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Polynomial and Non-Polynomial Splines with the Fourth Order of Approximation

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Abstract: This paper addresses the construction of polynomial and non-polynomial splines of the fourth order of approximation. Smooth non-polynomial splines of the minimal defect are constructed using preliminarily constructed Hermite-type splines. The estimates of the approximations are given and the constants they include are calculated.

Keywords: Non-polynomial local basis splines, polynomial local basis splines, Hermite interpolation, splines of the minimal defect

1 Introduction

Smooth local polynomial splines were first constructed in England at the end of the 19th century. The term spline was first used by Schoenberg (1946). Since then, a huge number of splines with various properties have been constructed. Nevertheless, the solution of newly arising problems requires new methods of solution. They can also often be solved more efficiently using spline approximations. For example the solution of the systems of integral and differential equations that are emerging nowadays often requires the development of new solving methods. The development of new methods often involves the use of splines with prescribed properties. Currently, the construction of splines with prescribed properties represents an important task which several investigation addresses.

The B-spline is a spline function that has minimal support with respect to a given degree, smoothness, and domain partition. B-splines are often used to solve a wide range of computational problems. We have selected several papers from a long list of works. In study [1], the authors tackle the cubic B-spline method to solve two-point boundary value problems. The cubic B-spline approximation equations, which are based on the quarter-sweep concept, are used to discretize the proposed problem and construct the linear system. Paper [2] handles the system of the cubic B-spline approximation equation which is generated by applying cubic B-spline discretization scheme in solving two-point boundary value problems. Non-polynomial splines are less known than polynomial B-splines, so they are rarely used. Smooth non-polynomial splines are less used. Non-polynomial quintic spline functions based algorithms are used in [3] for computing an approximation to the nonlinear boundary value problems arising in human physiology. Among non-polynomial splines, trigonometric splines are of significant interest. In 1964, Schoenberg introduced trigonometric spline functions and proved the existence of locally supported trigonometric spline and B-spline functions [4]. Recently, trigonometric splines have been increasingly used. An application of the quartic trigonometric B-spline for a finite element method is employed in [5]. A set of cubic trigonometric B-splines are used in [6]. In paper [7], the authors develop a numerical solution based on the non-polynomial B-spline (trigonometric B-spline) collocation method for solving time-dependent equations involving PHI-Four and Allen-Cahn equations. Paper [8] highlights the utilization of Radial Basis Functions (RBF) for reconstruction of damaged images. Paper [9] describes novel, fast, simple, and robust algorithm with $O(N)$ expected complexity. The proposed algorithm gives significant speed-up to applications, when medium and large data sets are processed.

This paper continues the series of papers on approximation with local polynomial and non-polynomial splines (see [10], [11], [12], [15]). The proposed paper offers non-polynomial splines of the Hermite type with the fourth order approximation of the first level (height),

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as well as smooth non-polynomial splines. The construction of these splines uses the functions of the Chebyshev system. These non-polynomial splines solve the interpolation problem.

The present paper is organized, as follows: After this introduction Section 2 involves information on polynomial and non-polynomial splines of the Hermite-type with the fourth-order approximation. Theorems on approximation errors by the Hermitian polynomial and non-polynomial splines are also covered. Section 3 shows the construction of polynomial splines of the minimum defect. Section 4 handles the construction of non-polynomial splines of the minimum defect. In Sections 3 and 4, estimates of the approximation error by these splines are presented. In Section 5, approximations of the first derivatives for the polynomial and non-polynomial case are constructed. Section 6 is devoted to research perspective.

2 Fourth-order spline approximation

Let function $u(x)$ be such that $u \in C^4([a,b])$. Let nodes $x_j \in [a,b]$, $j = 0,1,\dots,n$, be such that $\dots < x_{j-1} < x_j < x_{j+1} < \dots$. The formulas of the basis splines of the first level and the fourth order of approximation $w_{j,0}(x)$, $w_{j+1,0}(x)$, $w_{j,1}(x)$, $w_{j+1,1}(x)$ on an interval $[x_j, x_{j+1}]$ are obtained by solving the following system of equations:

$$\varphi_i(x_j)w_{j,0}(x) + \varphi_i(x_{j+1})w_{j+1,0}(x) + \varphi_i'(x_j)w_{j,1}(x) + \varphi_i'(x_{j+1})w_{j+1,1}(x) = \varphi_i(x), \quad i = 0,1,2,3. \tag{1}$$

Suppose the determinant is unequal to zero. The system of functions φ_i is the Chebyshev system on the interval $[\alpha,\beta]$, α, β are real numbers, $\beta > \alpha$. Based on different systems φ_i we will obtain different basis functions $w_{j,0}(x)$, $w_{j+1,0}(x)$, $w_{j,1}(x)$, $w_{j+1,1}(x)$. We construct the approximation of function $u(x)$ with these splines on the interval $[x_j, x_{j+1}]$ in the form:

$$U(x) = u(x_j)w_{j,0}(x) + u(x_{j+1})w_{j+1,0}(x) + u'(x_j)w_{j,1}(x) + u'(x_{j+1})w_{j+1,1}(x). \tag{2}$$

Let $\varphi_i(x) = x^i$, $i = 0,1,2,3$. Solving the system of linear algebraic equations, we obtain the basis polynomial functions

$$w_{j,0}(x_j + th) = 1 - 3t^2 + 2t^3, \quad w_{j+1,0}(x_j + th) = 3t^2 - 2t^3,$$
$$w_{j,1}(x_j + th) = ht^3 - 2ht^2 + th, \quad w_{j+1,1}(x_j + th) = t^3 - t^2.$$

Plots of the polynomial basis splines $w_{j,0}(x)$, $w_{j+1,0}(x)$ (when $x \in [x_j, x_{j+1}]$) are given in Figure 1. Plots of the polynomial basis splines $w_{j,1}(x)$, $w_{j+1,1}(x)$ (when $x \in [x_j, x_{j+1}]$) are given in Figure 2.

We construct the approximation of function $u(x)$ with these polynomial splines on the interval $[x_j, x_{j+1}]$ in the form:

$$U^P(x) = u(x_j)w_{j,0}(x) + u(x_{j+1})w_{j+1,0}(x) + u'(x_j)w_{j,1}(x) + u'(x_{j+1})w_{j+1,1}(x).$$

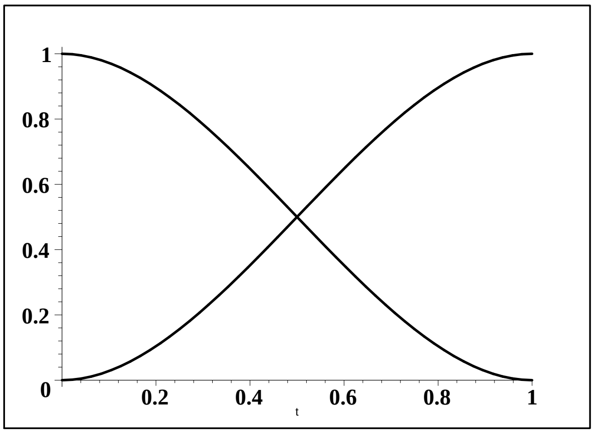

Fig. 1: Plots of the polynomial basis splines $w_{j,0}$, $w_{j+1,0}$.

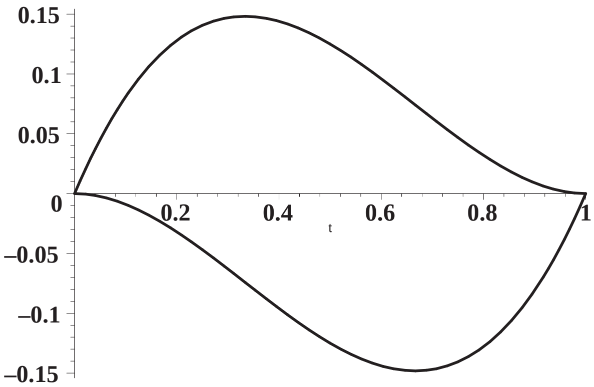

Fig. 2: Plots of the polynomial basis splines $w_{j,1}$, $w_{j+1,1}$.

The following theorem is valid when $\varphi_i(x) = x^i$, $i = 0,1,2,3$.

Theorem 2.1. Let function $u(x)$ be such that $u \in C^4([a,b])$, $\varphi_i(x) = x^i$, $i = 0,1,2,3$. Suppose ordered distinct nodes $\{x_k\}$ such that $x_{j+1} - x_j = h$. Then for $x \in [x_j, x_{j+1}]$, we have

$$| u(x) - U^P(x) | \le K_1 h^4 \|u^{IV}\|_{[x_j,x_{j+1}]},$$
$$K_1 = 1/(4! \cdot 16) \approx 0.0026.$$

Proof. On the interval $[x_j, x_{j+1}]$, we have the following relations $U^P(x_j) = u(x_j)$, $(U^P)'(x_j) = u'(x_j)$, $U^P(x_{j+1}) = u(x_{j+1})$, $(U^P)'(x_{j+1}) = u'(x_{j+1})$. Thus, we can construct the Hermite interpolation using (2) on

$[x_j, x_{j+1}]$ with basis functions:

$$w_{j,0}(x_j+th) = 1-3t^2+2t^3,$$
$$w_{j+1,0}(x_j+th) = 3t^2-2t^3,$$
$$w_{j,1}(x_j+th) = h(t-2t^2+t^3),$$
$$w_{j+1,1}(x_j+th) = h(t^3-t^2).$$

The error of the interpolation is, as follows:

$$U^P(x)-u(x) = \frac{u^{IV}(\xi)}{4!}(x-x_j)^2\left(x-x_{j+1}\right)^2.$$

Here $\xi = \xi(x)$, $\xi \in [x_j, x_{j+1}]$. If we put $x = x_j + th$, $t \in [0,1]$, we obtain $(x-x_j)^2\left(x-x_{j+1}\right)^2 = h^4t^2(t-1)^2$. It is easy to obtain $\max\limits_{t\in[0,1]} t^2(t-1)^2 = 1/16$. The proof is complete.

Remark. The formulas of the splines of the first level and the fourth order of approximation $w_{j,0}(x)$, $w_{j+1,0}(x)$, $w_{j,1}(x)$, $w_{j+1,1}(x)$ on the interval $[x_{j-1}, x_j]$ to the left of the node x_j are obtained by solving the following system of equations:

$$\varphi_i(x_{j-1})w_{j-1,0}(x)+\varphi_i(x_j)w_{j,0}(x)+\varphi_i'(x_{j-1})w_{j-1,1}(x)$$
$$+\varphi_i'(x_j)w_{j,1}(x) = \varphi_i(x), \quad i = 0,1,2,3.$$

Note that the supports of the basis splines $w_{j,0}$, $w_{j,1}$ occupy two adjacent grid intervals: supp $w_{j,0}$ = supp $w_{j,1} = [x_{j-1}, x_{j+1}]$.

Combining the formulas of basis splines obtained at the adjacent intervals $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$, we obtain formulas of local splines that can be used to solve various problems, for example, when solving differential equations by variational methods.

The graphs of the basis splines are presented in Figures 3, 4.

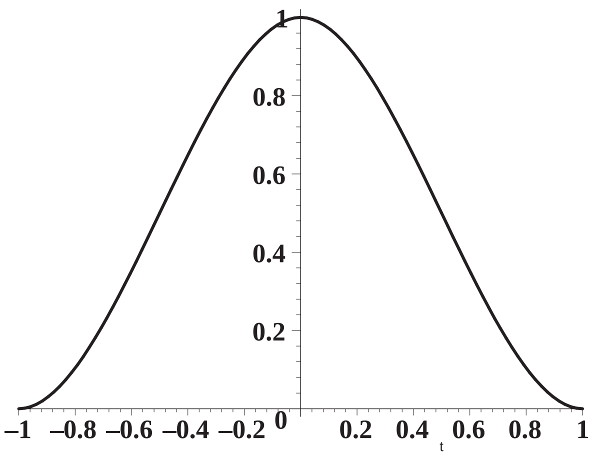

Fig. 3: Plot of the polynomial basis spline $w_{j,0}$.

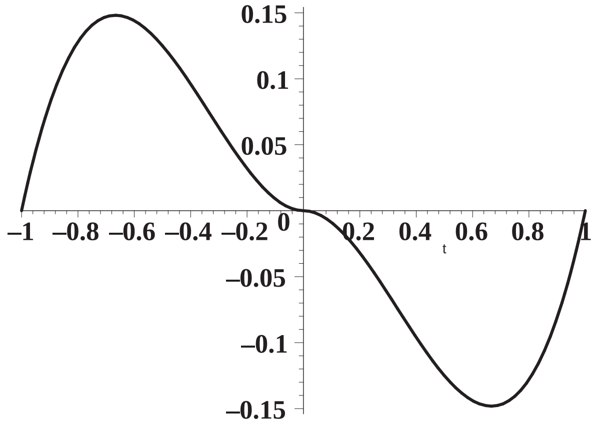

Fig. 4: Plot of the polynomial basis spline $w_{j,1}$.

Based on different systems φ_i, we will obtain different basis functions $w_{j,0}(x)$, $w_{j+1,0}(x)$, $w_{j,1}(x)$, $w_{j+1,1}(x)$. It is easy to prove that the system $\varphi_0(x) = 1$, $\varphi_1(x) = \sin(x)$, $\varphi_2(x) = \cos(x)$, $\varphi_3(x) = x$ is the Chebyshev system when $x \in [0, 1.5]$.

Let $x \in [x_j, x_{j+1}]$. Using (1), we construct the approximation of function $u(x)$ with these non-polynomial splines on the interval $[x_j, x_{j+1}]$ in the form:

$$U^T(x) = u(x_j)w^T_{j,0}(x)+u(x_{j+1})w^T_{j+1,0}(x)$$
$$+u'(x_j)w^T_{j,1}(x)+u'(x_{j+1})w^T_{j+1,1}(x).$$

The basis splines $w^T_{j,i}$ can be written, as follows:

$$w^T_{j,0}(x_j+th) = (\cos(h)-1+h\sin(h)-\sin(h)th$$
$$-\cos(th)+\cos(th-h))/Z(h),$$

$$w^T_{j+1,0}(x_j+th) = (\cos(h)-1+\sin(h)th+\cos(th)$$
$$-\cos(th-h))/Z(h),$$

$$w^T_{j,1}(x_j+th) = (\sin(h)-h\cos(h)-th-\sin(th)$$
$$+h\cos(th-h)+\sin(th-h)+th\cos(h))/Z(h),$$

$$w^T_{j+1,1}(x_j+th) = (-\sin(h)+h-th+\sin(th)$$
$$-\cos(th)h-\sin(th-h)+th\cos(h))/Z(h),$$

where $Z(h) = (2\cos(h)-2+h\sin(h))$.

The following Theorem is valid.

Theorem 2.2. Let function $u(x)$ be such that $u \in C^4([a,b])$. Suppose the basis splines $w^T_{j,0}(x)$, $w^T_{j+1,0}(x)$, $w^T_{j,1}(x)$, $w^T_{j+1,1}(x)$ are constructed when $\varphi_0(x) = 1$, $\varphi_1(x) = \sin(x)$, $\varphi_2(x) = \cos(x)$, $\varphi_3(x) = x$.

Let ordered distinct nodes $\{x_k\}$ be such that $x_{j+1} - x_j = h$, $h < 1.5$. Then, for $x \in [x_j, x_{j+1}]$, we have

$$|u(x)-U^T(x)| \leq K_2h^4\|u^{IV}+u''\|_{[x_j,x_{j+1}]}, \quad K_2 = 0.00267.$$

Proof. Using the results from paper [10], we get

$$u(x) = \int_{x_j}^{x} (u^{IV}(t) + u''(t))(\sin(t-x) + x - t)\,dt$$
$$+ c_1 + c_2 \sin(x) + c_3 \cos(x) + c_4 x,$$

where c_i, $i = 1,2,3,4$, are some arbitrary constants, $x \in [x_j, x_{j+1}]$. Using the expression $u(x)$ and derivative of it, expression $U^T(x)$ with $w^T_{j,i}$, we receive the estimation of the error of the approximation with the non-polynomial splines. The proof is complete.

Remark. When $h \to 0$, the next relations are valid:

$$w^T_{j,0}(x_j + th) = 1 - 3t^2 + 2t^3 + O(h),$$
$$w^T_{j+1,0}(x_j + th) = 3t^2 - 2t^3 + O(h),$$
$$w^T_{j,1}(x_j + th) = h(t - 2t^2 + t^3) + O(h^2),$$
$$w^T_{j+1,1}(x_j + th) = h(t^3 - t^2) + O(h^2).$$

We have obtained the function $U^P(x)$, $x \in [x_j, x_{j+1}]$, such that

$$u(x_j) = U^P(x_j),\; u'(x_j) = (U^P)'(x_j),$$
$$u(x_{j+1}) = U^P(x_{j+1}),\; u'(x_{j+1}) = (U^P)'(x_{j+1}),$$

using formula

$$U^P(x) = u(x_j)w_{j,0}(x) + u(x_{j+1})w_{j+1,0}(x)$$
$$+ u'(x_j)w_{j,1}(x) + u'(x_{j+1})w_{j+1,1}(x)$$

on every interval $[x_j, x_{j+1}]$ for the polynomial case.

Now, we can construct the piecewise function $\tilde{U}^P(x)$, $x \in [a,b]$, such that $\tilde{U}^P(x) = U^P(x)$ for $x \in [x_j, x_{j+1}]$. Also, we can construct the piecewise function $\tilde{U}^T(x)$, $x \in [a,b]$, such that $\tilde{U}^T(x) = U^T(x)$ for $x \in [x_j, x_{j+1}]$.

These piecewise functions $\tilde{U}^P(x)$ and $\tilde{U}^T(x)$ interpolate the function u and its first derivative u' at the nodes. Thus, $\tilde{U}^T(x)$ is a continuous function and its first derivative is also a continuous one.

The maximums of the theoretical errors $\left(\max_{x\in[-1,1]} |\tilde{U}^T(x) - u(x)|,\quad \max_{x\in[-1,1]} |\tilde{U}^P(x) - u(x)|\right)$ in absolute values are given in Table 1. The maximums of the actual errors in absolute values are presented in Table 2.

Plots of the error of the approximation of the function $\sin(2x)\cos(x)$ with the polynomial splines and with the non-polynomial splines are manifested in Figures 5, 6.

As mentioned above, we need the values of the first derivative of u at nodes for the construction of the approximation. Now, we aim to construct a piecewise function $\tilde{\tilde{U}}(x)$, so it will not only have the first, but also the second continuous derivative. Moreover, it will interpolate the function u in the nodes x_j, $j = 1, \ldots, n-1$. The way of constructing such piecewise functions in polynomial case is known (see, for example, papers of Kvasov B.I., Zavyalov Yu.S., Miroshnichenko V.L). We shall construct a piecewise function $\tilde{\tilde{U}}(x)$, $x \in [a,b]$, which is equal to $u(x_j)w_{j,0}(x) + u(x_{j+1})w_{j+1,0}(x) + c_j w_{j,1}(x) + c_{j+1} w_{j+1,1}(x)$ on every $[x_j, x_{j+1}]$. The parameters c_j, c_{j+1} are defined by the condition that the second derivative of $\tilde{\tilde{U}}(x)$ is continuous.

The piecewise approximation $\tilde{\tilde{U}}(x)$ will be such that it is continuous and the first two derivatives of the piecewise interpolation will also be continuous. Let c_{j-1}, c_j, c_{j+1} be some parameters to be defined, and $x = x_j + th$, $t \in [0,1]$. On every interval $[x_j, x_{j+1}]$, $j = 0, \ldots, n-1$, we construct the approximation in the form:

$$V(x_j + th) = c_j w_{j,1}(x_j + th) + c_{j+1} w_{j+1,1}(x_j + th)$$
$$+ u(x_j) w_{j,0}(x_j + th) + u(x_{j+1}) w_{j+1,0}(x_j + th). \quad (3)$$

Table 1: The theoretical errors of the polynomial and non-polynomial approximations of the Hermite type, $h = 0.1$, $[a,b] = [-1,1]$.

$u(x)$	Pol. app.	Non-pol.app.
$\sin(3x)$	$0.211 \cdot 10^{-4}$	$0.191 \cdot 10^{-4}$
$1 + x - \sin(x) - \cos(x) + \frac{x^2}{2}$	$0.368 \cdot 10^{-6}$	$0.267 \cdot 10^{-6}$
$\sin(7x) - \cos(9x)$	$0.233 \cdot 10^{-2}$	$0.234 \cdot 10^{-2}$
$\sin(2x) - \cos(x)$	$0.398 \cdot 10^{-5}$	$0.320 \cdot 10^{-5}$
$1/(1+25x^2)$	$0.391 \cdot 10^{-2}$	$0.399 \cdot 10^{-2}$

Table 2: The actual errors of the polynomial and non-polynomial approximations of the Hermite type, $h = 0.1$, $[a,b] = [-1,1]$.

$u(x)$	Pol.app.	Non-pol.app.
$\sin(3x)$	$0.192 \cdot 10^{-4}$	$0.170 \cdot 10^{-4}$
$1 + x - \sin(x) - \cos(x) + \frac{x^2}{2}$	$0.333 \cdot 10^{-6}$	$0.259 \cdot 10^{-6}$
$\sin(7x) - \cos(9x)$	$0.171 \cdot 10^{-2}$	$0.168 \cdot 10^{-2}$
$\sin(2x) - \cos(x)$	$0.391 \cdot 10^{-5}$	$0.292 \cdot 10^{-5}$
$1/(1+25x^2)$	$0.108 \cdot 10^{-2}$	$0.107 \cdot 10^{-2}$

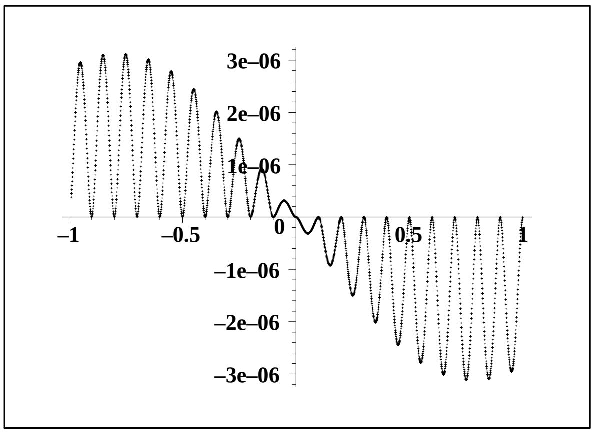

Fig. 5: Plot of the error of the approximation of the function $\sin(2x)\cos(x)$.

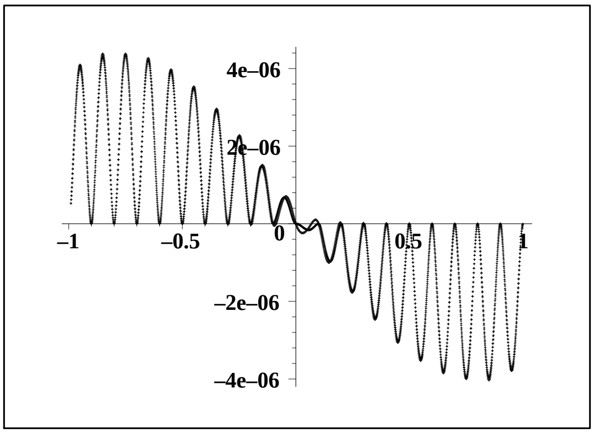

Fig. 6: Plot of the error of the approximation of the function $\sin(2x)\cos(x)$ with the non-polynomial splines.

We differentiate this expression twice and the similar one when $x \in [x_{j-1}, x_j]$. After that, set them equal to each other in the common node x_j, $j = 1,\dots,n-1$. Thus, we construct the system of equations:

$$c_{j-1}\tilde{w}''_{j-1,1}(1) + c_j\tilde{w}''_{j,1}(1) + u(x_{j-1})\tilde{w}''_{j-1,0}(1)$$
$$+ u(x_{j+1})\tilde{w}''_{j,0}(1) = c_j w''_{j,1}(0) + c_{j+1}w''_{j+1,1}(0)$$
$$+ u(x_j)w''_{j,0}(0) + u(x_{j+1})w''_{j+1,0}(0), \quad j = 1,\dots,n-1,$$

where functions $\tilde{w}_{j-1,1}(x_{j-1}+th)$, $\tilde{w}_{j,1}(x_{j-1}+th)$, $\tilde{w}_{j-1,0}(x_{j-1}+th)$, $\tilde{w}_{j,0}(x_{j-1}+th)$ are constructed when $x_{j-1}+th = x \in [x_{j-1}, x_j]$. We need two extra conditions at the ends of the interval $[a,b]$. Let $V'(a)$=$u'(a)$ and $V'(b)$=$u'(b)$. Taking into account the interpolation conditions $V(x_j) = u(x_j)$, $j = 0,\dots,n$, and the boundary conditions $V'(x_j)$=$u'(x_j)$, $j = 0,n$, we construct the piecewise function $\tilde{\tilde{U}}(x)$. This function and its first two derivatives will be continuous. It interpolates the function u in the nodes x_j, $j = 0,\dots,n$. Hence, we need the expressions for the second derivative of the basis functions $w_{j,i}$. The second derivative of the basis functions $w_{j,i}(t)$ can be easily obtained.

Thus, we have to solve the system of algebraic equations

$$GC = F,$$

with the square matrix $G = \{g_{ij}\}$, $G \in \mathbb{R}^{(n-1)\times(n-1)}$. The vector $F \in \mathbb{R}^{n-1}$ has the elements, as follows:

$$f_1 = -(u_2v_2 + u_0v_0) - u'_0 g_{1,2},$$
$$f_{n-1} = -(u_nv_n + u_{n-2}v_{n-2}) - u'_n g_{n-1,n},$$
$$f_j = -(u_{j+1}v_{j+1} + u_{j-1}v_{j-1}).$$

In the polynomial case, we have $v_j = 0$, $v_{j-1} = 3/(2h)$, $v_{j+1} = v_n = v_2 = -3/(2h)$, $v_0 = v_{n-2} = 3/(2h)$, $v_n = v_2 = -3/(2h)$. The matrix G has nonzero elements on only three adjacent diagonals. In the polynomial case with equidistant nodes x_j, the nonzero elements of the matrix G are as follows: $g_{j,j} = 2$, $g_{j,j+1} = 1/2$, $g_{j-1,j} = 1/2$, $j = 1,\dots,n-1$. In the non-polynomial case with equidistant nodes, we have:

$$g_{j,j} = h(2h\cos(h) - 2\sin(h))/z_1,$$
$$g_{j,j+1} = g_{j-1,j} = h(-h + \sin(h))/z_1,$$
$$v_{j-1} = -v_{j+1},\ v_j = 0,$$

where

$$v_{j+1} = h(1-\cos(h))/z_1,$$
$$z_1 = 4(2\cos(h) - 2 + h\sin(h)).$$

Solving the system with thridiagonal matrix G, we obtain the parameters c_j, $j = 1,\dots,n-1$. Using formula (3) on every $[x_j, x_{j+1}]$, $j = 1,\dots,n-1$, we construct the piecewise non-polynomial or polynomial approximation which is twice differential on $[a,b]$ and interpolates the function u in nodes x_j.

The degree of the polynomial spline with the first and second continuous derivatives is three.

The maximums of the actual errors in absolute values of the piecewise polynomial and non-polynomial approximations of the maximum smoothness with the fourth order of approximation are illustrated in Table 3.

Table 3: The actual errors of the smooth polynomial and non-polynomial approximations, $h = 0.1$, $[a,b] = [-1,1]$.

$u(x)$	Pol.app.	Non-pol.app.
$\sin(3x)$	$0.196\cdot10^{-4}$	$0.174\cdot10^{-4}$
$\sin(7x)-\cos(9x)$	$0.209\cdot10^{-2}$	$0.206\cdot10^{-2}$
$\frac{x^2}{2}+x+1-\sin(x)-\cos(x)$	$0.334\cdot10^{-6}$	$0.259\cdot10^{-6}$
$\sin(2x)-\cos(x)$	$0.393\cdot10^{-5}$	$0.294\cdot10^{-5}$
$1/(1+25x^2)$	$0.231\cdot10^{-2}$	$0.230\cdot10^{-2}$

3 Error of approximation with polynomial splines

In the case of the polynomial or non-polynomial basis functions, the system of equations $GC = F$ can be written in the form:

$$a_kc_{k-1} + b_kc_k + d_kc_{k+1} = f_k, \quad k = 1,\dots,n-1. \quad (4)$$

Let $\Delta_{n-1} = det(G)$.

Let us first estimate the errors of the approximation with twice continuously differentiable polynomial splines. In this case, system (4) is brought to the form

$$c_{k-1} + 4c_k + c_{k+1} = f_k, \quad k = 1,\dots,n-1,$$

where

$$f_k = 3(u(x_{k+1}) - u(x_{k-1}))/h, \quad k = 2,\dots,n-2,$$
$$f_1 = 3(u(x_2) - u(x_0))/h - u'(x_0),$$
$$f_{n-1} = 3(u(x_n) - u(x_{n-2}))/h - u'(x_n).$$

We also have the additional relations $c_0 = u'(x_0)$, $c_n = u'(x_n)$.

After changing to the unknowns $s_k = c_k - u'(x_k)$, system (4) is written as

$$s_0 = 0,\ s_n = 0, \qquad\qquad (5)$$
$$s_{k-1} + 4s_k + s_{k+1} = Q_k,\ \ k = 1,\dots,n-1.$$

where $Q_k = f_k - u'(x_{k-1}) - 4u'(x_k) - u'(x_{k+1})$.

Lemma 3.1. Let $c_0 = u'(x_0) = 0$, $c_n = u'(x_n) = 0$. The coefficients c_k determined by (4) satisfy the relation

$$|c_k - u'(x_k)| \le K_q,$$

where $K_q = Kh^4\|u^{(5)}\|$ with $K = 0.0333$.

Proof. As it is well-known (see [13]), $|s_k| \le \max_k |Q_k|$.

Representing $u'(x_{k-1})$, $u'(x_k)$, and $u'(x_{k+1})$ in expression (5) for Q_k as Taylor series expansions about the point x_k and combining terms, we obtain $|Q_k| \le 0.0333h^4\|u^{(5)}\|$.

Thus, the Lemma is proved for $K = 0.0333$ and $K_q = \max_k |Q_k|$.

Let

$$\tilde{G} = \begin{pmatrix} 4 & 1 & 0 & \dots & 0 & 0 \\ 1 & 4 & 1 & \dots & 0 & 0 \\ 0 & 1 & 4 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 4 \end{pmatrix},$$

$\tilde{\Delta}_{n-1} = det(\tilde{G})$, and C_n^k be the number of combinations of n elements taken k at a time. For convenience, we introduce $m = n-1$. It is easy to calculate the determinant of the system of equations.

Lemma 3.2 The following assertions are valid:

1.$\tilde{\Delta}_m = 4^m C^1_{m-1} 4^{m-2} + C^2_{m-2} 4^{m-4} - \dots.$

2.cond$(\tilde{G}) \le 3$.

Proof. 1. Decomposing $\tilde{\Delta}_m$ in terms of the elements of the first row yields the relation $\tilde{\Delta}_m = 4\tilde{\Delta}_{m-1} - \tilde{\Delta}_{m-2}$. Next, the required relation is derived in a similar manner to solve Problem 221 in [14].

2. By applying Gershgorin's theorem, it is easy to see that cond$(\tilde{\Delta}) \le 3$.

Theorem 3.1. Let $u \in C^4([a,b])$ and $\tilde{\tilde{U}}$ be a continuously differentiable approximation constructed with the help of polynomial basis splines. Then,

$$\|\tilde{\tilde{U}} - u\|_{[a,b]} \le h^4 \Big(0.0026\,\|u^{(4)}\|_{[a,b]} + 0.033\,\|u^{(5)}\|_{[a,b]}\Big) \qquad (6)$$

Proof. We have

$$\tilde{\tilde{U}}(x) - u(x) = c_j w_{j,1}(x) + c_{j+1}w_{j+1,1}(x) + u(x_j)w_{j,0}(x) + u(x_{j+1})w_{j+1,0}(x) - u(x).$$

Thus, we have

$$|\tilde{\tilde{U}}(x) - u(x)| \le |u'(x_j)w_{j,1}(x) + u'(x_{j+1})w_{j+1,1}(x) + u(x_j)w_{j,0}(x) + u(x_{j+1})w_{j+1,0}(x) - u(x)| + |(c_j - u'(x_j))w_{j,1}(x) + (c_{j+1} - u'(x_{j+1}))w_{j+1,1}(x)|.$$

Now, we have in view of Theorem 2.1 and Lemma 3.1

$$\|\tilde{\tilde{U}} - u\|_{[x_j,x_{j+1}]} \le h^4 \Big(0.0026\,\|u^{(4)}\|_{[x_j,x_{j+1}]} + 0.0333\,\|u^{(5)}\|_{[x_j,x_{j+1}]}\Big).$$

4 Error of Approximation with Non-polynomial Splines

Now, let us first estimate the errors of the twice continuously differentiable non-polynomial splines. In this case, system (4) is brought to the form

$$m_{k-1}c_{k-1} + m_k c_k + m_{k+1}c_{k+1} = f_k,\ k = 1,\dots,n-1,$$

where the m_{k-1}, m_k and m_{k+1} are the following:

$$m_{k-1} = m_{k+1} = \frac{h(\sin(h) - h)}{2(2\cos(h) - 2 + h\sin(h))},$$
$$m_k = \frac{h(-2\sin(h) + 2h\cos(h))}{2(2\cos(h) - 2 + h\sin(h))},$$

and the f_k are the following:

$$f_k = (Z_{k+1}u(x_{k+1}) + Z_{k-1}u(x_{k-1})),\ \ k = 1,\dots,n-1,$$
$$f_1 = (u(x_2)Z_{k+1} + u(x_0)Z_{k-1}) - u'(x_0)m_{k+1},$$
$$f_{n-1} = (u(x_n)Z_{k+1} + u(x_{n-2})Z_{k-1}) - u'(x_n)m_{k+1},$$

where

$$Z_{k+1} = -Z_{k-1} = \frac{h(1 - \cos(h))}{2(2\cos(h) - 2 + h\sin(h))}.$$

We also have $c_0 = u'(x_0)$, $c_n = u'(x_n)$.
Note that $m_k = 4 + O(h)$, $m_{k-1} = m_{k+1} = 1 + O(h)$.

After changing to the unknowns $s_k = c_k - u'(x_k)$, system (4) is written as

$$s_0 = 0,\ s_n = 0,$$
$$m_{k-1}s_{k-1} + m_k s_k + m_{k+1}s_{k+1} = Q_k,$$

where

$$Q_k = f_k - m_{k-1}u'(x_{k-1}) - m_k u'(x_k) - m_{k+1}u'(x_{k+1}),\ \ k = 1,\dots,n-1.$$

Lemma 4.1. Let $c_0 = u'(x_0) = 0$, $c_n = u'(x_n) = 0$. The coefficients c_k defined by (4) satisfy the relation $|c_k - U'(x_k)| \le Q_k$, where $Q_k = K_3\|u^{(3)}\| + K_5\|u^{(5)}\|$ with

$$K_3 = h^3 \frac{|-4h - h\cos(h) + 5\sin(h)|}{120|h\sin(h) + 2\cos(h) - 2|}, \qquad (7)$$

$$K_5 = h^3 \frac{|-2h - h\cos(h) + 3\sin(h)|}{6|h\sin(h) + 2\cos(h) - 2|}. \qquad (8)$$

The proof of the Lemma is similar to the proof given in Lemma 3.1.

Note that $K_3 = 0.0333h^4 + O(h^6)$, $K_5 = 0.0028h^4 + O(h^6)$.

Theorem 4.1 Let $u \in C^4[a,b]$ and $\tilde{\tilde{U}}$ be a twice continuously differentiable approximation constructed with the help of non-polynomial basis splines. Then,

$$\|\tilde{\tilde{U}} - u\|_{[a,b]} \le h^4 0.00267\ \|u^{II} + u^{IV}\|_{[a,b]} + K_3\ \|u^{III}\|_{[a,b]} + K_5\ \|u^{V}\|_{[a,b]}.$$

Proof. We have

$$\tilde{\tilde{U}}(x) - u(x) = c_j w^T_{j,1}(x) + c_{j+1} w^T_{j+1,1}(x) + u(x_j) w^T_{j,0}(x) + u(x_{j+1}) w^T_{j+1,0}(x) - u(x).$$

Thus, we have

$$|\tilde{\tilde{U}}(x) - u(x)| \le |u'(x_j) w^T_{j,1}(x) + u'(x_{j+1}) w^T_{j+1,1}(x) + u(x_j) w^T_{j,0}(x) + u(x_{j+1}) w^T_{j+1,0}(x) - u(x)| + |(c_j - u'(x_j)) w^T_{j,1}(x) + (c_{j+1} - u'(x_{j+1})) w^T_{j+1,1}(x)|.$$

Now we have in view of Theorem 2.2 and Lemma 4.1, the statement:

$$\|\tilde{\tilde{U}} - u\|_{[x_j,x_{j+1}]} \le h^4 0.00267\ \|u^{II} + u^{IV}\|_{[x_j,x_{j+1}]} + K_3\|u^{III}\|_{[x_j,x_{j+1}]} + K_5\|u^{V}\|_{[x_j,x_{j+1}]},$$

where K_3 and K_5 are given in (7) and (8).

The proof is complete.

The plot of the function K_3 is given in Figure 7. The plot of the function K_5 is given in Figure 8.

The plots of the error of the approximation of the function $\sin(7x) - \cos(9x)$, with the polynomial splines (thin line) and with the smooth polynomial splines (thick line) (when $h = 0.1$), are given in Figure 9. The plots of the error of the approximation of the function $\sin(7x) - \cos(9x)$, with the non-polynomial splines (thin line) and with the smooth non-polynomial splines (thick line) (when $h = 0.1$), are given in Figure 10.

The maximums of the theoretical errors in absolute values of the piecewise polynomial and non-polynomial approximations of the maximum smoothness with the fourth order of approximation are given in Table 4.

The maximums of the theoretical and actual errors in absolute values of the piecewise non-polynomial approximations of the maximum smoothness with the fourth order of approximation are given in Table 5.

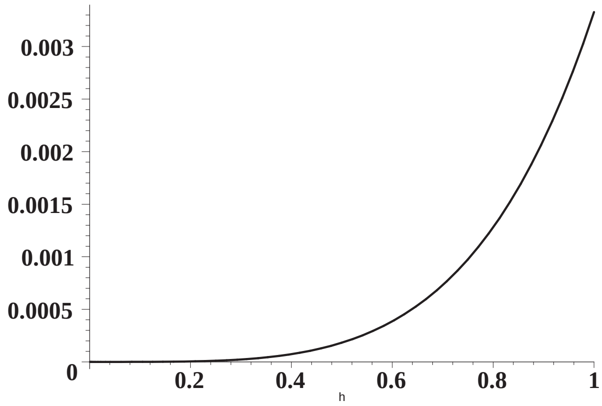

Fig. 7: The plots of the functions: K_3.

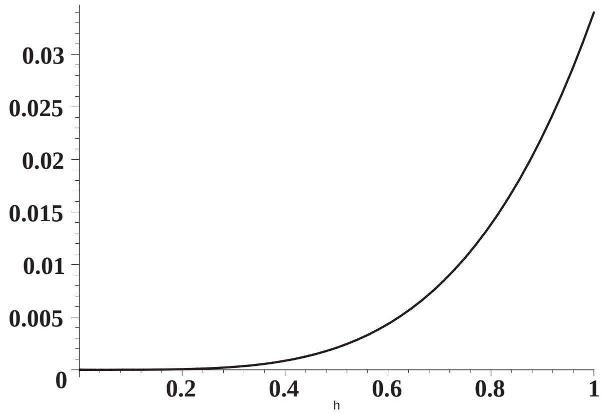

Fig. 8: The plots of the functions: K_5.

Table 4: The theoretical errors of the smooth polynomial and non-polynomial approximations, $h = 0.1$, $[a,b] = [-1,1]$.

$u(x)$	Pol.app.	Non-pol.app.
$\sin(3x)$	$0.110 \cdot 10^{-3}$	$0.177 \cdot 10^{-3}$
$\sin(7x) - \cos(9x)$	$0.585 \cdot 10^{-2}$	$0.271 \cdot 10^{-1}$
$\frac{x^2}{2} + x + 1 - \sin(x) - \cos(x)$	$0.503 \cdot 10^{-5}$	$0.538 \cdot 10^{-5}$
$\sin(2x) - \cos(x)$	$0.271 \cdot 10^{-4}$	$0.352 \cdot 10^{-4}$
$1/(1+25x^2)$	$0.583 \cdot 10^{-2}$	$0.940 \cdot 10^{-1}$

5 About formulas for the numerical differentiation

In this section, we return to the formulas for approximating the functions that contain the first derivatives. The advantage of this method is the presence of a local interpolation basis, and thereby, the possibility of using this method in real time. However, the disadvantage is the fact that the derivative is always unknown, so we need to replace it with the sum of the values of the function at the grid nodes. In this case, one must remember the fatal error of the numerical

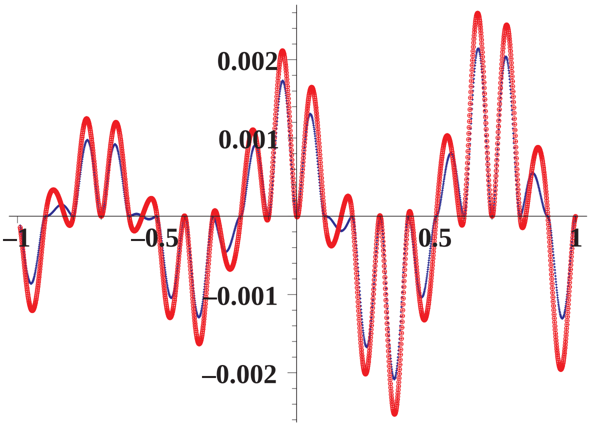

Fig. 9: The plots of the error of the approximation of the function $\sin(7x) - \cos(9x)$: with the polynomial splines (thin blue line) and with the smooth polynomial splines (thick red line).

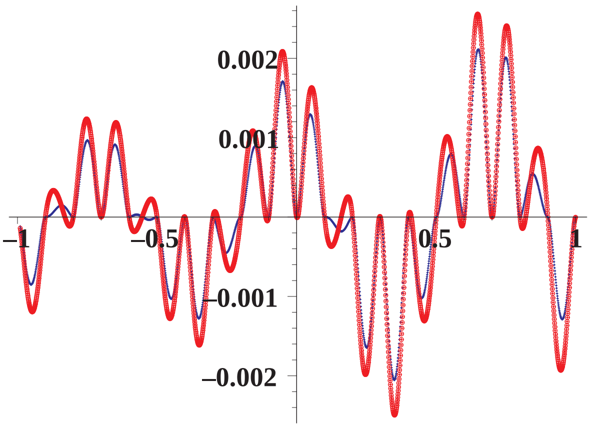

Fig. 10: The plots of the error of the approximation of the function $\sin(7x) - \cos(9x)$ with the non-polynomial splines (thin blue line) and with the smooth non-polynomial splines (thick red line).

Table 5: The theoretical and actual errors of the smooth non-polynomial approximations, $h = 0.01$, $[a,b] = [-1,1]$.

$u(x)$	Actual.err.	Theor.err.
$\sin(3x)$	$0.186 \cdot 10^{-8}$	$0.177 \cdot 10^{-7}$
$\sin(7x) - \cos(9x)$	$0.217 \cdot 10^{-6}$	$0.271 \cdot 10^{-5}$
$\frac{x^2}{2} + x + 1 - \sin(x) - \cos(x)$	$0.260 \cdot 10^{-10}$	$0.538 \cdot 10^{-9}$
$\sin(2x) - \cos(x)$	$0.311 \cdot 10^{-9}$	$0.352 \cdot 10^{-8}$
$1/(1+25x^2)$	$0.332 \cdot 10^{-6}$	$0.939 \cdot 10^{-5}$

differentiation formulas when a grid step is too small compared with the error of numbers rounding. Accordingly, we should not take too small grid step In numerical calculations, we need to remember the calculation error (calculation error associated with the data type used).

If we don't want to use the values of the first derivatives in the formulas of the approximation with polynomial or non-polynomial splines we could apply the formulas for the numerical differentiation. For the polynomial case, some of them are well-known.

We derive formulas for the first-order derivatives using splines of the Lagrangian type with the fifth order of approximation. We need formulas appropriate for approximation near the boundaries of a finite interval $[a,b]$, and in the middle of this interval. First, consider the polynomial case.

Lemma 5.1. Let $u \in C^{(5)}([a,b])$, $x_j = a + jh$, $h = \text{const}$. The following formulas are valid:

To approximate the first derivative of the function in the middle of the interval $[a,b]$, we can use the formulas:

$$1) u'(x_j) = \big(s_j^0 u(x_j) + s_{j+1}^0 u(x_{j+1}) + s_{j+2}^0 u(x_{j+2}) + s_{j-1}^0 u(x_{j-1}) + s_{j-2}^0 u(x_{j-2})\big) + R_1,$$

where $|R_1| \leq 0.05 \cdot h^4 \|u^{(5)}\|_{[x_{j-2},x_{j+2}]}$,

$$s_j^0 = 0, \;\; s_{j+1}^0 = 2/(3h),$$
$$s_{j+2}^0 = -1/(12h), \;\; s_{j-1}^0 = -2/(3h), \;\; s_{j-2}^0 = 1/(12h).$$

$$2) u'(x_{j+1}) = \big(s_j^1 u(x_j) + s_{j+1}^1 u(x_{j+1}) + s_{j+2}^1 u(x_{j+2}) + s_{j-1}^1 u(x_{j-1}) + s_{j-2}^1 u(x_{j-2})\big) + R_2,$$

where $|R_2| \leq 0.05 \cdot h^4 \|u^{(5)}\|_{[x_{j-2},x_{j+2}]}$,

$$s_j^1 = -3/(2h), \;\; s_{j+1}^1 = -3/(2h),$$
$$s_{j+2}^1 = 1/(4h), \;\; s_{j-1}^1 = 1/(2h), \;\; s_{j-2}^1 = -1/(12h).$$

To approximate the first derivative of the function on the left of the right end of the interval $[a,b]$, we can use the formulas

$$3) u'(x_j) = \big(s_j^0 u(x_j) + s_{j+1}^0 u(x_{j+1}) + s_{j-3}^0 u(x_{j-3}) + s_{j-1}^0 u(x_{j-1}) + s_{j-2}^0 u(x_{j-2})\big) + R_3,$$

where $|R_3| \leq 0.165 h^4 \|u^{(5)}\|_{[x_{j-3},x_{j+1}]}$,

$$s_j^0 = 5/(6h), S_{j+1}^0 = 1/(4h),$$
$$s_{j-3}^0 = -1/(12h), s_{j-1}^0 = -3/(2h), s_{j-2}^0 = 1/(2h),$$

$$4) u'(x_{j+1}) = \big(s_j^1 u(x_j) + s_{j+1}^1 u(x_{j+1}) + s_{j-3}^1 u(x_{j-3}) + s_{j-1}^1 u(x_{j-1}) + s_{j-2}^1 u(x_{j-2})\big) + R_4.$$

where $|R_4| \leq 0.165 \, h^4 \|u^{(5)}\|_{[x_{j-3},x_{j+1}]}$,

$$s_j^1 = -4/h, \; s_{j+1}^1 = -4/h,$$
$$s_{j-3}^1 = 1/(4h), \; s_{j-1}^1 = 3/h, \; s_{j-2}^1 = -4/(3h).$$

To approximate the first derivative of the function on the right of the left end of the interval $[a,b]$, we can use the formulas:

$$5) u'(x_j) = \big(s_j^0 u(x_j) + s_{j+1}^0 u(x_{j+1}) + s_{j+2}^0 u(x_{j+2}) + s_{j+3}^0 u(x_{j+3}) + s_{j+4}^0 u(x_{j+4})\big) + R_5,$$

where $|R_5| \le h^4 \|u^{(5)}\|_{[x_j, x_{j+4}]}$,

$$s_j^0 = -25/(12h),\ s_{j+1}^0 = 4/h,\ s_{j+2}^0 = -3/h,\ s_{j+3}^0 = 4/(3h),\ s_{j+4}^0 = -1/(4h).$$

$$6) u'(x_{j+1}) = \big(s_j^1 u(x_j) + s_{j+1}^1 u(x_{j+1}) + s_{j+2}^1 u(x_{j+2}) + s_{j+3}^1 u(x_{j+3}) + s_{j+4}^1 u(x_{j+4})\big) + R_6,$$

where $|R_6| \le h^4 \|u^{(5)}\|_{[x_j, x_{j+4}]}$,

$$s_j^1 = -1/(4h),\ s_{j+1}^1 = -1/(4h),\ s_{j+2}^1 = 3/(2h),\ s_{j+3}^1 = -1/(2h),\ s_{j+4}^1 = 1/(12h).$$

Proof. Let us prove the statement for case 6). The other cases can be proved similarly. First let us construct the basis functions W_k of the Lagrange type. The approximation for the function $u(x)$ on the interval $[x_j, x_{j+1}]$ can be constructed in the form:

$$\tilde{u}(x) = u(x_j)W_j(x) + u(x_{j+1})W_{j+1}(x) + u(x_{j+2})W_{j+2}(x) + u(x_{j+3})W_{j+3}(x) + u(x_{j+4})W_{j+4}(x)$$

using the relation

$$u(x) = \tilde{u}(x),\quad u = 1, x, x^2, x^3, x^4.$$

The system of equations that we have to solve is the following:

$$x_j^i W_j(x) + x_{j+1}^i W_{j+1}(x) + x_{j+2}^i W_{j+2}(x) + x_{j+3}^i W_{j+3}(x) + x_{j+4}^i W_{j+4}(x) = x^i,\quad i = 0,1,2,3,4.$$

We have the next relations for the first derivatives of W_k:

$$W_j'(x) + W_{j+1}'(x) + W_{j+2}'(x) + W_{j+3}'(x) + W_{j+4}'(x) = 0,$$
$$x_j^i W_j'(x) + x_{j+1}^i W_{j+1}'(x) + x_{j+2}^i W_{j+2}'(x) + x_{j+3}^i W_{j+3}'(x) + x_{j+4}^i W_{j+4}'(x) = i\, x^{i-1},\quad i = 1,2,3,4.$$

Differentiating the functions W_k, we obtain for $x = x_j + th$, $t \in [0,1]$:

$$W_j'(x_j + th) = (2t^3 - 15t^2 + 35t - 25)/(12h),$$
$$W_{j+1}'(x_j + th) = (24 - 52t + 27t^2 - 4t^3)/(6h),$$
$$W_{j+2}'(x_j + th) = (-6 + 19t - 12t^2 + 2t^3)/(2h),$$
$$W_{j+3}'(x_j + th) = (21t^2 + 8 - 28t - 4t^3)/(6h),$$
$$W_{j+4}'(x_j + th) = (-3 + 11t - 9t^2 + 2t^3)/(12h).$$

If we put $t = 0$, we get formulas 5). If we put $t = 1$, we get formulas 6).

We have to obtain the error of the approximation $|u'(x) - \tilde{u}'(x)|$, where

$$\tilde{u}'(x) \approx u(x_j)W_j'(x) + u(x_{j+1})W_{j+1}'(x) + u(x_{j+2})W_{j+2}'(x) + u(x_{j+3})W_{j+3}'(x) + u(x_{j+4})W_{j+4}'(x).$$

Using the Tailor expansion in the vicinity of x_j with the remainder in the integral form for $u(x_{j+i})$, $i = 1,2,3,4$, we get the error in the interval $[x_j, x_{j+1}]$: $|R_5| \le h^4 \|u^{(5)}\|_{[x_j, x_{j+4}]}$.

Now, consider the non-polynomial case.

Lemma 5.2. Let $u \in C^{(5)}([a,b])$, $x_j = a + jh$, $h = \text{const}$. The following formulas are valid:

To approximate the first derivative of the function in the middle of the interval $[a,b]$, we can use the formulas:

$$1) u'(x_j) = \big(s_j^0 u(x_j) + s_{j+1}^0 u(x_{j+1}) + s_{j+2}^0 u(x_{j+2}) + s_{j-1}^0 u(x_{j-1}) + s_{j-2}^0 u(x_{j-2})\big) + R_1,$$

where $|R_1| \le K\, h^4 \|u^{(5)}\|_{[x_{j-2}, x_{j+2}]}$, $K = 0.05$, $s_j^0 = 0$,

$$s_{j+1}^0 = (2\sin(h) - 6\sin(3h) + 4\sin(2h) - 3h\cos(3h) - 6h + 6h\cos(2h) + 2\sin(4h) + 3h\cos(h)) \,/\, (6h\sin(3h) + 14h\sin(h) - 14h\sin(2h) - h\sin(4h)),$$

$$s_{j+2}^0 = (\sin(3h) + 5\sin(h) - 4\sin(2h) - 6h - 2h\cos(2h) + 8h\cos(h)) \,/\, (12h\sin(3h) + 28h\sin(h) - 28h\sin(2h) - 2h\sin(4h)),$$

$$s_{j-1}^0 = (-4\sin(3h) - 4\sin(h) + 6\sin(2h) + \sin(4h) - 12h - 4h\cos(2h) + 16h\cos(h)) \,/\, (12h\sin(3h) + 28h\sin(h) - 28h\sin(2h) - 2h\sin(4h)),$$

$$s_{j-2}^0 = (-\sin(3h) - 5\sin(h) + 4\sin(2h) + 6h - 8h\cos(h) + 2h\cos(2h)) \,/\, (12h\sin(3h) + 28h\sin(h) - 28h\sin(2h) - 2h\sin(4h)).$$

$$2) u'(x_{j+1}) = \big(s_j^1 u(x_j) + s_{j+1}^1 u(x_{j+1}) + s_{j+2}^1 u(x_{j+2}) + s_{j-1}^1 u(x_{j-1}) + s_{j-2}^1 u(x_{j-2})\big) + R_2,$$

where $|R_2| \le 0.05 h^4 \|u^{(5)}\|_{[x_{j-2}, x_{j+2}]}$,

$$s_j^1 = (2\sin(h) - 6\sin(3h) + 4\sin(2h) - 3h\cos(3h) - 6h + 6h\cos(2h) + 2\sin(4h) + 3h\cos(h)) \,/\, (6h\sin(3h) + 14h\sin(h) - 14h\sin(2h) - h\sin(4h)),$$

$$s_{j+1}^1 = (2\sin(h) - 6\sin(3h) + 4\sin(2h) - 3h\cos(3h) - 6h + 6h\cos(2h) + 2\sin(4h) + 3h\cos(h)) \,/\, (6h\sin(3h) + 14h\sin(h) - 14h\sin(2h) - h\sin(4h)),$$

$s^1_{j+2} = (3\sin(3h)+15\sin(h)-12\sin(2h)-6h\cos(h)+2h$
$-2h\cos(3h)+6h\cos(2h))\,/\,(12h\sin(3h)+28h\sin(h)$
$-28h\sin(2h)-2h\sin(4h))\,,$

$s^1_{j-1} = (-16\sin(h)+10\sin(2h)-\sin(4h)-18h\cos(h)$
$+2h\cos(3h)+16h)\,/\,(12h\sin(3h)+28h\sin(h)$
$-28h\sin(2h)-2h\sin(4h))\,,$

$s^1_{j-2} = (\sin(3h)+5\sin(h)-4\sin(2h)-6h-2h\cos(2h)$
$+8h\cos(h))\,/\,(12h\sin(3h)+28h\sin(h)$
$-28h\sin(2h)-2h\sin(4h))\,.$

To approximate the first derivative of the function on the left of the right end of the interval $[a,b]$, we can use the formulas

$$3)\,u'(x_j) = \big(s^0_j u(x_j)+s^0_{j+1}u(x_{j+1})+s^0_{j-3}u(x_{j-3}) +s^0_{j-1}u(x_{j-1})+s^0_{j-2}u(x_{j-2})\big)+R_3,$$

where $|R_3| \le 0.18\, h^4\|u^{(5)}\|_{[x_{j-3},x_{j+1}]}$,

$s^0_j = (-2\sin(2h)-8\sin(h)+8\sin(3h)+10h\cos(h)$
$-3\sin(4h)-16h\cos(2h)+6h\cos(3h))\,/\,(28h\sin(h)$
$-28h\sin(2h)+12h\sin(3h)-2h\sin(4h))\,,$

$s^0_{j+1} = (-12\sin(2h)+15\sin(h)+3\sin(3h)+2h$
$+6h\cos(2h)-6h\cos(h)-2h\cos(3h))\,/\,(-28h\sin(2h)$
$+28h\sin(h)+12h\sin(3h)-2h\sin(4h))\,,$

$s^0_{j-3} = (5\sin(h)-4\sin(2h)+\sin(3h)-6\,h+8h\cos(h)$
$-2h\cos(2h))\,/\,(-28h\sin(2h)+28h\sin(h)$
$+12h\sin(3h)-2h\sin(4h))\,,$

$s^0_{j-1} = (4\sin(2h)+2\sin(h)-6\sin(3h)-6h+2\sin(4h)$
$+3h\cos(h)+6h\cos(2h)-3h\cos(3h))\,/\,(14h\sin(h)$
$-14h\sin(2h)+6h\sin(3h)-h\sin(4h))\,,$

$s^0_{j-2} = (10\sin(2h)-16\sin(h)-\sin(4h)-18h\cos(h)$
$+16h+2h\cos(3h))\,/\,(-28h\sin(2h)+28h\sin(h)$
$+12h\sin(3h)-2h\sin(4h))\,,$

$$4)\,u'(x_{j+1}) = \big(s^1_j u(x_j)+s^1_{j+1}u(x_{j+1})+s^1_{j-3}u(x_{j-3}) +s^1_{j-1}u(x_{j-1})+s^1_{j-2}u(x_{j-2})\big)+R_4,$$

where $|R_4| \le 0.18\, h^4\|u^{(5)}\|_{[x_{j-3},x_{j+1}]}$,

$s^1_j = (2\sin(2h)-20\sin(h)+12\sin(3h)-2h-5\sin(4h)$
$-16h\cos(3h)+12h\cos(2h)+6h\cos(4h))\,/\,(28h\sin(h)$
$-28h\sin(2h)+12h\sin(3h)-2h\sin(4h))\,,$

$s^1_{j+1} = (2\sin(2h)-20\sin(h)+12\sin(3h)-5\sin(4h)-2h$
$-16h\cos(3h)+12h\cos(2h)+6h\cos(4h))\,/\,(28h\sin(h)$
$-28h\sin(2h)+12h\sin(3h)-2h\sin(4h))\,,$

$s^1_{j-3} = (-12\sin(2h)+15\sin(h)+3\sin(3h)+6h\cos(2h)$
$-6h\cos(h)+2h-2h\cos(3h))\,/\,(-28h\sin(2h)$
$+28h\sin(h)+12h\sin(3h)-2h\sin(4h))\,,$

$s^1_{j-1} = (8\sin(2h)+4\sin(h)-12\sin(3h)-6h\cos(h)$
$+4\sin(4h)+3h+6h\cos(3h)-3h\cos(4h))\,/\,(14h\sin(h)$
$-14h\sin(2h)+6h\sin(3h)-h\sin(4h))\,,$

$s^1_{j-2} = (14\sin(2h)-28\sin(h)-3\sin(4h)+16h\cos(h)$
$-6h-12h\cos(2h)+2h\cos(4h)+4\sin(3h))\,/\,(28h\sin(h)$
$-28h\sin(2h)+12h\sin(3h)-2h\sin(4h))\,.$

To approximate the first derivative of the function on the right of the left end of the interval $[a,b]$, we can use the formulas:

$$5)\,u'(x_j) = \big(s^0_j u(x_j)+s^0_{j+1}u(x_{j+1})+s^0_{j+2}u(x_{j+2}) +s^0_{j+3}u(x_{j+3})+s^0_{j+4}u(x_{j+4})\big)+R_5,$$

where $|R_5| \le 1.05\, h^4\|u^{(5)}\|_{[x_j,x_{j+4}]}$,

$s^0_j = (2h\cos(4h)-5\sin(3h)+20\sin(2h)-25\sin(h)$
$+6h\cos(2h)-6h\cos(3h)-2h\cos(h))\,/\,(-2h\sin(4h)$
$+12h\sin(3h)-28h\sin(2h)+28h\sin(h))\,,$

$s^0_{j+1} = (5\sin(4h)-6h\cos(4h)-12\sin(3h)-2\sin(2h)$
$+20\sin(h)-12h\cos(2h)+2h$
$+16h\cos(3h))\,/\,(-2h\sin(4h)+12h\sin(3h)$
$-28h\sin(2h)+28h\sin(h))\,,$

$s^0_{j+2} = (12\sin(3h)-4\sin(4h)-4\sin(h)+3h\cos(4h)$
$-8\sin(2h)-3h-6h\cos(3h)+6h\cos(h))\,/\,(6h\sin(3h)$
$-h\sin(4h)-14h\sin(2h)+14h\sin(h))\,,$

$s^0_{j+3} = (3\sin(4h)+28\sin(h)-2h\cos(4h)-4\sin(3h)$
$-14\sin(2h)+6h+12h\cos(2h)$
$-16h\cos(h))\,/\,(12h\sin(3h)-2h\sin(4h)$
$-28h\sin(2h)+28h\sin(h))\,,$

$s^0_{j+4} = (-2h-3\sin(3h)+12\sin(2h)-15\sin(h)$
$-6h\cos(2h)+2h\cos(3h)+6h\cos(h))\,/\,(-2h\sin(4h)$
$+12h\sin(3h)-28h\sin(2h)+28h\sin(h))\,.$

$$6)\,u'(x_{j+1}) = \big(s^1_j u(x_j)+s^1_{j+1}u(x_{j+1})+s^1_{j+2}u(x_{j+2}) +s^1_{j+3}u(x_{j+3})+s^1_{j+4}u(x_{j+4})\big)+R_6,$$

where $|R_6| \leq 1.05\ h^4 \|u^{(5)}\|_{[x_j,x_{j+4}]}$,

$$s_j^1 = (-2h - 3\sin(3h) + 12\sin(2h) - 15\sin(h) -6h\cos(2h) + 2h\cos(3h) + 6h\cos(h)) / (-2h\sin(4h) +12h\sin(3h) - 28h\sin(2h) + 28h\sin(h)),$$

$$s_{j+1}^1 = (-2h - 3\sin(3h) + 12\sin(2h) - 15\sin(h) -6h\cos(2h) + 2h\cos(3h) + 6h\cos(h)) / (-2h\sin(4h) +12h\sin(3h) - 28h\sin(2h) + 28h\sin(h)),$$

$$s_{j+2}^1 = (-2\sin(4h) - 3h\cos(h) + 6\sin(3h) + 3h\cos(3h) -4\sin(2h) - 2\sin(h) - 6h\cos(2h) + 6h) / (-h\sin(4h) +6h\sin(3h) - 14h\sin(2h) + 14h\sin(h)),$$

$$s_{j+3}^1 = (\sin(4h) + 18h\cos(h) - 2h\cos(3h) - 10\sin(2h) +16\sin(h) - 16h) / (-2h\sin(4h) +12h\sin(3h) - 28h\sin(2h) + 28h\sin(h)),$$

$$s_{j+4}^1 = (4\sin(2h) - 8h\cos(h) - \sin(3h) - 5\sin(h) + 6h +2h\cos(2h)) / (-2h\sin(4h) + 12h\sin(3h) -28h\sin(2h) + 28h\sin(h)).$$

Proof. Let us prove the statement for case 6). The other cases can be proved similarly. First let us construct the basis functions $\tilde{W}_k$ of the Lagrange type. The approximation for the function $u(x)$ on the interval $[x_j, x_{j+1}]$ can be constructed in the form

$$\tilde{u}(x) = u(x_j)\tilde{W}_j(x) + u(x_{j+1})\tilde{W}_{j+1}(x) + u(x_{j+2})\tilde{W}_{j+2}(x) + u(x_{j+3})\tilde{W}_{j+3}(x) + u(x_{j+4})\tilde{W}_{j+4}(x)$$

using the relations $u(x) = \tilde{u}(x)$, $u(x) = \varphi_i(x)$, $i = 0,1,2,3,4$, where $\varphi_0(x) = 1$, $\varphi_1(x) = x$, $\varphi_2(x) = x^2$, $\varphi_3(x) = \sin(x)$, $\varphi_4(x) = \cos(x)$. The system of equations that we have to solve is the following:

$$\varphi_i(x_j)\tilde{W}_j(x) + \varphi_i(x_{j+1})\tilde{W}_{j+1}(x) + \varphi_i(x_{j+2})\tilde{W}_{j+2}(x) + \varphi_i(x_{j+3})\tilde{W}_{j+3}(x) + \varphi_i(x_{j+4})\tilde{W}_{j+4}(x) = \varphi_i(x), \quad i = 0,1,2,3,4.$$

Differentiating the functions $\tilde{W}_{j+i}$, we obtain for $x = x_j + th$, $t \in [0,1]$:

$$\tilde{W}_j'(x_j + th) = (25\sin(h) - 20\sin(2h) + 2h\cos(th - h) + 6h\cos(th - 3h) + 5\sin(3h) - 2h\cos(th - 4h) - 2t\sin(3h) - 6h\cos(th - 2h) - 10t\sin(h) + 8t\sin(2h))/(-28h\sin(h) + 28h\sin(2h) + 2h\sin(4h) - 12h\sin(3h)),$$

$$\tilde{W}_{j+1}'(x_j + th) = (20\sin(h) - 2\sin(2h) + 5\sin(4h) - 12\sin(3h) + 16h\cos(th - 3h) - 6h\cos(th - 4h) - 12h\cos(th - 2h) + 4t\sin(3h) - 2t\sin(4h) - 12t\sin(h) + 4t\sin(2h) + 2h\cos(th))/(28h\sin(h) - 28h\sin(2h) - 2h\sin(4h) + 12h\sin(3h)),$$

$$\tilde{W}_{j+2}'(x_j + th) = (-4\sin(h) - 8\sin(2h) - 4\sin(4h) + 12\sin(3h) + 6h\cos(th - h) - 6h\cos(th - 3h) + 3h\cos(th - 4h) - 6t\sin(3h) + 2t\sin(4h) + 2t\sin(h) + 4t\sin(2h) - 3h\cos(th))/(14h\sin(h) - 14h\sin(2h) - h\sin(4h) + 6h\sin(3h)),$$

$$\tilde{W}_{j+3}'(x_j + th) = (28\sin(h) - 14\sin(2h) + 3\sin(4h) - 4\sin(3h) - 16h\cos(th - h) - 2h\cos(th - 4h) + 12h\cos(th - 2h) + 4t\sin(3h) - 2t\sin(4h) - 12t\sin(h) + 4t\sin(2h) + 6h\cos(th))/(28h\sin(h) - 28h\sin(2h) - 2h\sin(4h) + 12h\sin(3h)),$$

$$\tilde{W}_{j+4}'(x_j + th) = (15\sin(h) - 12\sin(2h) + 3\sin(3h) - 6h\cos(th - h) - 2h\cos(th - 3h) + 6h\cos(th - 2h) - 2t\sin(3h) - 10t\sin(h) + 8t\sin(2h) + 2h\cos(th))/(-28h\sin(h) + 28h\sin(2h) + 2h\sin(4h) - 12h\sin(3h)).$$

If we put $t = 0$, we get formulas 5). If we put $t = 1$, we get formulas 6).

We have to obtain the error of the approximation $|u'(x) - \tilde{u}'(x)|$, where

$$\tilde{u}'(x) \approx u(x_j)\tilde{W}_j'(x) + u(x_{j+1})\tilde{W}_{j+1}'(x) + u(x_{j+2})\tilde{W}_{j+2}'(x) + u(x_{j+3})\tilde{W}_{j+3}'(x) + u(x_{j+4})\tilde{W}_{j+4}'(x).$$

Using the Tailor expansion in the vicinity of x_j with the remainder in the integral form for $u(x_{j+i})$, $i = 1,2,3,4$, we get the error in the interval $[x_j, x_{j+1}]$:

$$|R_5| \leq h^4 K \|u^{(5)}\|_{[x_j,x_{j+4}]}, \quad K = 1.05.$$

6 Application to the describing an experimental distribution law

An approximating function describing an experimental distribution law may be found with the help of the Chebyshev–Hermite polynomials. By applying regularization, the stability of the solution to the system can be improved. In [15], this problem was solved with the help of the polynomial integro-differential splines proposed by the author. The integrals which are in the approximations were approximately evaluated using the trapezoidal rule, and the first derivatives were replaced by difference relations with an $O(h^3)$ error. Let us solve the problem of obtaining an approximating function describing an experimental distribution law with the help of the polynomial splines proposed above. In practice, the probability approach as applied to measurement error estimation assumes primarily that an analytical model is known for the error distribution law and that distributions encountered in metrology are fairly diverse. In addition, according to a particular study, roughly half of the distributions were exponential, one fifth consisted of various bimodal distributions, and the others were flattened. Suppose that the bimodal distribution density $f = (f1 + f2)/2$, where

$$f_i = \frac{1}{\sqrt{2\pi}\sigma_i} exp(-(x-\alpha_i)^2/(2\sigma_i^2)),\ i = 1,2,$$

$\sigma_1 = 0.5$, $\sigma_2 = 0.8$, $\alpha_1 = -0.8$, $\alpha_2 = 1$ is approximated with the polynomial and non-polynomial splines on the interval [2, 3]. The first derivatives were replaced with the relations from section 4 about formulas for the numerical differentiation. We used results from Lemma 5.1. Formulas 1), 2) from Lemma 5.1 were used everywhere except for three intervals to the right and left of the boundaries of the interval $[a,b]$. Formulas 5), 6) were used to the right of the boundary a. Formulas 3), 4) were used to the left of the boundary b. Thus, we constructed an approximation with an approximation error of the order $O(h^4)$. Figure 11 shows the histogram, the density, and its approximation constructed with the polynomial splines in Maple when $h = 0.1$, $Digits = 15$. Figure 12 indicates the error of the approximation of the density.

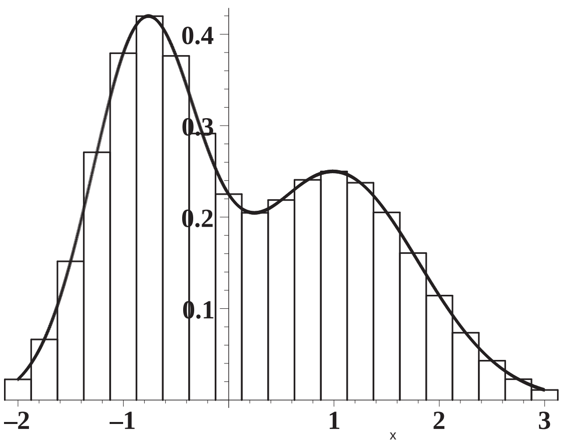

Fig. 11: The plots of the histogram, the density, and its approximation

If we calculate the error of the approximation using Theorem 2.1, we obtain the theoretical value $0.496 \cdot 10^{-5}$. When we approximate the first derivative using formulas from Lemma 5.1, we get the actual error of the approximation $0.637 \cdot 10^{-5}$. The order of the approximation is the same. We can construct other formulas for approximation of the first derivative. For example, in the middle of the interval $[a,b]$, we can use the formula $u'(x_j) = u(x_{j+3})/(30h) + u(x_{j-2})/(20h) + u(x_{j+1})/h - u(x_{j-1})/(2h) - u(x_j)/(3h) - u(x_{j+2})/(4h) +$

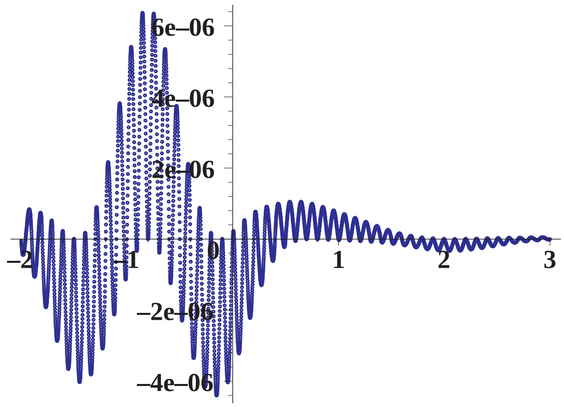

Fig. 12: The plot of the error of the approximation of the density.

$O(h^5)$. It can be constructed similar to Lemma 5.1. When we approximate the first derivative using this formula we obtain the actual error of the approximation $0.504 \cdot 10^{-5}$. Note that when applying the formulas of numerical differentiation, we should not take too small step commensurate with the errors in representing numbers on a computer.

7 Conclusion

In the present paper, we have constructed the non-polynomial interpolating spline which has the first continuous derivative. For its construction we used the basis functions and the values of the function and its first derivative. Using the basis functions we constructed another spline which has the first and second continuous derivatives. Interpolation with this new spline uses the values of the function in the nodes and the values of the first derivatives at the end of the interval $[a,b]$.

For small n, the approximation errors in the case of polynomial and trigonometric splines differ. In some cases, the trigonometric splines are superior as applied to the approximation of trigonometric functions. For weakly oscillating functions, the trigonometric splines are not advantageous. For sufficiently large n, these differences become insignificant. The theoretical error estimates agree with the actual ones. These splines could be used for the approximation functions of two variables with different properties in different directions. This area will be explored in details in the future work.

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