. Hence $f_1(x) = \frac{T_t^{\alpha} u(x,t_0)}{\alpha}$.

Again, applying the operator T_t^{α} twice to $u(x,t)$, one can obtain

$$T_t^{2\alpha} u(x,t) = (2\alpha)\alpha f_2(x) + (2\alpha)(3\alpha) f_3(x)(t-t_0)^{\alpha} + \dots,$$

while the substitution of $t = t_0$, it follows $f_2(x) = \frac{T_t^{2\alpha} u(x,t)}{2\alpha^2}$. If we follow this approach; apply the operator T_t^{α} k-times to $u(x,t)$ and evaluate $t = t_0$ in the resulting formula, we can easily see that $f_k(x) = \frac{T_t^{k\alpha} u(x,t_0)}{\alpha^k (k)!}$. The proof is completed.

The nth-partial sum of the multiple fractional PS of Eq. (9) can be given as

$$u_n(x,t) = \sum_{k=0}^{n} f_k(x)(t-t_0)^{k\alpha}, \quad x \in I,\ t_0 \leq t < t_0 + R^{1/\alpha}, R > 0. \tag{10}$$

Theorem 2.4: Let $\alpha \in (n-1,n]$, $T_t^{k\alpha} u(x,t)$ exist at a neighborhood of a point t_0 for $k = 0,1,2,\dots,n+1$, and $u(x,t)$ can be expressed by the multiple fractional PS (9) about $t = t_0$ such that $\left| T_t^{(n+1)\alpha} u(x,t) \right| \leq M(x)$, for some $n \in \mathbb{N}$. Then, for all $(\tau_0, \tau_0 + R^{1/\alpha})$, the reminder $\mathscr{R}_n(t)$ of the multiple fractional PS satisfies

$$|\mathscr{R}_n(x,t)| \leq \frac{M(x)}{\alpha^{n+1}(n+1)!}(t-t_0)^{(n+1)\alpha}, \tag{11}$$

where $\mathscr{R}_n(x,t) = \sum_{k=n+1}^{\infty} \frac{T_t^{k\alpha} u(x,t_0)}{\alpha^k (k)!}(t-t_0)^{k\alpha}$ $= u(x,t) - \sum_{k=0}^{n} \frac{T_t^{k\alpha} u(x,t_0)}{\alpha^k (k)!}(t-t_0)^{k\alpha}$.

Proof: From the assumption $\left| T_t^{(n+1)\alpha} u(x,t) \right| \leq M(x)$, it follows

$$-M(x) \leq T_t^{(n+1)\alpha} u(x,t) \leq M(x). \tag{12}$$

Thus, applying the operator $I_t^{(n+1)\alpha}$ to both sides of the inequality (11), we can get

$$\frac{-M(x)}{\alpha^{n+1}(n+1)!}(t-t_0)^{(n+1)\alpha} \leq I_t^{(n+1)\alpha} T_t^{(n+1)\alpha} u(x,t) \leq \frac{M(x)}{\alpha^{n+1}(n+1)!}(t-t_0)^{(n+1)\alpha},$$

so we complete the proof.

3 The conformable FPS method

This section aims to construct the fractional power series solutions for time-fractional KGEs (1) and (2) in terms of conformable fractional derivatives by substituting the Taylor series expansion for the truncated residual error functions.

Table 1: Numerical results of the 3^{rd} FPS for Example 4.1 at $\alpha = 1$, $x = 1$ and $t \in [0,1]$.

t	Exact solution	Approximate solution	Absolute Error	Relative Error
0.00	1.841470984807897	1.841470984807897	0.0	0.0
0.01	1.851521151892065	1.851521151474563	4.175014×10^{-10}	2.254910×10^{-10}
0.02	1.861672324834652	1.861672318141230	6.693422×10^{-9}	3.595382×10^{-9}
0.03	1.871925518761413	1.871925484807897	3.395352×10^{-8}	1.813828×10^{-8}
0.04	1.882281759000285	1.882281651474563	1.075257×10^{-7}	5.712520×10^{-8}
0.05	1.892742081183921	1.892741818141230	2.630427×10^{-7}	1.389744×10^{-7}
0.06	1.903307531353256	1.903306984807897	5.465454×10^{-7}	2.871556×10^{-7}
0.07	1.913979166062113	1.913978151474563	1.014588×10^{-6}	5.300933×10^{-7}
0.08	1.924758052482855	1.924756318141230	1.734342×10^{-6}	9.010699×10^{-7}
0.09	1.935645268513107	1.935642484807897	2.783705×10^{-6}	1.438128×10^{-6}
0.10	1.946641902883544	1.946637651474563	4251409×10^{-6}	2.183971×10^{-6}

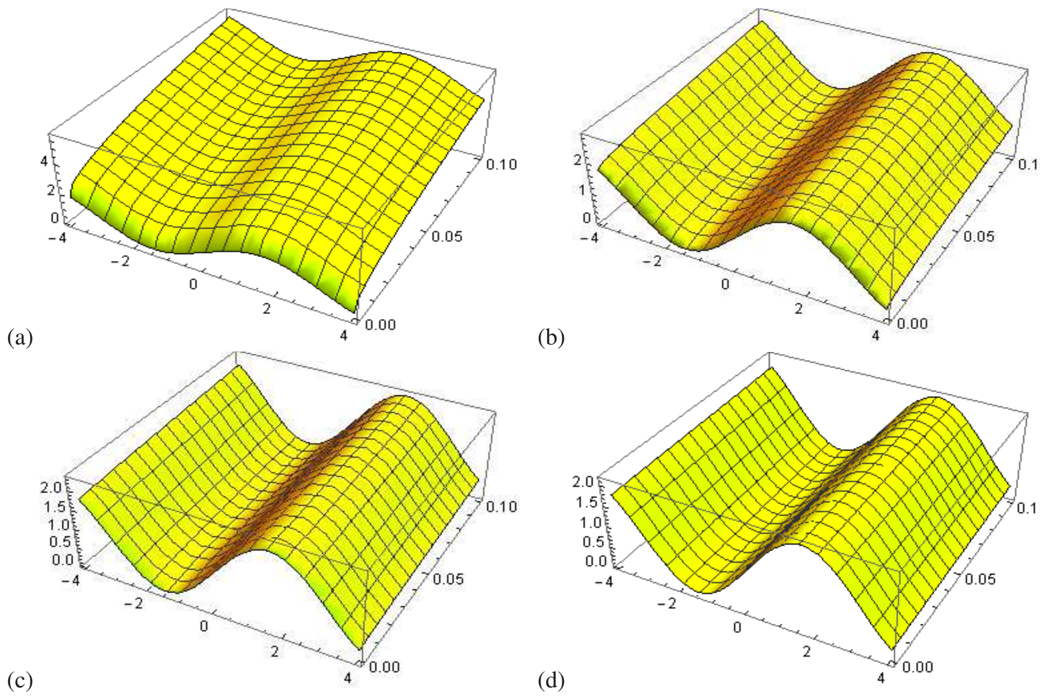

Fig. 1: Surface plot of the 3^{rd} CFPS approximation $\omega_3(x,t)$ for Example 4.1 with $x \in [-4,4]$ and $t \in [0,0.1]$ for different values of fractional order: (a) $\alpha = 0.9$ (b) $\alpha = 0.7$ (c) $\alpha = 0.5$ (d) $\alpha = 0.3$.

Following the procedure of the FPS method, the solution for time-fractional KGEs (1) and (2) about $t_0 = 0$, has a multiple FPS as follows:

$$\omega(x,t) = \sum_{n=0}^{\infty} \omega_n(x) \frac{t^{n\alpha}}{\alpha^n n!}, 0 < \alpha \leq 1, t \geq 0\,,\ x \in \mathbb{R}. \quad (13)$$

By applying the initial condition $\omega(x,0) = \omega_0(x)$ to Eq.(13), the expansion form of the solution can be written as

$$\omega(x,t) = \omega_0(x) + \sum_{n=1}^{\infty} \omega_n(x) \frac{t^{n\alpha}}{\alpha^n n!}. \quad (14)$$

To find out the multiple CFPS approximate solutions, let us assume that $\omega_n(x,t)$ indicates the nth-truncated series of $\omega(x,t)$, i.e.

$$\omega_n(x,t) = \omega_0(x) + \sum_{n=1}^{k} \omega_n(x) \frac{t^{n\alpha}}{\alpha^n n!}. \quad (15)$$

Define the nth-residual function as follows

$$\begin{aligned} Res_{\omega}^{n}(x,t) &= T_t^{\alpha} \omega_n(x,t) - (\omega_n(x,t))_{xx} \\ &\quad - \lambda \omega_n(x,t) - \mu \omega_n^2(x,t) - \gamma \omega_n^3(x,t). \end{aligned} \quad (16)$$

and the residual error function as follows

$$\begin{aligned} Res_{\omega}(x,t) &= \lim_{n\to\infty} Res_{\omega}^{n}(x,t) \\ &= T_t^{\alpha} \omega(x,t) - \omega_{xx}(x,t) - \lambda \omega(x,t) \\ &\quad - \mu \omega^2(x,t) - \gamma \omega^3(x,t). \end{aligned} \quad (17)$$

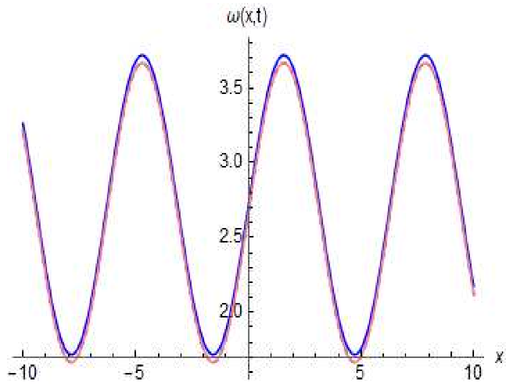

Fig. 2: The behavior of exact and approximate solutions when $\alpha = 1$ for Example 4.1 at $t = 0.01$ and $x \in [-10, 10]$: The exact solution is blue, and the approximate solution is red.

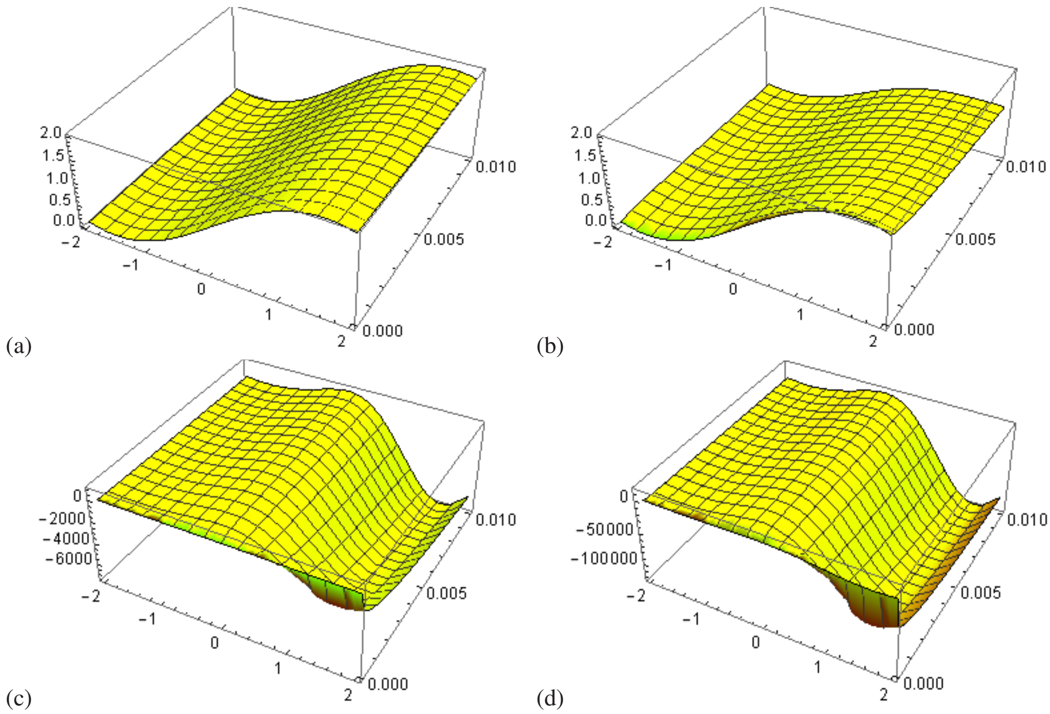

Fig. 3: Surface plot of the 3^{rd} CRPS approximation of Example 4.2 with $x \in [-2, 2]$ and $t \in [0, 0.1]$ for different values of fractional order: (a) $\alpha = 0.95$ (b) $\alpha = 0.5$ (c) $\alpha = 0.1$ (d) $\alpha = 0.01$..

From (17), it can be noted that $Res_{\omega}(x,t) = 0$ for each $x \in \mathbb{R}$ and $0 < t < R^{1/\alpha}$, where $R^{1/\alpha}$ is the convergence radius for the multiple CFPS (13). Similar to that in [[38]- [43]], it can be proved that $T_t^{j\alpha} Res_{\omega}(x,t) = 0$. Also, $T_t^{(j-1)\alpha} Res_{\omega}(x,t)_{|t=0} = T_t^{(j-1)\alpha} Res_{\omega}^{j}(x,t)_{|t=0}$ for each $j = 1, 2, \ldots, n$ because the fractional derivative of a constant in the conformable sense is zero.

Consequently, the following fractional equation that can be solved manually

$$T_t^{(j-1)\alpha} Res_{\omega}^{j}(x,t)_{|t=0} = 0,\ j = 1, 2, 3, \ldots, n, \tag{18}$$

assists us to obtain the desired values of the unknown coefficients $\omega_n(x)$ of Eq. (15) for $n = 1, 2, 3, \ldots$. Therefore, the approximate solutions $\omega_n(x,t)$ can be given respectively.

To define the 1^{st} unknown coefficient, $\omega_1(x)$, let $n=1$ in Eq. (15), and substitute the 1^{st} approximation $\omega_1(x,t) = \omega_0(x) + \omega_1(x)\frac{t^\alpha}{\alpha}$ into the 1^{st} truncated residual function $Res^1_\omega(x,t)$ of Eq. (16) as follows

$$\begin{aligned} Res^1_\omega(x,t) &= T_t^\alpha \omega_1(x,t) - (\omega_1(x,t))_{xx} - \lambda\omega_1(x,t) \\ &- \mu\omega_1^2(x,t) - \gamma\omega_1^3(x,t) \\ &= \omega_1(x) - \left(\omega_0''(x) + \omega_1''(x)\frac{t^\alpha}{\alpha}\right) \\ &- \lambda\left(\omega_0(x) + \omega_1(x)\frac{t^\alpha}{\alpha}\right) \\ &- \mu\left(\omega_0^2(x) + 2\omega_0(x)\omega_1(x)\frac{t^\alpha}{\alpha} + \omega_1^2(x)\frac{t^{2\alpha}}{\alpha^2}\right) \\ &- \gamma\left(\omega_0(x) + \omega_1(x)\frac{t^\alpha}{\alpha}\right)^3. \end{aligned} \tag{19}$$

Using the result $Res^1_\omega(x,0) = 0$, it gives

$$\omega_1(x) = \omega_0''(x) + \lambda\omega_0(x) + \mu\omega_0^2(x) + \gamma\omega_0^3(x). \tag{20}$$

Thus, the 1^{st} FPS approximate solution of Eqs. (1) and (2) can be expressed as follows

$$\begin{aligned} \omega_1(x,t) = \omega_0(x) + (\omega_0''(x) + \lambda\omega_0(x) \\ + \mu\omega_0^2(x) + \gamma\omega_0^3(x))\frac{t^\alpha}{\alpha}. \end{aligned} \tag{21}$$

Similarly, to determine the 2^{nd} coefficient, $\omega_2(x)$, substitute the 2^{nd} truncated series solution $\omega_2(x,t) = \omega_0(x) + \omega_1(x)\frac{t^\alpha}{\alpha} + \omega_2(x)\frac{t^{2\alpha}}{2\alpha^2}$ into the 2^{nd} truncated residual function $Res^2_\omega(x,t)$ of Eq. (16) as follows

$$\begin{aligned} Res^2_\omega(x,t) &= T_t^\alpha \omega_2(x,t) - (\omega_2(x,t))_{xx} - \lambda\omega_2(x,t) \\ &- \mu\omega_2^2(x,t) - \gamma\omega_2^3(x,t) \\ &= \left(\omega_1(x) + \omega_2(x)\frac{t^\alpha}{\alpha}\right) \\ &- \left(\omega_0''(x) + \omega_1''(x)\frac{t^\alpha}{\alpha} + \omega_2''(x)\frac{t^{2\alpha}}{2\alpha^2}\right) \\ &- \lambda\left(\omega_0(x) + \omega_1(x)\frac{t^\alpha}{\alpha} + \omega_2(x)\frac{t^{2\alpha}}{2\alpha^2}\right) \\ &- \mu\left(\omega_0(x) + \omega_1(x)\frac{t^\alpha}{\alpha} + \omega_2(x)\frac{t^{2\alpha}}{2\alpha^2}\right)^2 \\ &- \gamma\left(\omega_0(x) + \omega_1(x)\frac{t^\alpha}{\alpha} + \omega_2(x)\frac{t^{2\alpha}}{2\alpha^2}\right)^3. \end{aligned} \tag{22}$$

Operating T_t^α on both sides of Eq. (22) and equating the resulting equation to zero at $t=0$ yield

$$\begin{aligned} T_t^\alpha Res^2_\omega(x,t)_{|t=0} = \\ (\omega_2(x) - \left(\omega_1''(x) + \omega_2''(x)\frac{t^\alpha}{\alpha}\right) \\ - \lambda\left(\omega_1(x) + \omega_2(x)\frac{t^\alpha}{\alpha}\right) \\ - \mu(2\omega_0(x)\omega_1(x) + 2\omega_1^2(x)\frac{t^\alpha}{\alpha} \\ + \cdots + \omega_2^2(x)\frac{t^{3\alpha}}{\alpha^3}) \\ - \gamma(3\omega_0^2(x)\omega_1(x) + 3\omega_1^3(x)\frac{t^{2\alpha}}{\alpha^2} \\ + \cdots + 3\omega_2^3(x)\frac{t^{5\alpha}}{4\alpha^5}))_{|t=0} = 0. \end{aligned} \tag{23}$$

Thus, solving $T_t^\alpha Res^2_\omega(x,0) = 0$, we have

$$\begin{aligned} \omega_2(x) = \\ \omega_1''(x) + \lambda\omega_1(x) \\ + 2\mu\omega_0(x)\omega_1(x) \\ + 3\gamma\omega_0^2(x)\omega_1(x). \end{aligned}$$

Therefore, the 2^{nd} FPS approximate solution of Eqs. (1) and (2) can be expressed as follows

$$\begin{aligned} \omega_2(x,t) = \omega_0(x) \\ + \left(\frac{\omega_0''(x) + \lambda\omega_0(x) + \mu\omega_0^2(x) + \gamma\omega_0^3(x)}{\alpha}\right)t^\alpha \\ + \frac{t^{2\alpha}}{2\alpha^2}(\omega_1''(x) + \lambda\omega_1(x) + 2\mu\omega_0(x)\omega_1(x) \\ + 3\gamma\omega_0^2(x)\omega_1(x)). \end{aligned} \tag{24}$$

In the same manner, the CRPS procedure can be applied to find out the 3^{rd} unknown coefficient by substituting the 3^{rd} approximation $\omega_3(x,t) = \omega_0(x) + \sum_{n=1}^{3}\omega_n(x)\frac{t^{n\alpha}}{\alpha^n n!}$ into the 3^{rd} truncated residual function $Res^3_\omega(x,t)$ of Eq. (16) as follows

$$\begin{aligned} Res^3_\omega(x,t)(x,t) = T_t^\alpha\omega_3(x,t) - (\omega_3(x,t))_{xx} \\ - \lambda\omega_3(x,t) - \mu\omega_3^2(x,t) - \gamma\omega_3^3(x,t), \end{aligned}$$

and by computing the $T_t^{2\alpha}Res^3_\omega(x,t)$ and using the fact in Eq. (18), the desired coefficient can be obtained as follows

$$\begin{aligned} \omega_3(x) = \omega_2''(x) + \lambda\omega_2(x) \\ + 2\mu\left(\omega_1^2(x) + \omega_0(x)\omega_2(x)\right) \\ + 3\gamma\left(2\omega_0(x)\omega_1^2(x) + \omega_0^2(x)\omega_2(x)\right). \end{aligned} \tag{25}$$

Hence, the 3^{rd} FPS approximate solution of Eqs. (1) and (2) can be expressed as follows

$$\begin{aligned}\omega_3(x,t) &= \omega_0(x)+\omega_0''(x)+\lambda\omega_0(x)+\mu\omega_0^2(x)+ \\ &\gamma\omega_0^3(x)\frac{t^\alpha}{\alpha}+\left(\omega_1''(x)+\lambda\omega_1(x)+2\mu\omega_0(x)\omega_1(x)\right. \\ &\left.+3\gamma\omega_0^2(x)\omega_1(x)\right)\frac{t^{2\alpha}}{2\alpha^2}+\left(\omega_2''(x)+\lambda\omega_2(x)\right. \\ &+2\mu\left(\omega_1^2(x)+\omega_0(x)\omega_2(x)\right) \\ &\left.+3\gamma\left(2\omega_0(x)\omega_1^2(x)+\omega_0^2(x)\omega_2(x)\right)\right)\frac{t^{3\alpha}}{3!\alpha^3}.\end{aligned} \tag{26}$$

Similarly, the 4^{th} unknown coefficient $\omega_4(x)$ can be given using $T_t^{3\alpha}Res_\omega^4(x,0)=0$. Continuing in this approach up to arbitrary order n, the multiple CFPS solution $\omega_n(x,t)$ of Eqs. (1) and (2) will be given. Furthermore, high accuracy can be accomplished by calculating more components of the CFPS solution.

4 Numerical applications

To demonstrate the behavior, properties, efficiency and applicability of the proposed new method, three examples of both linear and nonlinear problems are presented numerically. All computations are performed using the Mathematica 10 package.

Example 4.1. Consider the following linear fractional Klein–Gordon equation [44]

$$T_t^\alpha\omega(x,t)=\omega_{xx}(x,t)+\omega(x,t),\ t\geq 0,\ x\in\mathbb{R},\ 0<\alpha\leq 1 \tag{27}$$

subject to the initial condition

$$\omega(x,0)=1+\sin(x). \tag{28}$$

The exact solution at $\alpha=1$ is $\omega(x,t)=\sin x+e^t$.

In particular, Eq. (27) is a special case of the time-fractional Klein–Gordon equation (1) when $\lambda=1$ and $\mu=\gamma=0$.

Following the description of the CRPS algorithm, by taking $\omega_0(x)=\omega(x,0)=1+\sin(x)$, the nth-truncated series of IVP (27) and (28) can be given as

$$\omega_n(x,t)=1+\sin(x)+\sum_{n=1}^{k}\omega_n(x)\frac{t^{n\alpha}}{\alpha^n n!}, \tag{29}$$

and the nth-residual function can be given as

$$Res_\omega^n(x,t)=T_t^\alpha\omega_n(x,t)-(\omega_n(x,t))_{xx}-\omega_n(x,t). \tag{30}$$

To find the coefficients $\omega_n(x), n=1,2,\ldots,k$, of Eq. (29), substitute the 1st truncated series $\omega_1(x,t)=1+\sin(x)+\omega_1(x)\frac{t^\alpha}{\alpha}$ into $Res_\omega^1(x,t)$ as follows

$$\begin{aligned}Res_\omega^1(x,t) &= T_t^\alpha\left(1+\sin(x)+\frac{\omega_1(x)t^\alpha}{\alpha}\right) \\ &-\left(1+\sin(x)+\frac{\omega_1(x)t^\alpha}{\alpha}\right)_{xx} \\ &-\left(1+\sin(x)+\frac{\omega_1(x)t^\alpha}{\alpha}\right) \\ &=\omega_1(x)-\frac{\omega_1''(x)t^\alpha}{\alpha}-\frac{\omega_1(x)t^\alpha}{\alpha}-1.\end{aligned} \tag{31}$$

Now, letting $t=0$ in $Res_\omega^1(x,t)$, we have $Res_\omega^1(x,0)=\omega_1(x)-1$. Thus, for $Res_\omega^1(x,0)=0$, it follows that $\omega_1(x)=1$. To define the 2nd unknown coefficient $\omega_2(x)$, substitute the 2nd truncated series $\omega_2(x,t)=1+\sin(x)+\frac{t^\alpha}{\alpha}+\omega_2(x)\frac{t^{2\alpha}}{2\alpha^2}$ into $Res_\omega^2(x,t)$, as follows,

$$\begin{aligned}Res_\omega^2(x,t) &= T_t^\alpha\left(1+\sin(x)+\frac{t^\alpha}{\alpha}+\frac{\omega_2(x)t^{2\alpha}}{2\alpha^2}\right) \\ &-\left(1+\sin(x)+\frac{t^\alpha}{\alpha}+\frac{\omega_2(x)t^{2\alpha}}{2\alpha^2}\right)_{xx} \\ &-\left(1+\sin(x)+\frac{t^\alpha}{\alpha}+\frac{\omega_2(x)t^{2\alpha}}{2\alpha^2}\right) \\ &=\frac{\omega_2(x)t^\alpha}{\alpha}-\frac{\omega_2''(x)t^{2\alpha}}{2\alpha^2}-\frac{t^\alpha}{\alpha}-\frac{\omega_2(x)t^{2\alpha}}{2\alpha^2}.\end{aligned} \tag{32}$$

Operating T_t^α on both sides of Eq. (32), we have

$$\begin{aligned}T_t^\alpha Res_\omega^2(x,t) &= T_t^\alpha\left(\frac{\omega_2(x)t^\alpha}{\alpha}-\frac{\omega_2''(x)t^{2\alpha}}{2\alpha^2}\right. \\ &\left.-\frac{t^\alpha}{\alpha}-\frac{\omega_2(x)t^{2\alpha}}{2\alpha^2}\right) \\ &=\omega_2(x)-\frac{\omega_2''(x)t^\alpha}{\alpha}-\frac{\omega_2(x)t^\alpha}{\alpha}-1,\end{aligned} \tag{33}$$

and equating $T_t^\alpha Res_\omega^2(x,t)$ to 0 for $t=0$, it follows $\omega_2(x)=1$. Therefore, the 2nd CFPS approximate solution of IVP (27) and (28) is

$$\omega_2(x,t)=1+\sin(x)+\frac{t^\alpha}{\alpha}+\frac{t^{2\alpha}}{2\alpha^2}.$$

Applying the same procedure for $n=3$, the 3^{rd} FPS approximation for Eqs. (27) and (28) can be given as

$$\omega_3(x,t)=1+\sin(x)+\frac{t^\alpha}{\alpha}+\frac{t^{2\alpha}}{2\alpha^2}+\frac{t^{3\alpha}}{6\alpha^3}. \tag{34}$$

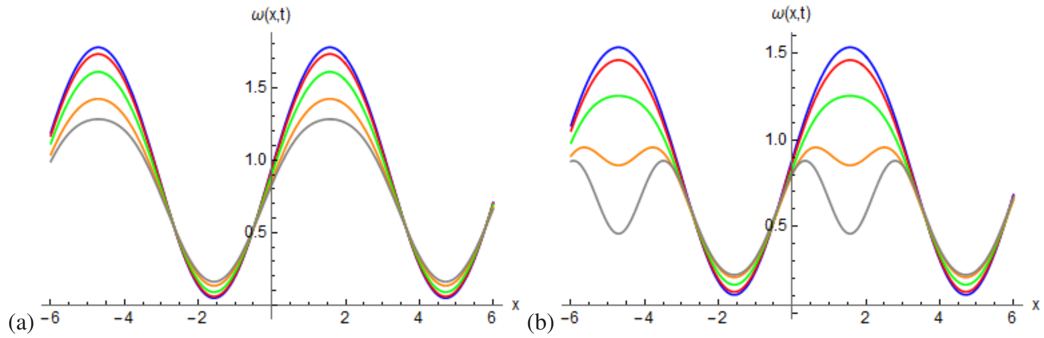

Fig. 4: The CRPS solutions of Example 4.2 for different values of α: (a) when $t = 0.04$, (b) when $t = 0.1$ such that blue for $\alpha = 0.95$, red for $\alpha = 0.9$, green for $\alpha = 0.8$, orange for $\alpha = 0.7$, and gray for $\alpha = 0.6$.

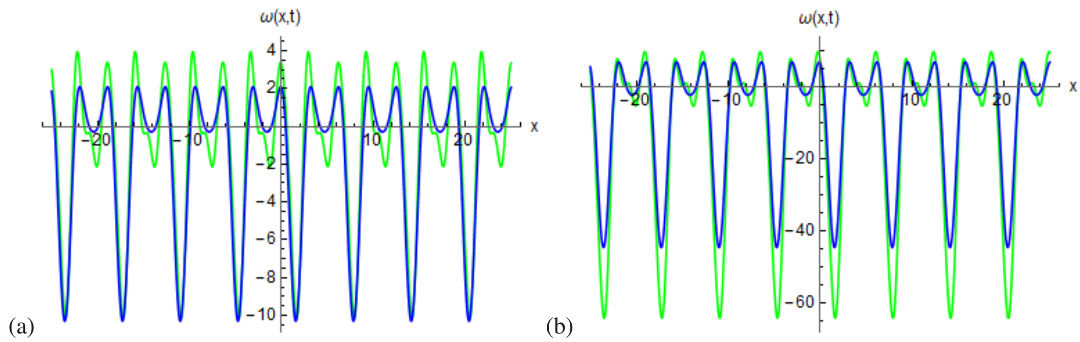

Fig. 5: Comparison between the CRPS and HPM for Example 4.2 when $t = 0.9$, and $x \in [-25, 25]$, where blue and green are used for CRPS and HPM [44] solutions, respectively: (a) $\alpha = 0.5$ (b) $\alpha = 1$.

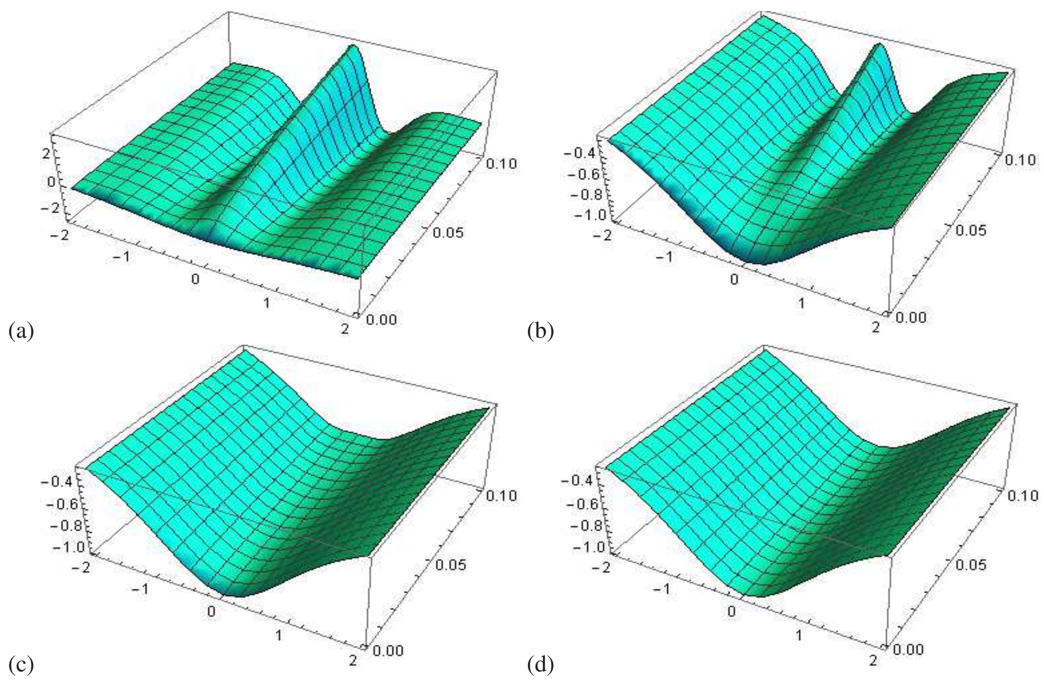

Fig. 6: The Phase schema of the CPS approximation $\omega_3(x,t)$ for Example 4.3 with $x \in [-2, 2]$ and $t \in [0, 0.1]$ at different values of fractional order: (a) $\alpha = 0.25$, (b) $\alpha = 0.5$, (c) $\alpha = 0.75$, (d) $\alpha = 1$.

Similarly, the multiple CFPS solution $\omega(x,t)$ in terms of infinite series is obtained as

$$\begin{aligned}\omega(x,t) &= 1+\sin(x)+\frac{t^{\alpha}}{\alpha}+\frac{t^{2\alpha}}{(2)!\alpha^{2}}+\frac{t^{3\alpha}}{(3)!\alpha^{3}} \\ &+\cdots+\frac{t^{k\alpha}}{(k)!\alpha^{k}}+\ldots \\ &= \sin(x)+\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{(n)!\alpha^{n}},\end{aligned} \tag{35}$$

which is consistent with the results contained in [44]. Furthermore, for $\alpha=1$, the CFPS solution will be $\omega(x,t)=\sin x+e^{t}$, which is fully compatible with the exact solution.

For numerical simulation, Table 1 shows the approximate solution $\omega_3(x,t)$, absolute error and relative error of Example 4.1 when $\alpha=1$ for $x=1$ and some selected grid points t with step size 0.01. To show the geometric behaviors of the 3^{rd} CFPS approximation of IVPs (27) and (28), the third dimensional surface plots of $\omega_3(x,t)$ are illustrated in Figure 1 for $x\in[-4,4]$ and $t\in[0,0.1]$ for different values of fractional order α such that $\alpha=\{0.3,0.5,0.7,0.9\}$. It manifests that the behavior of the CRPS approximate solutions depends continuously on the value of conformable derivative α. Thus, we conclude that the graphs almost have similar behaviors, and are consistent with each other, especially when considering the integer-order derivative.

While Figure 2 shows the comparison of surface plots between the exact solution $\omega(x,t)$ and CFPS solution $\omega_3(x,t)$ at $\alpha=1$, for $t=0.01$ and each $x\in[-10,10]$, where blue and red are used for the exact and approximate solutions, respectively. It exhibits that the approximate solution matches the exact solution during the spatial interval, which indicates efficiency of the proposed method.

Example 4.2. Consider the following nonlinear fractional Klein–Gordon equation [44]:

$$T_t^{\alpha}\omega(x,t)=\omega_{xx}(x,t)-\omega^{2}(x,t),\ 0<\alpha\leq 1, \tag{36}$$

subject to the initial condition

$$\omega(x,0)=1+\sin(x). \tag{37}$$

In particular, Eq. (36) is a special case of the time-fractional Klein–Gordon equation (1) when $\lambda=\gamma=0$ and $\mu=-1$.

In view of the CRPS approach, the nth-truncated series of IVPs (36) and (37) is given as

$$\omega_n(x,t)=1+\sin(x)+\sum_{n=1}^{k}\omega_n(x)\frac{t^{n\alpha}}{\alpha^{n}n!}, \tag{38}$$

and the nth-residual function of IVPs (36) and (37) is given as

$$Res_{\omega}^{n}(x,t)=T_t^{\alpha}\omega_n(x,t)-(\omega_n(x,t))_{xx}+\omega_n^{2}(x,t). \tag{39}$$

To determine the 1st unknown coefficient $\omega_1(x)$ of Eq. (38), substitute the 1st truncated series $\omega_1(x,t)=1+\sin(x)+\omega_1(x)\frac{t^{\alpha}}{\alpha}$ into the 1st residual function as follows

$$\begin{aligned}Res_{\omega}^{1}(x,t) =&\omega_1(x)+\sin(x)-\frac{\omega_1''(x)t^{\alpha}}{\alpha} \\ &+\left(1+\sin(x)+\frac{\omega_1(x)t^{\alpha}}{\alpha}\right)^{2}.\end{aligned} \tag{40}$$

Now, by letting $t=0$ in $Res_{\omega}^{1}(x,t)$ such that $Res_{\omega}^{1}(x,0)=\omega_1(x)+1+3\sin(x)+\sin^{2}(x)$ and making use of $Res_{\omega}^{1}(x,0)=0$, we observe

$$\omega_1(x)=-1-3\sin(x)-\sin^{2}(x).$$

Therefore, the 1st approximate CFPS solution is

$$\omega_1(x,t)=1+\sin(x)-\left(1+3\sin(x)+\sin^{2}(x)\right)\frac{t^{\alpha}}{\alpha}.$$

Similarly, to define the 2nd unknown coefficient $\omega_2(x)$, substitute the 2nd truncated series $\omega_2(x,t)=1+\sin(x)-\frac{(1+3\sin(x)+\sin^{2}(x))t^{\alpha}}{\alpha}+\omega_2(x)\frac{t^{2\alpha}}{2\alpha^{2}}$ into $Res_{\omega}^{2}(x,t)$ such that

$$\begin{aligned}Res_{\omega}^{2}(x,t)=&T_t^{\alpha}(1+\sin(x)-\frac{(1+3\sin(x)+\sin^{2}(x))t^{\alpha}}{\alpha} \\ &+\frac{\omega_2(x)t^{2\alpha}}{2\alpha^{2}})-(1+\sin(x) \\ &-\frac{(1+3\sin(x)+\sin^{2}(x))t^{\alpha}}{\alpha}+\frac{\omega_2(x)t^{2\alpha}}{2\alpha^{2}})_{xx} \\ &+(1+\sin(x)-\frac{(1+3\sin(x)+\sin^{2}(x))t^{\alpha}}{\alpha} \\ &+\frac{\omega_2(x)t^{2\alpha}}{2\alpha^{2}})^{2} \\ &=\left(-(1+3\sin(x)+\sin^{2}(x))+\frac{\omega_2(x)t^{\alpha}}{\alpha}\right) \\ &+(\sin(x)-(3\sin(x)+2\left(\sin^{2}(x)-\cos^{2}(x)\right) \\ &\frac{t^{\alpha}}{\alpha}-\frac{\omega_2''(x)t^{2\alpha}}{2\alpha^{2}})+(1+2\sin(x)+\sin^{2}(x) \\ &-2\left((1+\sin(x))\frac{(1+3\sin(x)+\sin^{2}(x))t^{\alpha}}{\alpha}\right) \\ &+\cdots+\frac{\omega_2^{2}(x)t^{4\alpha}}{4\alpha^{4}}).\end{aligned} \tag{41}$$

Operating T_t^{α} on both sides of Eq. (41) and equating to 0 for $t=0$, we have

$$
\begin{aligned}
T_t^{\alpha} Res_{\omega}^{2}(x,t)_{|t=0} &= T_t^{\alpha}\Big(\frac{\omega_2(x)t^{\alpha}}{\alpha} - \frac{\left(3\sin(x)+2\left(\sin^2(x)-\cos^2(x)\right)\right)t^{\alpha}}{\alpha} \\
&\quad - \frac{\omega_2''(x)t^{2\alpha}}{2\alpha^2} - 2((1+\sin(x)) \\
&\quad \frac{(1+3\sin(x)+\sin^2(x))t^{\alpha}}{\alpha}) \\
&\quad + \cdots + \frac{\omega_2^2(x)t^{4\alpha}}{4\alpha^4}\Big)_{t=0} \\
&= (\omega_2(x) - (3\sin(x)+2(\sin^2(x) \\
&\quad - \cos^2(x))) - \frac{\omega_2''(x)t^{\alpha}}{\alpha} \\
&\quad - 2(1+\sin(x))(1+3\sin(x)+\sin^2(x)) \\
&\quad + \cdots + \frac{\omega_2^2(x)t^{3\alpha}}{\alpha^3})_{t=0} \\
&= \omega_2(x) - 11\sin(x) - 12\sin^2(x) \\
&\quad - 2\sin^3(x) = 0.
\end{aligned} \tag{42}
$$

Thus, the 2nd unknown coefficient is

$$\omega_2(x) = 11\sin(x) + 12\sin^2(x) + 2\sin^3(x).$$

Therefore, the 2nd CFPS approximate solution of IVP (36) and (37) is

$$
\begin{aligned}
\omega_2(x,t) = 1+\sin(x) &- \frac{(1+3\sin(x)+\sin^2(x))t^{\alpha}}{\alpha} \\
&+ \frac{11\sin(x)+12\sin^2(x)+2\sin^3(x)t^{2\alpha}}{2\alpha^2}.
\end{aligned}
$$

By applying the same procedure for $n=3$, the 3rd unknown coefficient can be given as

$$
\begin{aligned}
\omega_2(x) = 22 - 33\sin(x) &- 116\sin^2(x) - 58\sin^3(x) \\
&- 6\sin^4(x),
\end{aligned}
$$

then the 3rd CFPS approximation for Eqs. (36) and (37) can be given as

$$
\begin{aligned}
\omega_3(x,t) = 1+\sin(x) &- \frac{(1+3\sin(x)+\sin^2(x))t^{\alpha}}{\alpha} \\
&+ \frac{11\sin(x)+12\sin^2(x)+2\sin^3(x)t^{2\alpha}}{2\alpha^2} \\
&+ \frac{t^{3\alpha}}{6\alpha^3}(22-33\sin(x)-116\sin^2(x) \\
&- 58\sin^3(x) - 6\sin^4(x)),
\end{aligned} \tag{43}
$$

which is consistent with the results in [44].

To show the accuracy of the CRPS algorithm in handling Example 4.2, the 3-dimension surface plots of the CFPS approximate solutions are given in Figure 3 when $x\in[-2,2]$ and $t\in[0,0.01]$ for different levels of fractional order α such that $\alpha=\{0.01,0.1,0.5,0.95\}$. From these graphs, the solution behavior indicates that an increase of the fractional parameter changes the nature of the solution with a smooth sense. The curves of the CRPS approximate solutions at different levels of the fractional order of α was drawn in Figure 4 when $t=0.4$ and $t=0.1$. Here we notice that the solution curves are consistent with each other and approach the exact curve with increasing fractional values to the integer-order value $\alpha=1$.

For further analysis, the comparison of the curves of CRPS and HPM [44] solutions are plotted in Figure 5 for $t=0.9$, $x\in[-25,25]$ when (a) $\alpha=0.5$ and (b) $\alpha=1$, where blue and green are used for CRPS and HPM solutions, respectively. From these curves, it can be noted that the approximate solutions obtained by CRPS and HPM are consistent with each other for different values of the fractional order α but the CRPS curve excels and similar to the closed solution more than HP solution.

Example 4.3. Consider the following nonlinear fractional Klein–Gordon equation [44]:

$$T_t^{\alpha}\omega(x,t) = \omega_{xx}(x,t) - \omega(x,t) + \omega^3(x,t),\ 0<\alpha\leq 1, \tag{44}$$

subject to the initial condition

$$\omega(x,0) = -\operatorname{sech}(x). \tag{45}$$

In this example, we consider $\lambda=-1$, $\mu=0$ and $\gamma=1$. In view of the CRPS algorithm, the nth-truncated series of IVPs (44) and (45) can be given as

$$\omega_n(x,t) = -\operatorname{sech}(x) + \sum_{n=1}^{k}\omega_n(x)\frac{t^{n\alpha}}{\alpha^n n!}, \tag{46}$$

and the nth-residual function can be given as

$$Res_{\omega}^{n}(x,t) = T_t^{\alpha}\omega_n(x,t) - \omega_n(x,t)_{xx} + \omega_n(x,t) - \omega_n^3(x,t). \tag{47}$$

By applying the former iteration process, the first few terms of the RPS (46) are given as follows

$$\omega_1(x) = \operatorname{sech}^3(x),$$

$$\omega_2(x) = (-5+4\cosh(2x))\operatorname{sech}^5(x),$$

$$\omega_3(x) = (117-112\cosh(2x)+8\cosh(4x))\operatorname{sech}^7(x),$$

$$\vdots$$

Therefore, the CFPS approximation of IVPs (44) and (45) is

$$\begin{aligned}\omega(x,t) = &-\operatorname{sech}(x) + \frac{1}{\alpha}\operatorname{sech}^3(x)\,t^{\alpha} \\ &+ \left(\frac{(4\cosh(2x) - 5)}{2\alpha^2}\right)\operatorname{sech}^5(x)\,t^{2\alpha} \\ &+ \left(\frac{(117 - 112\cosh(2x) + 8\cosh(4x))}{6\alpha^3}\right) \\ &\operatorname{sech}^7(x)\,t^{3\alpha} + \ldots.\end{aligned} \tag{48}$$

To demonstrate the efficacy of the proposed algorithm in solving Example 4.3, approximate behaviors of CRPS are plotted in Figure 6 with 3D-space graphs when $n = 3$, $x \in [-2,2]$ and $t \in [0,0.1]$ for different values of α such that $\alpha = \{0.25, 0.5, 0.75, 1\}$. From these graphs, one can see that the fractional order has strong geometrical effects on the model surface profiles, which tend to lead to unusual behaviors if they move away from the integer value as is evident from the drawing when $\alpha = 0.25$.

5 Conclusion

In this paper, the application of the conformable residual power series method has been successfully extended to obtain approximate analytical solutions to time-fractional nonlinear Kline-Gordon equations associated with conformable fractional derivative. The proposed method was used directly to solve these nonlinear fractional models without being linearized, discretized, or perturbation. Meanwhile, theoretical predictions and error analysis of the method have been discussed. To demonstrate consistency with the theoretical framework, three illustrative examples were presented. The approximate solutions are compared with the exact solutions and those in the literature to show validity and reliability of the CRPS method. Accurate solutions were introduced with the help of shapes and tables, which showed consistency with each other for different values of the fractional-order. Therefore, we can conclude that this method is an effective and simple tool for treating fractional partial differential equations with great potential in scientific applications.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

Data Availability: The data used to support the findings of this study are available from the corresponding author upon request.

References

[1] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, (1993).

[2] D. Baleanu, J.A.T. Machado, A.-C. Luo, *Fractional Dynamics and Control*, Springer Science and Business Media, New York, (2011).

[3] D. Baleanu. K. Diethelm, E. Scalas, J.J. Trujillo, *Fractional Calculus Models and Numerical Methods*, World Scientific, Singapore, (2009).

[4] B. Bira, T.R. Sekhar, D. Zeidan, Exact solutions for some time-fractional evolution equations using Lie group theory, *Mathematical Methods in the Applied Sciences*, **41** (16), 6717-6725 (2018).

[5] M. Al-Smadi, O. Abu Arqub, Computational algorithm for solving fredholm time-fractional partial integrodifferential equations of dirichlet functions type with error estimates, *Applied Mathematics and Computation*, **342**, 280-294 (2019).

[6] M. Al-Smadi, A. Freihat, H. Khalil, S. Momani, R.A. Khan, Numerical multistep approach for solving fractional partial differential equations, *International Journal of Computational Methods*, **14**, 1750029 (2017).

[7] D. Baleanu, A. K. Golmankhaneh, M.C. Baleanu, Fractional electromagnetic equations using fractional forms, *International Journal of Theoretical Physics* **48**, 3114-3123 (2009).

[8] M. Al-Smadi, Simplified iterative reproducing kernel method for handling time-fractional BVPs with error estimation, *Ain Shams Engineering Journal*, **9** (4), 2517-2525 (2018).

[9] Z. Altawallbeh, M. Al-Smadi, I. Komashynska, A. Ateiwi, Numerical Solutions of Fractional Systems of Two-Point BVPs by Using the Iterative Reproducing Kernel Algorithm, *Ukrainian Mathematical Journal*, **70** (5), 687–701 (2018).

[10] O. Abu Arqub, M. Al-Smadi, An adaptive numerical approach for the solutions of fractional advection–diffusion and dispersion equations in singular case under Riesz's derivative operator, *Physica A: Statistical Mechanics and its Applications*, **540**, 123257, 1-13, (2020). https://doi.org/10.1016/j.physa.2019.123257

[11] S. Hasan, A. El-Ajou, S. Hadid, M. Al-Smadi, S. Momani, Atangana-Baleanu fractional framework of reproducing kernel technique in solving fractional population dynamics system, *Chaos, Solitons and Fractals*, **133**, 109624 (2020). https://doi.org/10.1016/j.chaos.2020.109624

[12] A.M. Wazwaz, New travelling wave solutions to the Boussinesq and the Klein-Gordon equations, *Communications in Nonlinear Science and Numerical Simulation*, **13**(5), 889–901, (2008).

[13] E. Abuteen, A. Freihat, M. Al-Smadi, H. Khalil, R.A. Khan, Approximate Series Solution of Nonlinear, Fractional Klein-Gordon Equations Using Fractional Reduced Differential Transform Method, *Journal of Mathematics and Statistics*, **12**, 23-33, (2016).

[14] M. Al-Smadi, Solving fractional system of partial differential equations with parameters derivative by combining the GDTM and RDTM, *Nonlinear Studies*, **26(3)**, 587-601 (2019).

[15] A.M.A. El-Sayed, A. Elsaid, D. Hammad, A reliable treatment of homotopy perturbation method for solving the nonlinear Klein-Gordon equation of arbitrary (fractional) orders, *Journal of Applied Mathematics*, **2012**, 581481, (2012).

[16] K.A. Gepreel, M.S. Mohamed, Analytical approximate solution for nonlinear space-time fractional Klein Gordon equation, *Chinese Physics B*, **22**, 010201, (2013).
[17] E.A.B. Abdel-Salam, E.A. Yousif, Solution of nonlinear space-time fractional differential equations using the fractional Riccati expansion method, *Mathematical Problems in Engineering*, **2013**, 846283, (2013).
[18] M. Al-Smadi, O. Abu Arqub, S. Momani, Numerical computations of coupled fractional resonant Schrödinger equations arising in quantum mechanics under conformable fractional derivative sense, *Physica Scripta*, **95**, 075218 (2020). https://doi.org/10.1088/1402-4896/ab96e0
[19] M. Al-Smadi, O. Abu Arqub, S. Momani, A computational method for two-point boundary value problems of fourth-order mixed integrodifferential equations, *Mathematical Problems in Engineering*, **2013**, 832074, (2013). https://doi.org/10.1155/2013/832074
[20] E. Goncalves, D. Zeidan, Numerical study of turbulent cavitating flows in thermal regime, *International Journal of Numerical Methods for Heat and Fluid*, **27** (7), 1487-1503, (2017).
[21] G. Gumah, M. Naser, M. Al-Smadi, S.K.Q. Al-Omari, D. Baleanu, Numerical solutions of hybrid fuzzy differential equations in a hilbert space, *Applied Numerical Mathematics*, **151**, 402-412 (2020).
[22] M. Al-Smadi, O. Abu Arqub, N. Shawagfeh, S. Momani, Numerical investigations for systems of second-order periodic boundary value problems using reproducing kernel method, *Applied Mathematics and Computation*, **291**, 137-148 (2016).
[23] D. Zeidan, R. Touma, On the Computations of Gas-Solid Mixture Two-Phase Flow, *Advances in Applied Mathematics and Mechanics*, **6**, 49-74, (2014).
[24] E. Goncalves, D. Zeidan, Simulation of compressible two-phase flows using a void ratio transport equation, *Communications in Computational Physics*, **24**, 167-203, (2018).
[25] G.N. Gumah, M.F.M. Naser, M. Al-Smadi, S.K. Al-Omari, Application of reproducing kernel Hilbert space method for solving second-order fuzzy Volterra integro-differential equations, *Advances in Difference Equations*, **2018**, 475 (2018). https://doi.org/10.1186/s13662-018-1937-8
[26] M. Al-Smadi, Reliable Numerical Algorithm for Handling Fuzzy Integral Equations of Second Kind in Hilbert Spaces, *Filomat*, **33**, 583-597 (2019).
[27] D. Zeidan, A. Slaouti, E. Romenski, E.F. Toro, Numerical solution for hyperbolic conservative two-phase flow equations, *International Journal of Computational Methods*, **4** (2), 299-333, (2007).
[28] E. Goncalves, Y. Hoarau, D. Zeidan, Simulation of shock-induced bubble collapse using a four-equation model, *Shock Waves*, **29**, 221–234, (2019).
[29] A. Freihet, S. Hasan, M. Al-Smadi, M. Gaith, S. Momani, Construction of fractional power series solutions to fractional stiff system using residual functions algorithm, *Advances in Difference Equations*, **2019**, 95, (2019). https://doi.org/10.1186/s13662-019-2042-3
[30] S. Hasan, M. Al-Smadi, A. Freihet, S. Momani, Two computational approaches for solving a fractional obstacle system in Hilbert space, *Advances in Difference Equations*, **2019**, 55, (2019). https://doi.org/10.1186/s13662-019-1996-5
[31] S. Alshammari, M. Al-Smadi, M. Al Shammari, I. Hashim, M.A. Alias, Advanced analytical treatment of fractional logistic equations based on residual error functions, *International Journal of Differential Equations*, **2019**, Art. ID 7609879, 1-11 (2019). https://doi.org/10.1155/2019/7609879
[32] R. Saadeh, M. Alaroud, M. Al-Smadi, R.R. Ahmad, U.K. Salma Din, Application of fractional residual power series algorithm to solve Newell–Whitehead–Segel equation of fractional order, *Symmetry*, **11** (12), 1431, (2019).
[33] A. Freihet, S. Hasan, M. Alaroud, M. Al-Smadi, R.R. Ahmad, U.K. Salma Din, Toward computational algorithm for time-fractional Fokker Planck models, *Advances in Mechanical Engineering*, **11** (10), 1-10, (2019).
[34] A. Atangana, D. Baleanu, A. Alsaedi, New properties of conformable derivative, *Open Mathematics*, **13**, 889–898, (2015).
[35] T. Abdeljawad, On conformable fractional calculus, *Journal of Computational and Applied Mathematics*, **279**, 57–66, (2015).
[36] O. Abu Arqub, M. Al-Smadi, Fuzzy conformable fractional differential equations: novel extended approach and new numerical solutions, *Soft Computing*, **2020**, 1-22 (2020). https://doi.org/10.1007/s00500-020-04687-0
[37] M. Shqair, M. Al-Smadi, S. Momani, E. El-Zahar, Adaptation of Conformable Residual Power Series Scheme in Solving Nonlinear Fractional Quantum Mechanics Problems, *Applied Sciences*, **10** (3), 890, (2020).
[38] O. Abu Arqub, R. Edwan, M. Al-Smadi, S. Momani, Solving space-fractional Cauchy problem by modified finite-difference discretization scheme, *Alexandria Engineering Journal*, (2020) In Press. https://doi.org/10.1016/j.aej.2020.03.003
[39] M. Alshammari, M. Al-Smadi, O. Abu Arqub, I. Hashim, M.A. Alias, Residual Series Representation Algorithm for Solving Fuzzy Duffing Oscillator Equations, *Symmetry*, **12** (4), 572, (2020). https://doi.org/10.3390/sym12040572
[40] S. Alshammari, M. Al-Smadi, I. Hashim, M.A. Alias, Residual Power Series Technique for Simulating Fractional Bagley Torvik Problems Emerging in Applied Physics, *Applied Sciences*, **9** (23), 5029, (2019). https://doi.org/10.3390/app9235029
[41] O. Abu Arqub, M. Al-Smadi, Atangana–Baleanu fractional approach to the solutions of Bagley–Torvik and Painlevé equations in Hilbert space, *Chaos, Solitons and Fractals*, **117**, 161-167, (2018).
[42] O. Abu Arqub, Z. Odibat, M. Al-Smadi, Numerical solutions of time-fractional partial integrodifferential equations of Robin functions types in Hilbert space with error bounds and error estimates, *Nonlinear Dynamics*, **94**, 1819-1834, (2018).
[43] O. Abu Arqub, M. Al-Smadi, Numerical algorithm for solving time-fractional partial integrodifferential equations subject to initial and Dirichlet boundary conditions, *Numerical Methods for Partial Differential Equations*, **34**(5), 1577-1597, (2017).
[44] A.K. Golmankhaneh, A.K. Golmankhaneh, D. Baleanu, On nonlinear fractional Klein–Gordon equation, *Signal Processing*, **91**, 446–451, (2011).

Rasha Amryeen is a Ph.D student at Universiti Malaysia Terenganu (University of Terenganu in Malaysia "UTM") under Prof. Fatimah Noor Harun supervision. She received her B.Sc. degree in Mathematics from Al-Hussein Bin Talal University in 2008 and her M.Sc. degree in Mathematics from Mutah University (Jordan) in 2011. Then, she began work as a lecturer since 2011 at Hail University. Her research interests in applied mathematics, numerical methods and fractional differential equations.

Fatimah Noor Harun is a senior lecturer at Faculty of Ocean Engineering Technology and Informatics, Universiti Malaysia Terenganu. She obtained her B.Sc. in Computer Modeling and M.Sc. in Mathematics from Universiti Sains Malaysia. Then, in 2010, she received her Ph.D degree in Mathematics from University of Wollongong, Australia. Her research interests are in partial differential equations, applied mathematics and mathematical modelling particularly in linear and nonlinear waves. She is also a life member of the Malaysian Mathematical Sciences Society (PERSAMA). Her Scopus, Orcid and Publons Author IDs are 57204889542, 0000-0003-1088-7592 and AAF-7681-2019, respectively.

Mohammed Al-Smadi received his Ph.D. from the University of Jordan, Amman in 2011. He then began working as Assistant Professor of Applied Mathematics at Qassim University (2011-2012) and then at Tafila Technical University (2012-2013). Since Sep. 2013, Dr. Al-Smadi worked as an Assistant Professor of Applied Mathematics at Al-Balqa Applied University and he was promoted to Associate Professor in Sep. 2017. His research interests include Applied Mathematics, Numerical Analysis, and Fractional Calculus.

Azwani Alias is currently held a position as senior lecturer at Universiti Malaysia Terenganu. She obtained her B.Sc. and M.Sc. degrees in Mathematics from Universiti Teknologi Malaysia. She received her Ph.D degree in Mathematics from Loughborough University, United Kingdom in 2014. Her research interest in applied mathematics and mathematical modelling particularly in nonlinear waves. She is also a life member of the Malaysian Mathematical Sciences Society (PERSAMA). Her Scopus ID is 55797131800.