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Moments of Generalized Upper Record Values from Weibull- Power Function Distribution and Characterization

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Abstract: Tahir *et al.* (2014) proposed a new four parameter lifetime distribution called Weibull-power function distribution. In this article, some simple recurrence relations for the single and product moments as well as for inverse and quotient moments have been derived for the generalized upper record values ($k-$th upper record values). Moreover, these relations are deduced for moments of upper record values. Furthermore, conditional expectation, recurrence relations for the single and product moments and truncated moment have been used to characterize this distribution.

Keywords: Generalized upper record values, Weibull-power function distribution, single moments, product moments, inverse moments, quotient moments, conditional expectation, truncated moment and characterization.

1 Introduction

A random variable X is said to follow Weibull-power function distribution if its probability density function *(pdf)* has the form

$$f(x)=\frac{\theta\beta\gamma\alpha^{\beta}x^{\beta\theta-1}}{(\alpha^{\beta}-x^{\beta})^{\theta+1}}e^{-\gamma\left(\frac{x^{\beta}}{\alpha^{\beta}-x^{\beta}}\right)^{\theta}},\qquad 0<x<\alpha,\qquad \alpha,\ \beta,\theta,\gamma>0 \tag{1}$$

and the corresponding distribution function *(df)* is

$$\bar{F}(x)=e^{-\gamma\left(\frac{x^{\beta}}{\alpha^{\beta}-x^{\beta}}\right)^{\theta}},\qquad 0<x<\alpha,\qquad \alpha,\ \beta,\theta,\gamma>0 \tag{2}$$

where

$$\bar{F}(x)=1-F(x).$$

In view of (1) and (2), it is easy to see that

$$\theta\beta\gamma\bar{F}(x)=\sum_{u=0}^{\theta+1}(-1)^{u}\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}x^{\beta(u-\theta)+1}f(x). \tag{3}$$

The relation in (3) will be used to derive some simple recurrence relations for the moments of generalized upper record values from the distribution given in (2).

Tahir *et al.* (2014) introduced Weibull-power function distribution, which demonstrates the bathtub-shaped, J and U shaped hazard rate function and pointed out that this distribution is quite effective to provide consistently a better fit to real data sets than other competitive distributions. It also has wider applications in the fields of reliability engineering, mortality study, survival and lifetime data, insurance, hydrology and social sciences. It is an extension of power function distribution and is based on Weibull-G generator, pioneered by Bourguignon *et al.* (2014).

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The statistical study of record values started with the work of Chandler (1952), who formulated the theory of record values as a model of successive extremes in a sequence of independently and identically *(iid)* continuous random variables. Feller (1966) presented some examples of record values with respect to gambling problems. Resnick (1973) discussed the asymptotic theory of records. Later, Minimol and Thomas (2013) called the record values defined by Dziubdziela and Kopociński (1976) as the generalized record values, since the $r-$th member of the sequence of the ordinary record values is also known as the $r-$ th record value. If $k=1$, we obtain the ordinary record statistics.
Several applications of $k-$th record values can be found in pieces of literature. For instance, refer to the examples cited in Kamps (1995) as well as Danielak and Raqab (2004) in reliability theory. Suppose that a technical system or piece of equipment is subject to shocks, e.g. peaks of voltages. If the shocks are viewed as realizations of a sequence, the model of ordinary records is adequate. If it does not represent the records themselves, but second or third values are of special interest, the model of $k-$th record values is beneficial. Record statistics are applied in estimating materials strength, predicting natural disasters as well as sport achievements. For statistical inference based on ordinary records, serious difficulties arise if expected values of inter arrival time of records are infinite and occurrences of records are very rare in practice. This problem is avoided once we consider the model of generalized record statistics.
Let $\{X_n,\ n\geq 1\}$ be a sequence of *iid* random variables with *df* $F(x)$ and *pdf* $f(x)$. The $j-$th order statistic of a sample $X_1,X_2,\ldots,X_n$ is denoted by $X_{j:n}$. For a fixed positive integer k, we define the sequence $\{U_n^{(k)},\ n\geq 1\}$ of $k-$th upper record times of $\{X_n,\ n\geq 1\}$ as follows: $U_1^{(k)}=1$

$U_{n+1}^{(k)}=min\{j>U_n^{(k)}:X_{j:j+k-1}>X_{U_n^{(k)}:U_n^{(k)}+k-1}\}.$
The sequence $\{Y_n^{(k)},\ n\geq 1\}$, where $Y_n^{(k)}=X_{U_n^{(k)}}$ is called the sequence of generalized upper record values or $k-$th upper record values of $\{X_n,\ n\geq 1\}$. Note that for $k=1$, we have $Y_n^{(1)}=X_{U_n},\ n\geq 1$, which are the record values of $\{X_n,\ n\geq 1\}$ as defined in Ahsanullah (1995). Moreover, we see that $Y_0^{(k)}=0$ and $Y_1^{(k)}=min(X_1,X_2,...,X_k)=X_{1:k}$.
The *pdf* of $Y_n^{(k)}$ and the joint *pdf* of $Y_m^{(k)}$ and $Y_n^{(k)}$ are given by Dziubdziela and Kopociński (1976) as well as Grudzień (1982), respectively

$$f_{Y_n^{(k)}}(x)=\frac{k^n}{(n-1)!}\,[-ln\bar{F}(x)]^{n-1}\,[\bar{F}(x)]^{k-1}\,f(x),\ \ n\geq 1, \tag{4}$$

$$f_{Y_m^{(k)},Y_n^{(k)}}(x,y)=\frac{k^n}{(m-1)!(n-m-1)!}\,[-ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}$$
$$\times[ln\bar{F}(x)-ln\bar{F}(y)]^{n-m-1}[\bar{F}(y)]^{k-1}\,f(y),\ \ x<y,\ 1\leq m<n,\ n\geq 2, \tag{5}$$

and the conditional *pdf* of $Y_n^{(k)}$ given $Y_m^{(k)}=x$, is

$$f_{Y_m^{(k)}|Y_n^{(k)}}(y|x)=\frac{k^{n-m}}{(n-m-1)!}\,[ln\bar{F}(x)-ln\bar{F}(y)]^{n-m-1}\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{k-1}\frac{f(y)}{\bar{F}(x)},\quad x<y. \tag{6}$$

For some recent developments on generalized upper record values with special reference to those arising from exponential, Gumble, Pareto, generalized Pareto, Burr, Weibull, Gompertz, Makeham, modified Weibull, exponential-Weibull, additive Weibull distributions and Kumaraswamy-log-logistic distribution see Grudzień and Szynal (1983, 1997), Pawlas and Szynal (1998, 1999, 2000), Minimol and Thomas (2013, 2014), Khan and Khan (2016) and Khan *et al.* (2015, 2017) and Singh *et al.* (2019), respectively. In this article, we investigate the moment properties of generalized upper record values arising from the Weibull-power function distribution. Furthermore, conditional expectation and recurrence relations for single moments of generalized upper record values and truncated moment are used to characterize this distribution.

2 Relations for Single Moments

In this section, we derive simple recurrence relations between moments of generalized upper record values from the given distribution.
Theorem 2.1. For the distribution given in (2), fix a positive integer $k\geq 1$, $n\geq 1$ and $j=0,1,...$

$$E(Y_n^{(k)})^j=E(Y_{n-1}^{(k)})^j+\frac{j}{k\theta\beta\gamma}\sum_{u=0}^{\theta+1}(-1)^u\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}E(Y_n^{(k)})^{j+\beta(u-\theta)}. \tag{7}$$

Proof. In view of Khan *et al.* (2017), note that

$$E(Y_n^{(k)})^j - E(Y_{n-1}^{(k)})^j = \frac{jk^{n-1}}{(n-1)!}\int_\alpha^\beta x^{j-1}[-ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^k dx. \tag{8}$$

On substituting (3) in (8), we get

$$E(Y_n^{(k)})^j - E(Y_{n-1}^{(k)})^j = \frac{jk^{n-1}}{\theta\beta\gamma(n-1)!}\sum_{u=0}^{\theta+1}(-1)^u\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}$$
$$\times\int_0^\alpha x^{j+\beta(u-\theta)}[-ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)dx.$$

which upon simplification yields (7).
Corollary 2.1. Setting $n=1$ in (7), we deduce the following result for Weibull-power function distribution in terms of moments of minimal order statistics

$$E(X_{1:k})^j = \frac{j}{k\theta\beta\gamma}\sum_{u=0}^{\theta+1}(-1)^u\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}E(X_{1:k})^{j+\beta(u-\theta)}$$

and subsequently

$$E(X) = \frac{1}{\theta\beta\gamma}\sum_{u=0}^{\theta+1}(-1)^u\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}E(X)^{\beta(u-\theta)}.$$

Corollary 2.2. Putting $k=1$ in (7), the recurrence relations for single moments of upper record values from the Weibull-power function distribution has the form

$$E(X_{U_n})^j = E(X_{U_{n-1}})^j + \frac{j}{\theta\beta\gamma}\sum_{u=0}^{\theta+1}(-1)^u\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}E(X_{U_n})^{j+\beta(u-\theta)}.$$

3 Relations for Product Moments

In this section the recurrence relations for product moments of generalized upper record values are presented.
Theorem 3.1. For the distribution given in (2) and $m \geq 1$ and $i,j = 0,1,2,\ldots$

$$E[(Y_m^{(k)})^i(Y_{m+1}^{(k)})^j] = E[(Y_m^{(k)})^{i+j}] + \frac{j}{k\theta\beta\gamma}\sum_{u=0}^{\theta+1}(-1)^u\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}$$
$$\times E[(Y_m^{(k)})^i(Y_{m+1}^{(k)})^{j+\beta(u-\theta)}]. \tag{9}$$

and for $1 \leq m \leq n-2$, $i,j = 0,1,2,\ldots$

$$E[(Y_m^{(k)})^i(Y_n^{(k)})^j] = E[(Y_m^{(k)})^i(Y_{n-1}^{(k)})^j] + \frac{j}{k\theta\beta\gamma}\sum_{u=0}^{\theta+1}(-1)^u\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}$$
$$\times E[(Y_m^{(k)})^i(Y_n^{(k)})^{j+\beta(u-\theta)}]. \tag{10}$$

Proof. From Khan *et al.* (2017), we have

$$E[(Y_m^{(k)})^i(Y_n^{(k)})^j] - E[(Y_m^{(k)})^i(Y_{n-1}^{(k)})^j] = \frac{jk^{n-1}}{(m-1)!(n-m-1)!}\int_\alpha^\beta\int_x^\beta x^i y^{j-1}$$
$$\times[-ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}[-ln\bar{F}(y)+ln\bar{F}(x)]^{n-m-1}[\bar{F}(y)]^k dydx. \tag{11}$$

Using (3) in (11), we have

$$E[(Y_m^{(k)})^i(Y_n^{(k)})^j] - E[(Y_m^{(k)})^i(Y_{n-1}^{(k)})^j] = \frac{jk^{n-1}}{(m-1)!(n-m-1)!\theta\beta\gamma}\sum_{u=0}^{\theta+1}(-1)^u\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}$$

$$\times \int_0^{\alpha} \int_x^{\alpha} x^i y^{j+\beta(u-\theta)} [-ln\bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} [-ln\bar{F}(y) + ln\bar{F}(x)]^{n-m-1} [\bar{F}(y)]^{k-1} f(y) dy dx,$$

which establishes (10).
Now substituting $n = m+1$ in (10) and noting that $E[(Y_m^{(k)})^i (Y_m^{(k)})^j] = E[(Y_m^{(k)})^{i+j}]$, we get (9).

One can also note that Theorem 2.1 can be deduced from Theorem 3.1 through setting $i = 0$.

Corollary 3.1. At $k = 1$ in (10), we see that the recurrence relations for product moments of upper record values from the Weibull-power function distribution have the form

$$E[(X_{U_m})^i (X_{U_n})^j] = E[(X_{U_m})^i (X_{U_{n-1}})^j] + \frac{j}{\theta\beta\gamma} \sum_{u=0}^{\theta+1} (-1)^u \binom{\theta+1}{u} \alpha^{\beta(\theta-u)} \times E[(X_{U_m})^i (X_{U_n})^{j+\beta(u-\theta)}].$$

4 Relations for inverse moments

In this section, we present the recurrence relations for inverse moments of the generalized upper record values from the aforementioned distribution.
Theorem 4.1. For the distribution given in (2), fix a positive integer $k \geq 1$ for $n \geq 1$ and $j \geq 1$

$$E\left(\frac{1}{Y_n^{(k)}}\right)^j = \frac{j}{k\theta\beta\gamma} \sum_{u=0}^{\theta+1} (-1)^u \binom{\theta+1}{u} \alpha^{\beta(\theta-u)} E\left(\frac{1}{Y_n^{(k)}}\right)^{j-\beta(u-\theta)} - E\left(\frac{1}{Y_{n-1}^{(k)}}\right)^j. \tag{12}$$

Proof. Using (4), we see that

$$E\left(\frac{1}{Y_n^{(k)}}\right)^j = \frac{k^n}{(n-1)!} \int_0^{\alpha} x^{-j} [-ln\bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) dx. \tag{13}$$

Integrating (13) by parts, taking $'[\bar{F}(x)]^{k-1} f(x)'$ as the part to be integrated and rest of the integrand for differentiation, then using (3) and simplifying the resulting expression, we obtain the result given in (12).
Corollary 4.1. Setting $k = 1$ in (12), we deduce the recurrence relations for the inverse moments based on upper record values from the Weibull-power function distribution as

$$E\left(\frac{1}{X_{U_n}}\right)^j = \frac{j}{\theta\beta\gamma} \sum_{u=0}^{\theta+1} (-1)^u \binom{\theta+1}{u} \alpha^{\beta(\theta-u)} E\left(\frac{1}{X_{U_n}}\right)^{j-\beta(u-\theta)} - E\left(\frac{1}{X_{U_{n-1}}}\right)^j.$$

5 Relations for Quotient moments

This section involves the recurrence relations for the Quotient moments of generalized upper record values from this distribution.

Theorem 5.1. For the distribution given in (2), fix positive integer $k \geq 1$, for $1 \leq m \leq n$ and $i, j = 0, 1, \ldots$

$$E\left(\frac{(Y_m^{(k)})^i}{(Y_n^{(k)})^j}\right) = \frac{j}{k\theta\beta\gamma} \sum_{u=0}^{\theta+1} (-1)^u \binom{\theta+1}{u} \alpha^{\beta(\theta-u)} E\left(\frac{(Y_m^{(k)})^i}{(Y_n^{(k)})^{j-\beta(u-\theta)}}\right) - E\left(\frac{(Y_m^{(k)})^i}{(Y_{n-1}^{(k)})^j}\right). \tag{14}$$

Proof. From (5), we have

$$E\left(\frac{(Y_m^{(k)})^i}{(Y_n^{(k)})^j}\right) = \frac{k^n}{(m-1)!(n-m-1)!} \int_0^{\alpha} x^i [-ln\bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} I(x) dx, \tag{15}$$

where

$$I(x) = \int_x^{\alpha} y^{-j} [-ln\bar{F}(y) + ln\bar{F}(x)]^{n-m-1} [\bar{F}(y)]^{k-1} f(y) dy. \tag{16}$$

Integrating $I(x)$ by parts, taking $'[\bar{F}(y)]^{k-1}f(y)'$ as the part to be integrated and rest of the integrand for differentiation, then substituting the resulting expression in (15) and simplifying, we get the result given in (14).

Corollary 5.1. Under the assumptions as stated in Theorem 5.1 with $n=m+1$, the following relation holds

$$E\left(\frac{(Y_m^{(k)})^i}{(Y_{m+1}^{(k)})^j}\right)=\frac{j}{k\theta\beta\gamma}\sum_{u=0}^{\theta+1}(-1)^u\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}E\left(\frac{(Y_m^{(k)})^i}{(Y_{m+1}^{(k)})^{j-\beta(u-\theta)}}\right)-E\left((Y_m^{(k)})^{i-j}\right). \tag{17}$$

Corollary 5.2. Setting $k=1$ in (14), we deduce the recurrence relations for the quotient moments based on upper record values from the Weibull-Power function distribution as

$$E\left(\frac{(X_{U_m})^i}{(X_{U_n})^j}\right)=\frac{j}{\theta\beta\gamma}\sum_{u=0}^{\theta+1}(-1)^u\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}E\left(\frac{(X_{U_m})^i}{(X_{U_n})^{j-\beta(u-\theta)}}\right)-E\left(\frac{(X_{U_m})^i}{(X_{U_{n-1}})^j}\right). \tag{18}$$

6 Characterizations

The following theorems involve characterization results for this distribution based on recurrence relations for single and product moments, conditional moments as well as truncated moment.
Theorem 6.1. Fix a positive integer $k\geq 1$ and let j be a non-negative integer. A necessary and sufficient condition for a random variable X to be distributed with *pdf* given by (1) is that

$$E(Y_n^{(k)})^j=E(Y_{n-1}^{(k)})^j+\frac{j}{k\theta\beta\gamma}\sum_{u=0}^{\theta+1}(-1)^u\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}E(Y_n^{(k)})^{j+\beta(u-\theta)}. \tag{19}$$

for $n=1,2,...,n\geq k$.
Proof. The necessary part follows from (7). On the other hand if the recurrence relation in (19) is satisfied, then using Khan *et al.* (2017), we have

$$\frac{jk^{n-1}}{(n-1)!}\int_0^\alpha x^{j-1}[-ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^k dx$$
$$=\frac{jk^{n-1}}{(n-1)!\theta\beta\gamma}\sum_{u=0}^{\theta+1}(-1)^u\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}\int_0^\alpha x^{j+\beta(u-\theta)}[-ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)dx,$$

which implies

$$\int_0^\alpha x^{j-1}[-ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}\left\{\bar{F}(x)-\frac{1}{\theta\beta\gamma}\sum_{u=0}^{\theta+1}(-1)^u\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}x^{\beta(u-\theta)+1}f(x)\right\}dx=0. \tag{20}$$

Now applying a generalization of the Müntz-Szász theorem (see for example Hwang & Lin, 1984) to (20), we get

$$\theta\beta\gamma\bar{F}(x)=\sum_{u=0}^{\theta+1}(-1)^u\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}x^{\beta(u-\theta)+1}f(x),$$

which proves that $f(x)$ has the form as given in (3).
Corollary 6.1. If $k=1$ in (19), we obtain the characterizing result based on upper record values for the Weibull-power function distribution as

$$E(X_{U_n})^j=E(X_{U_{n-1}})^j+\frac{j}{\theta\beta\gamma}\sum_{u=0}^{\theta+1}(-1)^u\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}E(X_{U_n})^{j+\beta(u-\theta)}.$$

Theorem 6.2. For a positive integer k , let i and j be non-negative integers. A necessary and sufficient condition for a random variable X to be distributed with *pdf* given by (1) is that

$$E[(Y_m^{(k)})^i(Y_n^{(k)})^j]=E[(Y_m^{(k)})^i(Y_{n-1}^{(k)})^j]+\frac{j}{k\theta\beta\gamma}\sum_{u=0}^{\theta+1}(-1)^u\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}E[(Y_m^{(k)})^i(Y_n^{(k)})^{j+\beta(u-\theta)}]. \tag{21}$$

Proof. The necessary part follows from (9). On the other hand if the relation in (21) is satisfied, then using Khan *et al.* (2017), (21) can be expressed as

$$\frac{jk^{n-1}}{(m-1)!(n-m-1)!}\int_0^\alpha\int_x^\alpha x^i y^{j-1}[-ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}[-ln\bar{F}(y)+ln\bar{F}(x)]^{n-m-1}$$

$$\times[\bar{F}(y)]^k dydx=\frac{jk^{n-1}}{(m-1)!(n-m-1)!\theta\beta\gamma}\sum_{u=0}^{\theta+1}(-1)^u\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}$$

$$\times\int_0^\alpha\int_x^\alpha x^i y^{j+\beta(u-\theta)}[-ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}[-ln\bar{F}(y)+ln\bar{F}(x)]^{n-m-1}[\bar{F}(y)]^{k-1}f(y)dydx.$$

which implies that

$$\int_0^\alpha\int_x^\alpha x^i y^{j-1}[-ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}[-ln\bar{F}(y)+ln\bar{F}(x)]^{n-m-1}[\bar{F}(y)]^k$$

$$\times\left\{\bar{F}(y)-\frac{1}{\theta\beta\gamma}\sum_{u=0}^{\theta+1}(-1)^u\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}y^{\beta(u-\theta)+1}f(y)\right\}dydx=0. \tag{22}$$

Applying a generalization of the Müntz-Szász Theorem, (see for example Hwang and Lin, (1984)) to (22), we get

$$\theta\beta\gamma\bar{F}(y)=\sum_{u=0}^{\theta+1}(-1)^u\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}y^{\beta(u-\theta)+1}f(y),$$

which proves that $f(y)$ has the form as given in (3).

Corollary 6.2. If we assume $k=1$ in (21), we obtain the following characterizing result for the Weibull-power function distribution based on upper record values

$$E[(X_{U_m})^i(X_{U_n})^j]=E[(X_{U_m})^i(X_{U_{n-1}})^j]+\frac{j}{\theta\beta\gamma}\sum_{u=0}^{\theta+1}(-1)^u\binom{\theta+1}{u}\alpha^{\beta(\theta-u)}E[(X_{U_m})^i(X_{U_n})^{j+\beta(u-\theta)}].$$

Theorem 6.3. Let X be a non-negative random variable having an absolutely continuous *df* $F(x)$ with $F(0)=0$ and $0\le F(x)\le 1$ for all $0<x<\alpha$, then

$$E[\xi(Y_n^{(k)})\mid(Y_l^{(k)})=x]=e^{-\gamma\left(\frac{x^\beta}{\alpha^\beta-x^\beta}\right)^\theta},\left(\frac{k}{k+1}\right)^{n-l},\ \ l=m,\ \ m+1 \tag{23}$$

if and only if

$$F(x)=1-e^{-\gamma\left(\frac{x^\beta}{\alpha^\beta-x^\beta}\right)^\theta},\qquad 0<x<\alpha,\quad \alpha,\ \beta,\theta,\gamma>0.$$

where

$$\xi(y)=e^{-\gamma\left(\frac{y^\beta}{\alpha^\beta-y^\beta}\right)^\theta}.$$

Proof. From (4) and (5), we have

$$E[\xi(Y_n^{(k)})\mid(Y_m^{(k)})=x]=\frac{k^{n-m}}{(n-m-1)!}\int_x^\alpha e^{-\gamma\left(\frac{y^\beta}{\alpha^\beta-y^\beta}\right)^\theta}[ln\bar{F}(x)-ln\bar{F}(y)]^{n-m-1}$$

$$\times\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{k-1}\frac{f(y)}{\bar{F}(x)}dy. \tag{24}$$

By setting $u = \frac{\bar{F}(y)}{\bar{F}(x)} = \frac{e^{-\gamma\left(\frac{y^\beta}{\alpha^\beta - y^\beta}\right)^\theta}}{e^{-\gamma\left(\frac{x^\beta}{\alpha^\beta - x^\beta}\right)^\theta}}$ in (24), we have

$$E[\xi(Y_n^{(k)}) \mid (Y_m^{(k)}) = x] = \frac{k^{n-m}}{(n-m-1)!} e^{-\gamma\left(x^\beta/(\alpha^\beta - x^\beta)\right)^\theta} \int_0^1 u^k(-lnu)^{n-m-1} du. \tag{25}$$

In view of Gradshteyn and Ryzhik (2007, p.551), note that

$$\int_0^1 (-lnx)^\mu x^{\nu-1} dx = \frac{\Gamma\mu}{\nu^\mu}, \mu > 0, \nu > 0. \tag{26}$$

Substituting (26) in (25), we have the result given in (23).
To prove the sufficient part, we have

$$\frac{k^{n-m}}{(n-m-1)!} \int_x^\alpha e^{-\gamma\left(y^\beta/(\alpha^\beta - y^\beta)\right)^\theta} [ln\bar{F}(x) - ln\bar{F}(y)]^{n-m-1} [\bar{F}(y)]^{k-1} f(y) dy = [\bar{F}(x)]^k g_{n|m}(x), \tag{27}$$

where

$$g_{n|m}(x) = e^{-\gamma\left(x^\beta/(\alpha^\beta - x^\beta)\right)^\theta} \left(\frac{k}{k+1}\right)^{n-m}.$$

Differentiating both the sides of (27) with respect to x, we get

$$-\frac{k^{n-m} f(x)}{(n-m-1)!\bar{F}(x)} \int_x^\infty e^{-\gamma\left(y^\beta/(\alpha^\beta - y^\beta)\right)^\theta} [ln\bar{F}(x) - ln\bar{F}(y)]^{n-m-2}$$

$$\times [\bar{F}(y)]^{k-1} f(y) dy = g'_{n|m}(x)[\bar{F}(x)]^k - k\, g_{n|m}(x)[\bar{F}(x)]^{k-1} f(x)$$

or

$$-k\, g_{n|m+1}(x)[\bar{F}(x)]^{k-1} f(x) = g'_{n|m}(x)[\bar{F}(x)]^k - k\, g_{n|m}(x)[\bar{F}(x)]^{k-1} f(x).$$

Consequently,

$$\frac{f(x)}{\bar{F}(x)} = -\frac{g'_{n|m}(x)}{k[g_{n|m+1}(x) - g_{n|m}(x)]} = \frac{\theta\beta\gamma\alpha^\beta x^{\beta\theta-1}}{(\alpha^\beta - x^\beta)^{\theta+1}}. \tag{28}$$

where

$$g'_{n|m}(x) = -\alpha^\beta \theta\beta\gamma x^{-\beta-1} \left(\frac{x^\beta}{\alpha^\beta - x^\beta}\right)^{\theta+1} e^{-\gamma\left(x^\beta/(\alpha^\beta - x^\beta)\right)^\theta} \left(\frac{k}{k+1}\right)^{n-m},$$

$$g_{n|m+1}(x) - g_{n|m}(x) = \frac{1}{k} e^{-\gamma\left(x^\beta/(\alpha^\beta - x^\beta)\right)^\theta} \left(\frac{k}{k+1}\right)^{n-m}.$$

Now integrating both sides of (28) with respect to x between $(0, y)$, the sufficiency part is proved.
Theorem 6.4 Suppose that an absolutely continuous (with respect to Lebesgue measure) random variable X has the *df* $F(x)$ and *pdf* $f(x)$ for $0 < x < \alpha$, such that $f'(x)$ and $E(X|X \le x)$ exist for all x, $0 < x < \alpha$, then

$$E(X|X \le x) = g(x)\eta(x), \tag{29}$$

where

$$\eta(x) = \frac{f(x)}{F(x)}$$

and

$$g(x) = -\frac{(\alpha^\beta - x^\beta)^{\theta+1}}{\gamma\theta\beta\alpha^\beta x^{\beta\theta-2}} + \frac{(\alpha^\beta - x^\beta)^{\theta+1} e^{\gamma\left(x^\beta/(\alpha^\beta - x^\beta)\right)^\theta}}{\gamma\theta\beta\alpha^\beta x^{\beta\theta-1}} \int_0^x e^{-\gamma\left(u^\beta/(\alpha^\beta - u^\beta)\right)^\theta} du.$$

if and only if

$$f(x) = \frac{\theta\beta\gamma\alpha^\beta x^{\beta\theta-1}}{(\alpha^\beta - x^\beta)^{\theta+1}} e^{-\gamma\left(\frac{x^\beta}{\alpha^\beta - x^\beta}\right)^\theta}, \quad 0 < x < \alpha, \quad \alpha,\ \beta, \theta, \gamma > 0. \tag{30}$$

Proof. From (29), we have

$$E(X|X\leq x)=\frac{1}{F(x)}\int_0^x uf(u)du$$

$$=\frac{\theta\beta\gamma\alpha^\beta}{F(x)}\int_0^x u\frac{u^{\beta\theta-1}}{(\alpha^\beta-u^\beta)^{\theta+1}}e^{-\gamma\left(u^\beta/(\alpha^\beta-u^\beta)\right)^\theta}du. \tag{31}$$

Integrating (31) by parts, taking ' $\frac{u^{\beta\theta-1}}{(\alpha^\beta-u^\beta)^{\theta+1}}e^{-\gamma\left(u^\beta/\alpha^\beta-u^\beta\right)^\theta}$ ' as the part to be integrated and the rest of the integrand for differentiation, we get

$$E(X|X\leq x)=\frac{1}{F(x)}\left\{-xe^{-\gamma(x^\beta/(\alpha^\beta-x^\beta))^\theta}+\int_0^x e^{-\gamma(u^\beta/(\alpha^\beta-u^\beta))^\theta}du\right\}. \tag{32}$$

Multiplying and dividing (32) by $f(x)$, we have the result given in (29).
To prove sufficient part, we have from (29)

$$\int_0^x uf(u)du=g(x)f(x). \tag{33}$$

Differentiating (33) on both sides with respect to x, we find that

$$xf(x)=g'(x)f(x)+g(x)f'(x).$$

Therefore,

$$\frac{f'(x)}{f(x)}=\frac{x-g'(x)}{g(x)} \qquad \text{[Ahsanullah } et\ al.\text{ (2016)]}$$

or

$$=-\frac{(1-\beta\theta)}{x}+\frac{\beta(\theta+1)x^{\beta-1}}{(\alpha^\beta-x^\beta)}+\frac{\theta\beta\gamma\alpha^\beta x^{\beta\theta-1}}{(\alpha^\beta-x^\beta)^{\theta+1}}, \tag{34}$$

where

$$g'(x)=x+g(x)\left(\frac{(1-\beta\theta)}{x}-\frac{\beta(\theta+1)x^{\beta-1}}{(\alpha^\beta-x^\beta)}+\frac{\theta\beta\gamma\alpha^\beta x^{\beta\theta-1}}{(\alpha^\beta-x^\beta)^{\theta+1}}\right).$$

Integrating both the sides (34) with respect to x, we have

$$f(x)=\frac{cx^{\beta\theta-1}}{(\alpha^\beta-x^\beta)^{\theta+1}}e^{-\gamma\left(x^\beta/\alpha^\beta-x^\beta\right)^\theta}.$$

Now, using the condition $\int_0^\alpha f(x)dx=1$, one obtains

$$\frac{1}{c}=\int_0^\alpha\frac{x^{\beta\theta-1}}{(\alpha^\beta-x^\beta)^{\theta+1}}e^{-\gamma\left(x^\beta/\alpha^\beta-x^\beta\right)^\theta}dx=\frac{1}{\beta\theta\gamma\alpha^\beta},$$

which proves that

$$f(x)=\frac{\theta\beta\gamma\alpha^\beta x^{\beta\theta-1}}{(\alpha^\beta-x^\beta)^{\theta+1}}e^{-\gamma\left(x^\beta/\alpha^\beta-x^\beta\right)^\theta}, \qquad 0<x<\alpha, \qquad \alpha,\ \beta,\theta,\gamma>0.$$

7 Conclusion

In this study, recurrence relations for the single and product moments as well as for inverse and quotient moments for the generalized upper record values ($k-$th upper record values) from the Weibull-power function distribution have been established and some particular cases are also addressed, Furthermore, conditional expectation and recurrence relation for single moments as well as product moments of $k-$th record values have been utilized to obtain the characterizing results of the Weibull-power function distribution.

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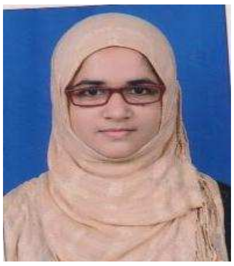

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