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Continuity with Respect to Fractional Order of the Time a Linear Fractional Pseudo-Parabolic Equation

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Abstract: In this paper, we study the linear pseudo-parabolic equation with Caputo derivative. In practice, many models related to time fractional diffusion equations depend on fractional orders. Our main goal is to investigate the initial value problem for fractional diffusion equation and discuss continuity with respect to the fractional derivative order.

Keywords: Caputo derivative, pseudoparabolic equation, well–posedness, regularity estimates.

1 Introduction

Fractional PDEs are now widely used to describe phenomena in domains such as physics, biology, and chemistry, where integer-order PDEs are ineffective under specific conditions [1–10].Fractional derivatives with some models are a generalization of equations with integer-order partial derivatives and a hot topic in both academia and industry.

The following time-fractional PDE is considered in this paper

$$\begin{cases} \partial_t^\alpha\big(u(x,t)+k\mathscr{L}u(x,t)\big)+\mathscr{L}u(x,t)=\mathscr{F}(x,t), & (x,t)\in\Omega\times(0,T], \\ u(x,t)=0, & (x,t)\in\partial\Omega\times(0,T], \\ u(x,0)=u_0(x), & x\in\Omega, \\ u_t(x,0)=0, & x\in\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in R^n with sufficiently smooth boundary $\partial\Omega$, $1<\alpha<2$, ∂_t^α is Caputo fractional derivative operator of order α. When the function u is absolutely continuous in time, the definition in [11] reduces to the traditional form

$$\partial_t^\alpha u(x,t)=\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}\frac{\partial^2 u(x,s)}{\mathrm{ds}}\mathrm{ds} \tag{2}$$

where $\Gamma(z)=\int\limits_0^\infty \mathrm{e}^{-t}t^{z-1}\mathrm{dt}$ and $\frac{\partial^2 u}{ds}$ is the function $u(s)$'s second order integer derivative with respect to the independent variable s. The $\mathscr{L}$ operator is a symmetrical, uniformly elliptic operator on Ω. $\mathscr{F},u_0(x)$ are given functions that satisfy some later-specified assumption. Equation (1) describes mass transfer processes in media and systems with a fractal structure.

When $\alpha=1$, the (1) model transforms into the well-known pseudo-parabolic with an integer order derivative.

$$u_t+k\mathscr{L}u_t+\mathscr{L}u=\mathscr{F}(u). \tag{3}$$

The pseudo-parabolic equation (3) has several practical applications, such as seepage of homogeneous fluids through a fissured rock [12], unidirectional propagation of nonlinear dispersive long waves [13,14], and population aggregation [15].

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There are many works on well-posedness of the pseudo-parabolic equation with classical derivative, we can see [16–27] and the references therein. The scientific community has long sought to understand the existence, uniqueness, and stability of fractional differential equations, particularly in fractional calculus.

The present paper aims is to present well-posedness and continuity of the mild solution of a fractional order pseudo-parabolic partial differential equation (FPPDE for short). There have only been a few papers on the fractional order pseudo-parabolic partial differential equation up until now. The Ulam-Hyers and generalized Ulam-Hyers-Rassias stabilities for the solution of an FPPDE were recently explored by the authors, see [28].

In practice, some models on time-space fractiona PDEs depend on fractional order. However these fractional parameters are not known a priori in modelling problems. Hence continuity of the solutions with respect to these parameters is important for modelling purposes. This paper comes from the motivation of [32] for considering the continuity of the solutions on fractional order. Assume that a sequence satisfies $\alpha_k \to \alpha$ when $k \to \infty$. Since α is unknown, in real life applications, we only get computed approximation u_{α_k} of u_α. Then our main goal in this paper is to answer the following question:

$$\textit{Does } u_{\alpha_k} \to u_\alpha \textit{ in an appropriate sense as } k \to \infty. \tag{4}$$

This paper is organized as follows: In section 2, we collect some preliminaries. In section 3, we consider the problem of continuity with respect to fractional order.

2 Preliminaries

2.1 The Mittag-Leffler function

The Mittag-Leffler function is a function which is denoted and defined as folows

$$E_{a,b}(\xi) = \sum_{n=1}^{\infty} \frac{\xi^n}{\Gamma(na+b)}$$

$(\xi \in \mathbb{C})$, for $a > 0$ and $b \in \mathbb{R}$. The following lemmas will be relevant for the primary analysis of Section 3 and Section 4, respectively (see, for example, [33], [**?**], and [34]).

Lemma 1.*If $1 < a < 2$, then for any $\xi > 0$, $\mathfrak{M}$ is a positive constant that exclusively depends on $alpha$, such that*

$$|E_{a,1}(-\xi)| \le \mathfrak{M}, \qquad |E_{a,a}(-\xi)| \le \mathfrak{M}.$$

We now have the Lemma below.

Lemma 2.*If $\lambda > 0$ and $1 < a < 2$, then the following identities hold for all $t > 0$.*

$$\begin{aligned} &\partial_t E_{a,1}(-\lambda t^a) = -\lambda t^{a-1} E_{a,a}(-\lambda t^a), \\ &\partial_t (t^{a-1} E_{a,a}(-\lambda t^a)) = t^{a-2} E_{a,a-1}(-\lambda t^a). \end{aligned} \tag{5}$$

*Proof.*Applying Lemma (2.2) in [35].

Lemma 3.*(see [36]) Let $1 < a < 2$, for all $j \in \mathbb{N}$ then $E_{a,1}(-\lambda_j T^a) \neq 0$,where T is a large enough number. Moreover, exists two constant $\mathfrak{n}_a$ and $\mathfrak{N}_a$ such that*

$$\frac{\mathfrak{n}_a}{1+\lambda_j T^a} \le \left| E_{a,1}(-\lambda_j T^a) \right| \le \frac{\mathfrak{N}_a}{1+\lambda_j T^a}. \tag{6}$$

The following lemma is derived from Lemma 2.3 [37].

Lemma 4.*Let $1 < \delta < \nu < 2$ and $a \in (\delta, \nu)$. There are three positive constants: $\mathfrak{N}1$, $\mathfrak{N}2$ and $\mathfrak{N}3$ that only depend on δ, ν such that for any $\xi > 0$ we obtain*

$$\frac{\mathfrak{N}_1(\delta,\nu)}{1+\xi} \le |E_{a,1}(-\xi)| \le \frac{\mathfrak{N}_2(\delta,\nu)}{1+\xi}, \qquad |E_{a,a}(-\xi)| \le \frac{\mathfrak{N}_3(\delta,\nu)}{1+\xi}. \tag{7}$$

Lemma 5. *Assume that $0<\delta<\alpha<\alpha'<\nu$ and $0<t\le T$. Then there always exists $\mathfrak{N}_\varepsilon$ such that*

$$|t^{\alpha}-t^{\alpha'}|\le \max(T^{\nu+2\varepsilon},1)\mathfrak{N}_\varepsilon(\alpha'-\alpha)^{\varepsilon}t^{\alpha-\varepsilon}. \tag{8}$$

where ε is a positive number which is independent of α.

Proof. It can be found in lemma 3.2 [38]

Lemma 6. *Let $1<\delta<\alpha<\alpha'<\nu<2$ and $\varepsilon>0$. Then there exists a positive constant $\mathscr{A}(\delta,\nu,\varepsilon,\gamma,T)$*

$$|E_{\alpha,1}(-\lambda_n t^{\alpha})-E_{\alpha',1}(-\lambda_n t^{\alpha'})|\le \mathscr{A}(\delta,\nu,\varepsilon,\gamma,T)\lambda_n^{\gamma-1}t^{-\nu(1-\gamma)-\varepsilon}\Big[(\alpha'-\alpha)^{\varepsilon}+(\alpha'-\alpha)\Big] \tag{9}$$

for any $\gamma\in[0,1]$ and $t\in(0,T]$.

Proof. See section 3 in [38]

Lemma 7. *Assume that $1<\delta<\alpha<\alpha'<\nu<2$. For any $0\le\gamma\le 1$ and $\varepsilon>0$ there exists a positive constant $\mathscr{B}(\delta,\nu,\varepsilon,\gamma,T)$*

$$\begin{aligned}&\Big|t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^{\alpha})-t^{\alpha'-1}E_{\alpha',\alpha'}(-\lambda_n t^{\alpha'})\Big|\\ &\qquad\le \mathscr{B}(\delta,\nu,\varepsilon,\gamma,T)\lambda_n^{\gamma-1}t^{\delta\gamma-\varepsilon-1}\Big[(\alpha'-\alpha)^{\varepsilon}+(\alpha'-\alpha)\Big].\end{aligned} \tag{10}$$

Proof. Using lemma 3.4 from section 3 in [38].

2.2 Relevant notations and a representation of solution

We define some proper Sobolev spaces and standardize some notation. $\mathbb{L}^2(\Omega), H_0^1(\Omega), H^2(\Omega)$ denote usual sobolev spaces. The symmetric uniform elliptic operator $\mathscr{L}:\mathbb{L}^2(\Omega)\to\mathbb{L}^2(\Omega)$ is defined by

$$\mathscr{L}u(x)=-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\mathscr{L}_{ij}(x)\frac{\partial}{\partial x_j}u(x)\right)+l(x)u(x,t), x\in\overline{\Omega}$$

where $D(\mathscr{L})=H_0^1(\Omega)\cap H^2(\Omega)$.Assume that $l(x)\in C(\overline{\Omega},[0,\infty)$, $\mathscr{L}_{ij}\in C^1(\overline{\Omega})$, $\mathscr{L}_{ij}=\mathscr{L}_{ji}$, $1\le i,j\le n$, and there exists a positive constant $\widetilde{L}>0$, for $x\in\overline{\Omega}$, $\xi=(\xi_1,\xi_2,\cdots,\xi_n)\subset\mathbb{R}^n$, such that

$$\widetilde{L}\sum_{i=1}^{n}\xi_i^2\le\sum_{1\le i,j\le n}\xi_i\mathscr{L}_{ij}(x)\xi_j$$

see e.g. [39] Let us remind ourselves of the spectral problem.

$$\begin{cases}\mathscr{L}\phi_n(x)=\lambda_n\phi_n(x), & \text{in } \Omega,\\ \varphi_n(x)=0, & \text{on } \partial\Omega,\end{cases} \tag{11}$$

takes a series of eigenvalues (see e.g [40]), for all $1\le i<j\le n$ such that

$$0<\lambda_i\le\lambda_j \text{ and } \lim_{n\to+\infty}\lambda_n=+\infty.$$

Furthermore, $\mathscr{L}^s u$ is defined as the following operator:

$$\mathscr{L}^q u:=\sum_{n=1}^{\infty}\langle u,\varphi_n\rangle\lambda_n^q\varphi_n,\qquad v\in\mathbb{D}(\mathscr{L}^q)=\left\{v\in\mathbb{L}^2(\Omega):\sum_{n=1}^{\infty}|\langle u,\varphi_n\rangle|^2\lambda_n^{2q}<\infty\right\}. \tag{12}$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product of $\mathbb{L}^2(\Omega)$ and the notation $\|\cdot\|_S$ represents the norm in the Banach space S. The domain $\mathbb{D}(\mathscr{A}^s)$ is Banach space equipped with the norm.

$$\|u\|^2_{\mathbb{D}(\mathscr{L}^q)}:=\sum_{n=1}^{\infty}\lambda_n^{2q}\big|\langle u,\varphi_n\rangle\big|^2. \tag{13}$$

If $q=1$ then $\mathbb{D}(\mathscr{A}^1)=H^2(\Omega)$.
For a given positive number p, the Hilbert space

$$\mathbb{H}^p(\Omega)=\left\{v\in\mathbb{L}^2(\Omega):\sum_{j=1}^{\infty}|\langle v,\varphi_n\rangle|^2\lambda_n^{2p}<\infty\right\}. \tag{14}$$

where

$$\|v\|^2_{\mathbb{H}^p(\Omega)}=\sum_{n=1}^{\infty}\lambda_n^{2p}|\langle v,\varphi_n\rangle|^2.$$

If $q=0$ then $\mathbb{H}^0(\Omega)=L^2(\Omega)$. Let us denote by $\mathscr{C}((0,T];\mathbb{H}^p(\Omega))$ the space of all continuous functions which map $(0,T]$ into $\mathbb{H}^p(\Omega)$. For a given number $0<\beta<1$, we define by $\mathscr{C}^\beta((0,T];\mathbb{H}^p(\Omega))$ the subspace of $\mathscr{C}((0,T];\mathbb{H}^p(\Omega))$ such that

$$\sup_{0<t\le T} t^\beta\|v(t)\|_{\mathbb{H}^p(\Omega)}<\infty,\qquad \text{for all } v\in\mathscr{C}((0,T];\mathbb{H}^p(\Omega)),$$

which is endowed with the norm, see [41],

$$\|u\|_{\mathscr{C}^\beta((0,T];\mathbb{H}^p(\Omega))}:=\sup_{0<t\le T}t^\beta\|v(t)\|_{\mathbb{H}^p(\Omega)}.$$

We denote the Banach space of real-valued functions $w:(0;T)\to D$ measurable by $L^q(0,T;D), 1\le q\le\infty$, provided that

$$\|v\|_{L^p(0,T;D)}=\left(\int_0^T\|v(t)\|_D^p\mathrm{d}t\right)^{\frac{1}{p}},\quad \text{for } 1\le p<\infty; \tag{15}$$

while

$$\|v\|_{L^\infty(0,T;D)}=esssup_{t\in(0,T)}\|v(t)\|_D,\quad \text{for } p=\infty. \tag{16}$$

Assume that problem (1) has a unique solution, and then determine its form. Let $u(x,t)=\sum_{n=1}^{\infty}\langle u(t,\cdot),\phi_n(\cdot)\rangle\,\phi_n(x)$ be the Fourier series in $\mathbb{L}^2(\Omega)$ with. From (1), taking the inner product of both sides of first equation of (1) with $\phi_n(x)$, we get

$$\langle\partial_t^\alpha u(\cdot,t),\phi_n\rangle+k\langle\partial_t^\alpha\mathscr{L}u(\cdot,t),\phi_n\rangle+\langle\mathscr{L}u(\cdot,t),\phi_n\rangle=\langle\mathscr{F}(\cdot,t),\phi_n\rangle \tag{17}$$

Using (11), we have

$$(1+k\lambda_n)\partial_t^\alpha\langle u(\cdot,t),\phi_n\rangle+\lambda_n\langle u(\cdot,t),\phi_n\rangle=\langle\mathscr{F}(\cdot,t),\phi_n\rangle \tag{18}$$

and

$$\partial_t^\alpha\langle u(\cdot,t),\phi_n\rangle+\frac{\lambda_n}{1+k\lambda_n}\langle u(\cdot,t),\phi_n\rangle=\frac{1}{1+k\lambda_n}\langle\mathscr{F}(\cdot,t),\phi_n\rangle. \tag{19}$$

We can obtain a unique function u_n as follows (see for example, [42], [43])

$$u_n(t)=E_{\alpha,1}\left(\frac{-\lambda_n t^\alpha}{1+k\lambda_n}\right)u_{0,n}+\frac{1}{1+k\lambda_n}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{-(t-s)^\alpha\lambda_n}{1+k\lambda_n}\right)\mathscr{F}_n(s)\mathrm{d}s, \tag{20}$$

where $u_{0,n}=\langle u_0,\phi_n\rangle$ and $\mathscr{F}_n(t)=\langle\mathscr{F}(t),\phi_n\rangle$. Hence, the Fourier series $u(x,t)=\sum_{n=1}^{\infty}\langle u(\cdot,t),\phi_n(\cdot)\rangle\,\phi_n(x)$ can be used to represent the solution (1). Therefore, we can get

$$u(x,t)=\sum_{n=1}^{\infty}E_{\alpha,1}\left(\frac{-\lambda_n t^\alpha}{1+k\lambda_n}\right)u_{0,n}\phi_n+\sum_{n=1}^{\infty}\frac{1}{1+k\lambda_n}\left[\int_0^t(t-s)^\alpha E_{\alpha,\alpha}\left(\frac{-(t-s)^{\alpha-1}\lambda_n}{1+k\lambda_n}\right)\mathscr{F}_n(s)\mathrm{d}s\right]\phi_n. \tag{21}$$

3 Continuity with respect to fractional order of the time a linear fractional pseudo-parabolic equation.

In this section, we present the continuous dependence of the solution of Problem (1)) on the input data (the fractional order α, and the initial condition u_0).

Theorem 1.*Given a number $p < q < p+1$. Assume that $1 < \delta < \alpha < \alpha' < \nu < 2$ and let $u_0 \in \mathbb{H}^q(\Omega)$, $\mathscr{F} \in \mathscr{L}^\infty(0,T;\mathbb{H}^q(\Omega))$. Let $\mathbf{u}_\alpha$ and $\mathbf{u}_{\alpha'}$ be the solutions of Problem (1) with respect to the fractional orders α and α'. If the numbers γ, ε satisfy $0<\gamma<1$, and $0<\varepsilon<\delta\gamma$, then*

$$\|\mathbf{u}(x,t)\|_{\mathscr{C}^{\alpha p}(0,T,\mathbb{H}^q(\Omega))} \le \mathfrak{N}_2(\delta,\nu)(\lambda_1^{-1}+k)^p\|u_0\|_{\mathbb{H}^q(\Omega)} + \mathfrak{N}_3(\delta,\nu)\Big(\lambda_1^{-1}+k\Big)^{p-1}\lambda_1^{-1}\|\mathscr{F}(\cdot,s)\|_{\mathbb{H}^q(\Omega)}\frac{T^\alpha}{\alpha-\alpha p}, \tag{22}$$

and

$$\begin{aligned}&\|\mathbf{u}_{\alpha'}(x,t)-\mathbf{u}_\alpha(x,t)\|_{\mathscr{C}^{\nu(1-\gamma)+\varepsilon}(0,T,\mathbb{H}^q(\Omega))}\\ &\le \mathscr{A}(\delta,\nu,\varepsilon,\gamma,T)\Big[(\alpha'-\alpha)^\varepsilon+(\alpha'-\alpha)\Big](\lambda_1^{-}+k)^{(1-\gamma)}\|u_0\|_{\mathbb{H}^q(\Omega)}\\ &+\mathscr{B}(\delta,\nu,\varepsilon,\gamma,T)\Big[(\alpha'-\alpha)^\varepsilon+(\alpha'-\alpha)\Big]\lambda_1^{-1}k^{-\gamma}\frac{T^{(\delta-\nu)\gamma+\nu}}{\delta\gamma-\varepsilon}\|\mathscr{F}(\cdot,s)\|_{\mathscr{L}^\infty(0,T,\mathbb{H}^q(\Omega))},\end{aligned} \tag{23}$$

*Proof.*The proof comes from the results given in Section 2.1 and some estimates of Mittag-Leffler functions. From (21) we obtain the solution of Problem 1

$$\mathbf{u}_\alpha(x,t) = \mathscr{M}_\alpha(t)u_0 + \int_0^t \mathscr{N}_\alpha(t-s)\mathscr{F}(\cdot,s)\mathrm{d}s, \tag{24}$$

where

$$\mathscr{M}_\alpha(t)w = \sum_{n=1}^\infty E_{\alpha,1}\left(\frac{-\lambda_n t^\alpha}{1+k\lambda_n}\right)\langle w,\phi_n\rangle\phi_n,$$

$$\mathscr{N}_\alpha(t-s)w = \sum_{n=1}^\infty \frac{1}{1+k\lambda_n}\left[(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{-(t-s)^\alpha\lambda_n}{1+k\lambda_n}\right)\langle w,\phi_n\rangle\right]\phi_n.$$

Therefore, we arrived at the below estimate:

$$\mathbf{u}_{\alpha'}(x,t) = \mathscr{M}_{\alpha'}(t)u_0 + \int_0^t \mathscr{N}_{\alpha'}(t-s)\mathscr{F}(\cdot,s)\mathrm{d}s. \tag{25}$$

Lemma 8.*Let $1<\alpha<2, p<q<p+1$ and $w\in\mathbb{H}^q(\Omega)$.we get several inequalities as follows:*

$$\|\mathscr{M}_\alpha(t)w\|_{\mathbb{H}^q(\Omega)} \le \mathfrak{N}_2(\delta,\nu)t^{-\alpha p}(\lambda_1^{-1}+k)^p\|w\|_{\mathbb{H}^q(\Omega)}. \tag{26}$$

$$\|\mathscr{N}_\alpha(t-s)w\|_{\mathbb{H}^\eta(\Omega)} \le (t-s)^{(\alpha-1-\alpha p)}\mathfrak{N}_3(\delta,\nu)\Big(\lambda_1^{-1}+k\Big)^{p-1}\lambda_1^{-1}\|w\|_{\mathbb{H}^q(\Omega)}. \tag{27}$$

*Proof.*From lemma (4), with 0, we deduce

$$\begin{aligned}\|\mathscr{M}_\alpha(t)w\|^2_{\mathbb{H}^q(\Omega)} &= \sum_{n=1}^\infty \lambda_n^{2q}E^2_{\alpha,1}\left(\frac{-\lambda_n t^\alpha}{1+k\lambda_n}\right)|\langle w,\phi_n\rangle|^2 \le \sum_{n=1}^\infty \lambda_n^{2q}\left(\frac{\mathfrak{N}_2(\delta,\nu)}{1+\frac{\lambda_n t^\alpha}{1+k\lambda_n}}\right)^2|\langle w,\phi_n\rangle|^2,\\ &\le \sum_{n=1}^\infty \lambda_n^{2q}\frac{\mathfrak{N}_2^2(\delta,\nu)}{\left(1+\frac{\lambda_n t^\alpha}{1+k\lambda_n}\right)^p}|\langle w,\phi_n\rangle|^2 \le \sum_{n=1}^\infty \lambda_n^{2q}\mathfrak{N}_2^2(\delta,\nu)\left(\frac{\lambda_n t^\alpha}{1+k\lambda_n}\right)^{-2p}|\langle w,\phi_n\rangle|^2\\ &\le \mathfrak{N}_2^2(\delta,\nu)t^{-2\alpha p}(1+k\lambda_1^{-1})^{2p}\sum_{n=1}^\infty \lambda_n^{2q}|\langle w,\phi_n\rangle|^2\\ &\le \mathfrak{N}_2^2(\delta,\nu)t^{-2\alpha p}(\frac{1}{\lambda_1}+k)^{2p}\|w\|^2_{\mathbb{H}^q(\Omega)}.\end{aligned}$$

This implies that

$$\|\mathscr{M}_\alpha(t)w\|_{\mathbb{H}^q(\Omega)} \leq \mathfrak{N}_2(\delta,\nu)t^{-\alpha p}(\frac{1}{\lambda_1}+k)^p\|w\|_{\mathbb{H}^q(\Omega)}. \tag{28}$$

Similarly, applying lemma (4), we also get

$$\begin{aligned}
\|\mathscr{N}_\alpha(t-s)w\|^2_{\mathbb{H}^q(\Omega)} &= \sum_{n=1}^{\infty}\frac{\lambda_n^{2q}}{(1+k\lambda_n)^2}(t-s)^{2(\alpha-1)}E^2_{\alpha,\alpha}\left(\frac{-\lambda_n(t-s)^\alpha}{1+k\lambda_n}\right)|\langle w,\varphi_j\rangle|^2,\\
&\leq \sum_{n=1}^{\infty}\frac{\lambda_n^{2q}}{(1+k\lambda_n)^2}(t-s)^{2(\alpha-1)}\left(\frac{\mathfrak{N}_3(\delta,\nu)}{1+\frac{\lambda_n(t-s)^\alpha}{1+k\lambda_n}}\right)^2|\langle w,\varphi_n\rangle|^2,\\
&\leq (t-s)^{2(\alpha-1-\alpha p)}\mathfrak{N}_3^2(\delta,\nu)\sum_{n=1}^{\infty}\lambda_n^{2q}\left(\frac{1}{\lambda_n}+k\right)^{2(p-1)}\lambda_n^{-2}|\langle w,\varphi_n\rangle|^2,\\
&\leq (t-s)^{2(\alpha-1-\alpha p)}\mathfrak{N}_3^2(\delta,\nu)\left(\lambda_1^{-1}+k\right)^{2(p-1)}\lambda_1^{-2}\sum_{n=1}^{\infty}\lambda_n^{2q}|\langle w,\varphi_n\rangle|^2,\\
&\leq (t-s)^{2(\alpha-1-\alpha p)}\mathfrak{N}_3^2(\delta,\nu)\left(\frac{1}{\lambda_1+k}\right)^{2(p-1)}\lambda_1^{-2}\|w\|^2_{\mathbb{H}^q(\Omega)}.
\end{aligned} \tag{29}$$

Therefore, we deduce that

$$\|\mathscr{N}_\alpha(t-s)w\|_{\mathbb{H}^\eta(\Omega)} \leq (t-s)^{(\alpha-1-\alpha p)}\mathfrak{N}_3(\delta,\nu)\left(\frac{1}{\lambda_1+k}\right)^{p-1}\lambda_1^{-1}\|w\|_{\mathbb{H}^q(\Omega)}. \tag{30}$$

Thus, the proof of the lemma is complete. (8).

Lemma 9. *Let $1<\alpha<2$, $q>0$; $0<\gamma<1$ and $w\in\mathbb{H}^q(\Omega)$. The following inequalities are present:*

$$\begin{aligned}
&\left\|\left[\mathscr{M}_{\alpha'}(t)-\mathscr{M}_\alpha(t)\right]w\right\|_{\mathbb{H}^q(\Omega)}\\
&\quad\leq \mathscr{A}(\delta,\nu,\varepsilon,\gamma,T)\left[(\alpha'-\alpha)^\varepsilon+(\alpha'-\alpha)\right]t^{-\nu(1-\gamma)-\varepsilon}(\lambda_1^{-}+k)^{(1-\gamma)}\|w\|_{\mathbb{H}^q(\Omega)}.
\end{aligned} \tag{31}$$

$$\begin{aligned}
&\left\|\left[\mathscr{N}_{\alpha'}(t-s)-\mathscr{N}_\alpha(t-s)\right]w\right\|_{\mathbb{H}^q(\Omega)}\\
&\quad\leq \mathscr{B}(\delta,\nu,\varepsilon,\gamma,T)\left[(\alpha'-\alpha)^\varepsilon+(\alpha'-\alpha)\right]\lambda_1^{-1}k^{-\gamma}(t-s)^{\delta\gamma-\varepsilon-1}\|w\|_{\mathbb{H}^q(\Omega)}.
\end{aligned} \tag{32}$$

Proof. We get

$$\left\|\left[\mathscr{M}_{\alpha'}(t)-\mathscr{M}_\alpha(t)\right]w\right\|^2_{\mathbb{H}^q(\Omega)} = \sum_{n=1}^{\infty}\lambda_n^{2q}\left[E_{\alpha',1}\left(\frac{-\lambda_n t^{\alpha'}}{1+k\lambda_n}\right)-E_{\alpha,1}\left(\frac{-\lambda_n t^\alpha}{1+k\lambda_n}\right)\right]^2|\langle w,\varphi_n\rangle|^2. \tag{33}$$

Using Lemma (6), we get

$$\begin{aligned}
&\left\|\left[\mathscr{M}_{\alpha'}(t)-\mathscr{M}_\alpha(t)\right]w\right\|^2_{\mathbb{H}^q(\Omega)}\\
&\quad\leq \sum_{n=1}^{\infty}\lambda_n^{2q}\mathscr{A}^2(\delta,\nu,\varepsilon,\gamma,T)\left[(\alpha'-\alpha)^\varepsilon+(\alpha'-\alpha)\right]^2\left(\frac{\lambda_n}{1+k\lambda_n}\right)^{2(\gamma-1)}t^{-2\nu(1-\gamma)-2\varepsilon}|\langle w,\varphi_n\rangle|^2,\\
&\quad\leq \mathscr{A}^2(\delta,\nu,\varepsilon,\gamma,T)\left[(\alpha'-\alpha)^\varepsilon+(\alpha'-\alpha)\right]^2t^{-2\nu(1-\gamma)-2\varepsilon}(\lambda_1^{-}+k)^{2(1-\gamma)}\sum_{n=1}^{\infty}\lambda_n^{2q}|\langle w,\varphi_n\rangle|^2,\\
&\quad\leq \mathscr{A}^2(\delta,\nu,\varepsilon,\gamma,T)\left[(\alpha'-\alpha)^\varepsilon+(\alpha'-\alpha)\right]^2t^{-2\nu(1-\gamma)-2\varepsilon}(\lambda_1^{-}+k)^{2(1-\gamma)}\|w\|^2_{\mathbb{H}^q(\Omega)}.
\end{aligned}$$

So, we obtain that

$$\begin{aligned}
&\left\|\left[\mathscr{M}_{\alpha'}(t)-\mathscr{M}_\alpha(t)\right]w\right\|_{\mathbb{H}^q(\Omega)}\\
&\quad\leq \mathscr{A}(\delta,\nu,\varepsilon,\gamma,T)\left[(\alpha'-\alpha)^\varepsilon+(\alpha'-\alpha)\right]t^{-\nu(1-\gamma)-\varepsilon}(\lambda_1^{-}+k)^{(1-\gamma)}\|w\|_{\mathbb{H}^q(\Omega)}.
\end{aligned} \tag{34}$$

where $0 < \gamma < 1$. We get the following approximation using the lemma (7)

$$\begin{aligned}
&\left\|\left[\mathscr{N}_{\alpha'}(t-s)-\mathscr{N}_{\alpha}(t-s)\right]w\right\|^2_{\mathbb{H}^q(\Omega)},\\
&=\sum_{n=1}^{\infty}\frac{\lambda_n^{2q}}{(1+k\lambda_n)^2}\left[(t-s)^{\alpha'-1}E_{\alpha',\alpha'}\left(\frac{-\lambda_n}{1+k\lambda_n}(t-s)^{\alpha'}\right)-(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{-\lambda_n}{1+k\lambda_n}(t-s)^{\alpha}\right)\right]^2|\langle w,\varphi_n\rangle|^2,\\
&\le\sum_{n=1}^{\infty}\mathscr{B}^2(\delta,\nu,\varepsilon,\gamma,T)\left(\frac{\lambda_n}{1+k\lambda_n}\right)^{2(\gamma-1)}t^{2\delta\gamma-2\varepsilon-2}\left[(\alpha'-\alpha)^{\varepsilon}+(\alpha'-\alpha)\right]^2\frac{\lambda_n^{2q}}{(1+k\lambda_n)^2}|\langle w,\varphi_n\rangle|^2,\\
&\le\mathscr{B}^2(\delta,\nu,\varepsilon,\gamma,T)\left[(\alpha'-\alpha)^{\varepsilon}+(\alpha'-\alpha)\right]^2\lambda_1^{-2}(t-s)^{2\delta\gamma-2\varepsilon-2}\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}+k\right)^{-2\gamma}\lambda_n^{2q}|\langle w,\varphi_n\rangle|^2,\\
&\le\mathscr{B}^2(\delta,\nu,\varepsilon,\gamma,T)\left[(\alpha'-\alpha)^{\varepsilon}+(\alpha'-\alpha)\right]^2\lambda_1^{-2}k^{-2\gamma}(t-s)^{2\delta\gamma-2\varepsilon-2}\|w\|^2_{\mathbb{H}^q(\Omega)}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left\|\left[\mathscr{N}_{\alpha'}(t-s)-\mathscr{N}_{\alpha}(t-s)\right]w\right\|_{\mathbb{H}^q(\Omega)}\\
&\qquad\le\mathscr{B}(\delta,\nu,\varepsilon,\gamma,T)\left[(\alpha'-\alpha)^{\varepsilon}+(\alpha'-\alpha)\right]\lambda_1^{-1}k^{-\gamma}(t-s)^{\delta\gamma-\varepsilon-1}\|w\|_{\mathbb{H}^q(\Omega)}. \qquad (35)
\end{aligned}$$

All estimations of lemma (9) are obtained. The proof is now complete.

Now, we need to estimate the term $\|u(\cdot,t)\|_{\mathbb{H}^p(\Omega)}$. From (24) and applying lemma (8), we obtain

$$\begin{aligned}
\|\mathbf{u}(x,t)\|_{\mathbb{H}^q(\Omega)}&\le\left\|\mathscr{M}_{\alpha}(t)u_0\right\|_{\mathbb{H}^q(\Omega)}+\left\|\int_0^t\mathscr{N}_{\alpha}(t-s)\mathscr{F}(\cdot,s)\mathrm{d}s\right\|\\
&\le\mathfrak{N}_2(\delta,\nu)t^{-\alpha p}(\lambda_1^{-1}+k)^p\|u_0\|_{\mathbb{H}^q(\Omega)}\\
&+\int_0^t(t-s)^{(\alpha-1-\alpha p)}\mathfrak{N}_3(\delta,\nu)\left(\lambda_1^{-1}+k\right)^{p-1}\lambda_1^{-1}\|\mathscr{F}(\cdot,s)\|_{\mathbb{H}^q(\Omega)}ds. \qquad (36)
\end{aligned}$$

Assume $0 < p < 1$ then $\alpha-\alpha p-1 > -1$.By the assumption $\mathscr{F}\in\mathscr{L}^{\infty}(0,T;\mathbb{H}^q(\Omega))$, we obtain

$$\begin{aligned}
\|\mathbf{u}(x,t)\|_{\mathbb{H}^q(\Omega)}&\le\mathfrak{N}_2(\delta,\nu)t^{-\alpha p}(\lambda_1^{-1}+k)^p\|u_0\|_{\mathbb{H}^q(\Omega)}\\
&+\mathfrak{N}_3(\delta,\nu)\left(\lambda_1^{-1}+k\right)^{p-1}\lambda_1^{-1}\|\mathscr{F}(\cdot,s)\|_{\mathbb{H}^q(\Omega)}\int_0^t(t-s)^{(\alpha-1-\alpha p)}ds\\
&\le\mathfrak{N}_2(\delta,\nu)t^{-\alpha p}(\lambda_1^{-1}+k)^p\|u_0\|_{\mathbb{H}^q(\Omega)}\\
&+\mathfrak{N}_3(\delta,\nu)\left(\lambda_1^{-1}+k\right)^{p-1}\lambda_1^{-1}\|\mathscr{F}(\cdot,s)\|_{\mathbb{H}^q(\Omega)}\frac{t^{\alpha-\alpha p}}{\alpha-\alpha p}. \qquad (37)
\end{aligned}$$

Multiplying both sides of the latter estimate to $t^{\alpha p}$, we get the following

$$\begin{aligned}
t^{\alpha p}\|\mathbf{u}(x,t)\|_{\mathbb{H}^q(\Omega)}&\le\mathfrak{N}_2(\delta,\nu)(\lambda_1^{-1}+k)^p\|u_0\|_{\mathbb{H}^q(\Omega)}\\
&+\mathfrak{N}_3(\delta,\nu)\left(\lambda_1^{-1}+k\right)^{p-1}\lambda_1^{-1}\|\mathscr{F}(\cdot,s)\|_{\mathbb{H}^q(\Omega)}\frac{t^{\alpha}}{\alpha-\alpha p}. \qquad (38)
\end{aligned}$$

Therefor, we obtain the following estimates

$$\begin{aligned}
\|\mathbf{u}(x,t)\|_{\mathscr{C}^{\alpha p}(0,T,\mathbb{H}^q(\Omega))}&\le\mathfrak{N}_2(\delta,\nu)(\lambda_1^{-1}+k)^p\|u_0\|_{\mathbb{H}^q(\Omega)}\\
&+\mathfrak{N}_3(\delta,\nu)\left(\lambda_1^{-1}+k\right)^{p-1}\lambda_1^{-1}\|\mathscr{F}(\cdot,s)\|_{\mathbb{H}^q(\Omega)}\frac{T^{\alpha}}{\alpha-\alpha p}. \qquad (39)
\end{aligned}$$

From (24)-(25), by applying Lemma 9, we get the following estimate

$$\begin{aligned}
&\|\mathbf{u}_{\alpha'}(x,t)-\mathbf{u}_{\alpha}(x,t)\|_{\mathbb{H}^q(\Omega)} \\
&\le \left\|\left[\mathscr{M}_{\alpha'}-\mathscr{M}_{\alpha}(t)\right](t)u_0\right\|_{\mathbb{H}^q(\Omega)}+\left\|\int_0^t\left[\mathscr{N}_{\alpha'}(t-s)-\mathscr{N}_{\alpha}(t-s)\right]\mathscr{F}(\cdot,s)\mathrm{d}s\right\| \\
&\le \mathscr{A}(\delta,\nu,\varepsilon,\gamma,T)\left[(\alpha'-\alpha)^{\varepsilon}+(\alpha'-\alpha)\right]t^{-\nu(1-\gamma)-\varepsilon}(\lambda_1^-+k)^{(1-\gamma)}\|u_0\|_{\mathbb{H}^q(\Omega)} \\
&+\int_0^t\mathscr{B}(\delta,\nu,\varepsilon,\gamma,T)\left[(\alpha'-\alpha)^{\varepsilon}+(\alpha'-\alpha)\right]\lambda_1^{-1}k^{-\gamma}(t-s)^{\delta\gamma-\varepsilon-1}\|\mathscr{F}(\cdot,s)\|_{\mathbb{H}^q(\Omega)}ds.
\end{aligned} \tag{40}$$

Since the assumption $0<\varepsilon<\delta\gamma$, we know that $\delta\gamma-1-\varepsilon>-1$. It follows from $\mathscr{F}\in\mathscr{L}^{\infty}(0,T;\mathbb{H}^q(\Omega))$ that

$$\begin{aligned}
&\|\mathbf{u}_{\alpha'}(x,t)-\mathbf{u}_{\alpha}(x,t)\|_{\mathbb{H}^q(\Omega)} \\
&\le \mathscr{A}(\delta,\nu,\varepsilon,\gamma,T)\left[(\alpha'-\alpha)^{\varepsilon}+(\alpha'-\alpha)\right]t^{-\nu(1-\gamma)-\varepsilon}(\lambda_1^-+k)^{(1-\gamma)}\|u_0\|_{\mathbb{H}^q(\Omega)} \\
&+\mathscr{B}(\delta,\nu,\varepsilon,\gamma,T)\left[(\alpha'-\alpha)^{\varepsilon}+(\alpha'-\alpha)\right]\lambda_1^{-1}k^{-\gamma}\|\mathscr{F}(\cdot,s)\|_{\mathscr{L}^{\infty}(0,T,\mathbb{H}^q(\Omega))}\int_0^t(t-s)^{\delta\gamma-\varepsilon-1}ds. \\
&\le \mathscr{A}(\delta,\nu,\varepsilon,\gamma,T)\left[(\alpha'-\alpha)^{\varepsilon}+(\alpha'-\alpha)\right]t^{-\nu(1-\gamma)-\varepsilon}(\lambda_1^-+k)^{(1-\gamma)}\|u_0\|_{\mathbb{H}^q(\Omega)} \\
&+\mathscr{B}(\delta,\nu,\varepsilon,\gamma,T)\left[(\alpha'-\alpha)^{\varepsilon}+(\alpha'-\alpha)\right]\lambda_1^{-1}k^{-\gamma}\|\mathscr{F}(\cdot,s)\|_{\mathscr{L}^{\infty}(0,T,\mathbb{H}^q(\Omega))}\frac{t^{\delta\gamma-\varepsilon}}{\delta\gamma-\varepsilon}.
\end{aligned} \tag{41}$$

We get the following estimate by multiplying both sides by $t^{\nu(1-\gamma)+\varepsilon}$

$$\begin{aligned}
&t^{\nu(1-\gamma)+\varepsilon}\|\mathbf{u}_{\alpha'}(x,t)-\mathbf{u}_{\alpha}(x,t)\|_{\mathbb{H}^q(\Omega)} \\
&\le \mathscr{A}(\delta,\nu,\varepsilon,\gamma,T)\left[(\alpha'-\alpha)^{\varepsilon}+(\alpha'-\alpha)\right](\lambda_1^-+k)^{(1-\gamma)}\|u_0\|_{\mathbb{H}^q(\Omega)} \\
&+\mathscr{B}(\delta,\nu,\varepsilon,\gamma,T)\left[(\alpha'-\alpha)^{\varepsilon}+(\alpha'-\alpha)\right]\lambda_1^{-1}k^{-\gamma}\|\mathscr{F}(\cdot,s)\|_{\mathscr{L}^{\infty}(0,T,\mathbb{H}^q(\Omega))}\frac{t^{(\delta-\nu)\gamma+\nu}}{\delta\gamma-\varepsilon}
\end{aligned}$$

Hence, we obtain that

$$\begin{aligned}
&\|\mathbf{u}_{\alpha'}(x,t)-\mathbf{u}_{\alpha}(x,t)\|_{\mathscr{C}^{\nu(1-\gamma)+\varepsilon}(0,T,\mathbb{H}^q(\Omega))} \\
&\le \mathscr{A}(\delta,\nu,\varepsilon,\gamma,T)\left[(\alpha'-\alpha)^{\varepsilon}+(\alpha'-\alpha)\right](\lambda_1^-+k)^{(1-\gamma)}\|u_0\|_{\mathbb{H}^q(\Omega)} \\
&+\mathscr{B}(\delta,\nu,\varepsilon,\gamma,T)\left[(\alpha'-\alpha)^{\varepsilon}+(\alpha'-\alpha)\right]\lambda_1^{-1}k^{-\gamma}\frac{T^{(\delta-\nu)\gamma+\nu}}{\delta\gamma-\varepsilon}\|\mathscr{F}(\cdot,s)\|_{\mathscr{L}^{\infty}(0,T,\mathbb{H}^q(\Omega))}
\end{aligned}$$

We finish Theorem (1) .

Conflict of Interest

The authors declare that they have no conflict of interest.

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