J. Stat. Appl. Pro. **10**, No. 3, 847-864 (2021)

Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/100321

The Odd Inverse Pareto-Moyal Distribution

M. Ibrahem[1,*], *M. M. Nassar*[2], *N. L. Bassily*[2] *and M. Abdel-Aziz*[1]

[1]Department of Basic Sciences, Faculty of Computer and Information Sciences, Ain Shams University, Abbassia, Cairo, Egypt
[2]Department of Mathematics, Faculty of Science, Ain Shams University, Abbassia, Cairo, Egypt

Received: 16 Jun. 2020, Revised: 21 Aug. 2020, Accepted: 26 Aug. 2020
Published online: 1 Nov. 2021

Abstract: In this paper, a four parameter generalization of Moyal distribution is obtained, with the purpose of obtaining a more flexible model relative to the behaviour of hazard rate functions. Various statistical properties of this distribution including the density, hazard rate functions, quantile function, mode, moments, incomplete moments, moment generating functions, mean deviation, Lorenz, Bonferroni and Zenga curves, Rényi and continuous entropies and distribution of r^{th} order statistics have been derived. The method of maximum likelihood estimation has been used to estimate the parameters of the generalized Moyal distribution and the observed information matrix is derived. Two real data sets are presented to demonstrate the effectiveness of the new model.

Keywords: Moyal distribution, inverse Pareto distribution, moment generating function, entropy and maximum likelihood estimation.

1 Introduction

The procedure of adding new shape parameters to a family of distributions to generate new distributions that are more flexible is a well-known technique in the statistical literature. Some well-known generators are the Beta-G distributions by Eugene et al. [1], Jones [2], Nadarajah and Kotz [3], [4], Cordeiro and Lemonte [5], Cordeiro et al. [6], Nassar and Nada [7], [8], [9], Nassar and Elmasry [10] and Mahmoud et al. [11], Kumaraswamy-G by Jones [12], Cordeiro et al. [13], [14], Cordeiro and de Castro [15], Elbatal and Elgarhy [16], Nassar [17] and Gamma-G distributions by Zografos and Balakrishnan [18].

In this paper, we introduce a four parameter model, called the odd inverse Pareto - Moyal (OIP-Mo) distribution, to extend the Moyal model for its importance and usefulness in many practical situations.

The Moyal distribution was proposed by J. E. Moyal [19] as an approximation to the Landau distribution. It was also shown that it remains valid taking into account quantum resonance effects and details of atomic structure of the absorber. The Moyal distribution is a universal form for the energy loss by ionization for a fast charged particle and the number of ion pairs produced in this process. From the Topp Leone-Moyal distribution proposed by Nassar and Ibrahem [20], we write $X \sim Mo(\mu,\, \sigma)$ to denote a random variable X having Moyal (Mo) distribution with cumulative distribution function (cdf), depending on the lower incomplete gamma function $\gamma(\alpha,x) = \int_0^x t^{\alpha-1}e^{-t}dt$, given by

$$G(x) = 1 - \frac{\gamma\left(\frac{1}{2},\ \frac{1}{2}e^{-\left(\frac{x-\mu}{\sigma}\right)}\right)}{\Gamma\left(\frac{1}{2}\right)} = 1 - \operatorname{erf}\left(\frac{\sqrt{e^{-\left(\frac{x-\mu}{\sigma}\right)}}}{\sqrt{2}}\right), \quad -\infty < x,\mu < \infty,\ \sigma > 0. \tag{1}$$

where $\gamma\left(\frac{1}{2},x\right) = \sqrt{\pi}erf\left(\sqrt{x}\right)$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and the error function $erf\left(x\right)$ is defined by $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$.

The corresponding probability density function (pdf) is

$$g\left(x\right) = \frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-1}{2}\left[\left(\frac{x-\mu}{\sigma}\right)+e^{-\left(\frac{x-\mu}{\sigma}\right)}\right]}, \quad -\infty<x,\mu<\infty,\ \sigma>0. \tag{2}$$

* Corresponding author e-mail: mohamedibrahem1993@yahoo.com

Let Z be a random variable following the Moyal standard pdf given by

$$g(z)=\frac{1}{\sqrt{2\pi}}e^{\frac{-1}{2}(z+e^{-z})}\quad,\qquad -\infty<z<\infty\ . \tag{3}$$

The cdf of the standard Moyal distribution (3) is

$$G(z)=1-\frac{\gamma\left(\frac{1}{2}\,,\ \frac{1}{2}e^{-z}\right)}{\Gamma\left(\frac{1}{2}\right)}=1-\operatorname{erf}\left(\frac{\sqrt{e^{-z}}}{\sqrt{2}}\right),\quad -\infty<z<\infty. \tag{4}$$

Two generalizations of the Moyal distribution was the four – parameter Beta Moyal distribution proposed by Cordeiro et al. [6] and the three - parameter Topp Leone-Moyal distribution proposed by Nassar and Ibrahem [20].

2 The Odd Inverse Pareto-Moyal Distribution

In this section, we introduce the odd inverse Pareto-Moyal (OIP-Mo) distribution. Some reliability functions corresponding to the OIP-Mo distribution are also discussed. The pdf of the inverse Pareto (IP) distribution, is given by

$$f(x;\alpha,\beta)=\frac{\alpha\beta x^{\alpha-1}}{(\beta+x)^{\alpha+1}}\ ,x>0\ ,\alpha,\beta>0. \tag{5}$$

The cdf of the inverse Pareto (IP) distribution (5) is

$$F(x;\alpha,\beta)=\left(\frac{x}{x+\beta}\right)^{\alpha},x>0\ ,\ \alpha,\beta>0. \tag{6}$$

The odd inverse Pareto-G (OIP-G) is based on the inverse Pareto distribution and the family of distributions G, proposed by Alzaatreh et al. [21]. Then, the cdf and pdf of OIP-G family are

$$F_{OIP-G}(x)=\alpha\beta\int_0^{\frac{G(x)}{1-G(x)}}\frac{t^{\alpha-1}}{(\beta+t)^{\alpha+1}}dt=\frac{[G(x)]^{\alpha}}{[\beta(1-G(x))+G(x)]^{\alpha}}\ ,\alpha>0,\beta>0. \tag{7}$$

and

$$f_{OIP-G}(x)=\alpha\beta\frac{g(x)\,[G(x)]^{\alpha-1}}{(1-G(x))^{\alpha+1}}\left(\beta+\frac{G(x)}{1-G(x)}\right)^{-\alpha-1}=\frac{\alpha\beta g(x)\,[G(x)]^{\alpha-1}}{[\beta(1-G(x))+G(x)]^{\alpha+1}},\alpha>0,\beta>0. \tag{8}$$

where G(x) is the baseline distribution function and g(x)= $\partial G(x)/\partial x$ is the baseline density function. Inserting Equation (1) in Equation (7), we obtain the odd inverse Pareto- Moyal (OIP-Mo) distribution with cdf given by

$$F(x;\alpha,\beta,\mu,\sigma)=\left[1+\frac{\beta\operatorname{erf}\left(\frac{\sqrt{e^{-\left(\frac{x-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)}{1-\operatorname{erf}\left(\frac{\sqrt{e^{-\left(\frac{x-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)}\right]^{-\alpha},-\infty<x,\mu<\infty,\ \sigma>0,\alpha>0\ and\ \beta>0. \tag{9}$$

The pdf corresponding to Equation (9) will be as follows

$$f(x;\alpha,\beta,\mu,\sigma)=\frac{\alpha\beta\ \mathrm{e}^{-\frac{1}{2}\left(\mathrm{e}^{-\left(\frac{x-\mu}{\sigma}\right)}+\frac{x-\mu}{\sigma}\right)}\left[1-erf\left(\frac{\sqrt{e^{-\left(\frac{x-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)\right]^{\alpha-1}}{\sqrt{2\pi}\,\sigma\left[1-erf\left(\frac{\sqrt{e^{-\left(\frac{x-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)+\beta\ erf\left(\frac{\sqrt{e^{-\left(\frac{x-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)\right]^{\alpha+1}} \tag{10}$$

where $-\infty<x,\mu<\infty$, $\sigma>0$,$\alpha>0$ and $\beta>0$. For $\mu=0$ and $\sigma=1$, we obtain the standard OIP-Mo cdf given by

$$F(x)=\left[1+\frac{\beta\operatorname{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}{1-\operatorname{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}\right]^{-\alpha} \tag{11}$$

and the standard OIP-Mo density function given by

$$f(x) = \frac{\alpha\beta \, \mathrm{e}^{-\frac{1}{2}\left(\mathrm{e}^{-x}+x\right)} \left[1 - erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^{\alpha-1}}{\sqrt{2\pi}\left[1 - erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right) + \beta \; erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^{\alpha+1}} \tag{12}$$

Plots of the density function (10) for selected values of the OIP-Mo distribution are given in Figure 1. It is illustrated that these plots show great flexibility of the OIP-Mo for different values of the shape parameters α and β, including the special case in Equation (12). It is also illustrated that when the value of μ changes from -ve to +ve value, $f(x)$ is displaced to the +ve side of x-axis.

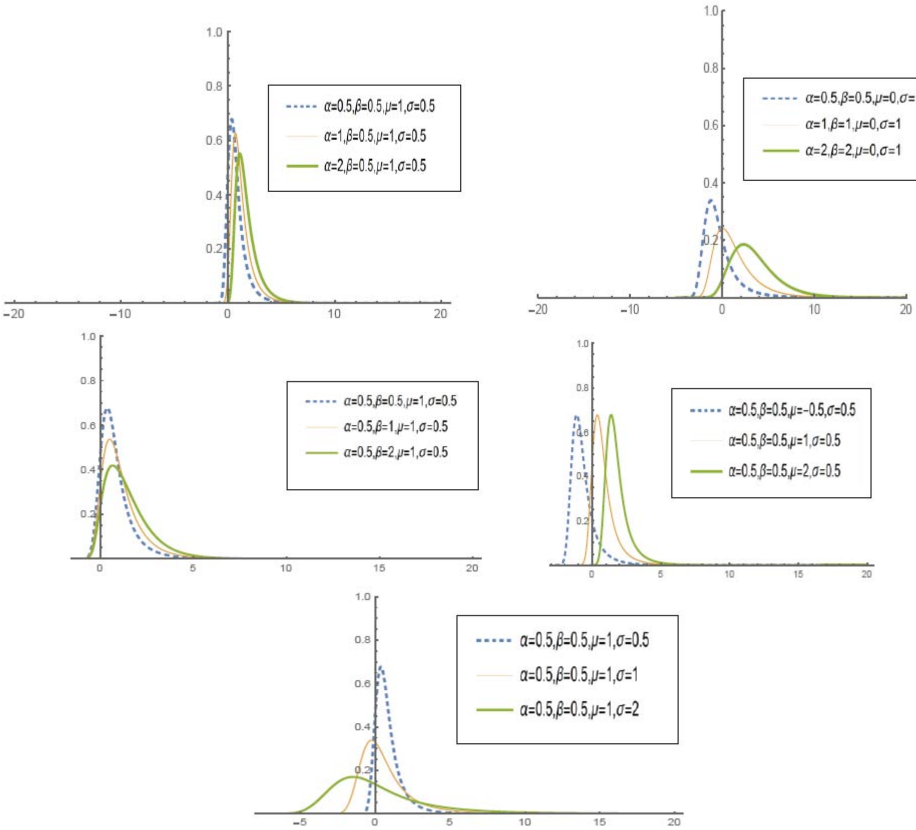

Fig. 1: Plots of the pdf of OIP-Mo distribution for some parameter values.

We define the hazard rate function of the OIP-Mo distribution as follows,

$$h(x) = \frac{f(x)}{R(x)},$$

where $R(x)$ is the reliability function of the OIP-Mo distribution, by the following

$$R(x) = 1 - F(x) = 1 - \left[1 + \frac{\beta \mathrm{erf}\left(\frac{\sqrt{e^{-\left(\frac{x-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)}{1 - \mathrm{erf}\left(\frac{\sqrt{e^{-\left(\frac{x-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)}\right]^{-\alpha}.$$

Then the hazard rate function of the OIP-Mo distribution (10) is given by

$$h(x)=\frac{\alpha\beta\ \mathrm{e}^{-\frac{1}{2}\left(\mathrm{e}^{-\left(\frac{x-\mu}{\sigma}\right)}+\frac{x-\mu}{\sigma}\right)}\left[1-erf\left(\frac{\sqrt{e^{-\left(\frac{x-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)\right]^{\alpha-1}}{\sqrt{2\pi}\,\sigma\left[1-erf\left(\frac{\sqrt{e^{-\left(\frac{x-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)+\beta\ erf\left(\frac{\sqrt{e^{-\left(\frac{x-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)\right]^{\alpha+1}\left(1-\left[1+\frac{\beta\mathrm{erf}\left(\frac{\sqrt{e^{-\left(\frac{x-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)}{1-\mathrm{erf}\left(\frac{\sqrt{e^{-\left(\frac{x-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)}\right]^{-\alpha}\right)} \tag{13}$$

Also, the hazard rate function of the standard OIP-Mo distribution is as follows,

$$h(x)=\frac{\alpha\beta\ \mathrm{e}^{-\frac{1}{2}\left(\mathrm{e}^{-x}+x\right)}\left[1-erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^{\alpha-1}}{\sqrt{2\pi}\left[1-erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)+\beta\ erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^{\alpha+1}\left(1-\left[1+\frac{\beta\mathrm{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}{1-\mathrm{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}\right]^{-\alpha}\right)} \tag{14}$$

Plots of the hazard rate function (13) of the OIP-Mo distribution (10) are given in Figure 2 for selected values of the parameters.

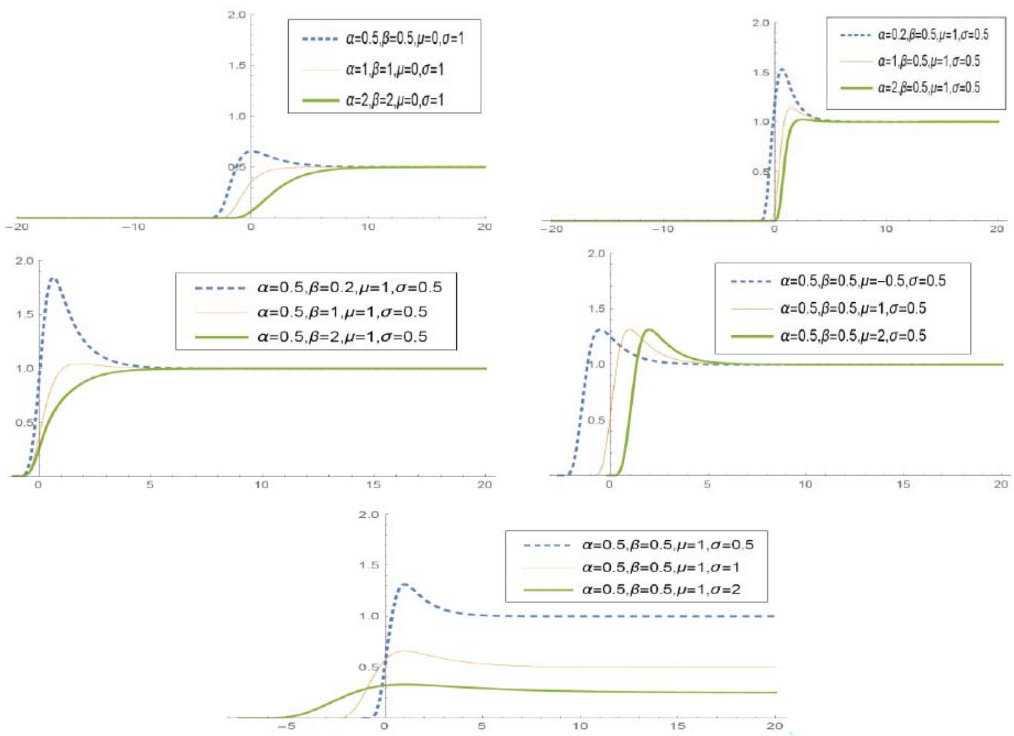

Fig. 2: Plots of the hazard rate function of OIP-Mo distribution for some parameter

3 Expansions for the OIP-Mo Distribution

The pdf of OIP-Mo in Equation (12) can be expressed as

$$f(x)=\frac{\alpha\beta\, e^{-\frac{1}{2}\left(e^{-x}+x\right)}}{\sqrt{2\pi}\left[1-erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^{2}}\left[1+\beta\frac{erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}{1-erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}\right]^{-\alpha-1} \tag{15}$$

Consider the power series

$$(1+dz)^{-q}=\sum_{k=0}^{\infty}\binom{-q}{k}d^{k}z^{k} \tag{16}$$

Using the expansion (16) in Equation (15), the pdf of OIP-Mo is written as

$$f(x)=\frac{\alpha\beta\, e^{-\frac{1}{2}\left(e^{-x}+x\right)}}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\binom{-\alpha-1}{k}\left[\beta\, erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^{k}\left[1-erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^{-k-2}$$

For $|z|<1$, the power series holds

$$(1-z)^{-q}=\sum_{k=0}^{\infty}\binom{-q}{k}(-z)^{k} \tag{17}$$

Applying the expansion (17) in the last equation, the pdf of OIP-Mo is written as follows

$$f(x)=\frac{\alpha\, e^{-\frac{1}{2}\left(e^{-x}+x\right)}}{\sqrt{2}}\sum_{k,j=0}^{\infty}\binom{-\alpha-1}{k}\binom{-k-2}{j}\frac{\beta^{k+1}(-1)^{j}}{\left(\sqrt{\pi}\right)^{k+j+1}}\gamma^{k+j}\left(\frac{1}{2},\frac{e^{-x}}{2}\right). \tag{18}$$

where $\gamma\left(\frac{1}{2},x\right)=\sqrt{\pi}erf\left(\sqrt{x}\right)$.

4 Statistical Properties of the OIP-Mo Distribution

If X is a random variable with density function (10), we write X : OIP-Mo$(\alpha,\beta,\mu,\sigma)$.

Without loss of generality, for simplicity, we will take $\mu=0,$ *and* $\sigma=1$.

In this section, we discuss some statistical properties of the proposed distribution such as quantile function, mode, n^{th} moment, moment generating functions, incomplete moment, mean deviation, Lorenz, Bonferroni and Zenga curves, Rényi of entropy and continuous entropy.

4.1 Quantile function

Theorem 1.*Let X be a random variable following OIP-Mo distribution and let $u\in(0,1)$ where $F(x)=u$ and F(x) is the cdf of the OIP-Mo distribution. Then the quantile function is given by*

$$x=\ln\left(\frac{1}{2\left[erf^{-1}\left(\frac{1-u^{\frac{1}{\alpha}}}{1+\beta u^{\frac{1}{\alpha}}-u^{\frac{1}{\alpha}}}\right)\right]^{2}}\right) \tag{19}$$

*Proof.*The quantile function of OIP-Mo distribution $x=F^{-1}(u)$, $u\in(0,1)$ can be obtained by inverting Equation(11) as

$$F(x)=\left[\frac{1-\mathrm{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}{1-\mathrm{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)+\beta\mathrm{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}\right]^{\alpha}=u$$

or

$$\frac{1-\mathrm{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}{1-\mathrm{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)+\beta\mathrm{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}=u^{\frac{1}{\alpha}}$$

Then

$$\mathrm{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)=\frac{1-u^{\frac{1}{\alpha}}}{1+\beta u^{\frac{1}{\alpha}}-u^{\frac{1}{\alpha}}}$$

i.e.

$$e^{-x}=2\left[\mathrm{erf}^{-1}\left(\frac{1-u^{\frac{1}{\alpha}}}{1+\beta u^{\frac{1}{\alpha}}-u^{\frac{1}{\alpha}}}\right)\right]^{2}$$

Therefore, the quantile function of order u of the OIP-Mo distribution is the solution of Equation (19).
The median of the OIP-Mo distribution can be defined at u= 0.5 in Equation (19).
The OIP-Mo distribution is easily simulated from $F(x)$ in Equation (9) using the form of the quantile function in Equation (19).

4.2 Mode

We consider the density function of OIP-Mo distribution given in (12) and solve $\frac{df(x)}{dx}=0\ for\ x$, to obtain the mode of odd inverse Pareto- Moyal distribution as follows

$$\frac{df(x)}{dx}=\frac{-\alpha\beta\,\mathrm{e}^{-\frac{1}{2}\left(\mathrm{e}^{-x}+x\right)}\left[1-erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^{\alpha-1}\left(1-e^{-x}\right)}{2\sqrt{2\pi}\left[1-erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)+\beta\ erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^{\alpha+1}}+$$
$$\frac{\alpha\beta\,\mathrm{e}^{-\left(\mathrm{e}^{-x}+x\right)}\left[1-erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^{\alpha-1}}{2\pi\left[1-erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)+\beta\ erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^{\alpha+2}}\left(\beta(\alpha+1)+\frac{\beta(\alpha-1)erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}{1-erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}-2\right)=0$$

But we cannot obtain an explicit form so we calculate the mode numerically for different values of $\alpha\ and\ \beta$.

Table 1: Mode for some chosen different values of α *and* β.

The values of α and β	**Mode**
α=0.2 and $\beta=0.2$	-2.1998
α= 0.5 and $\beta=0.5$	-1.24314
α= 1 and $\beta=1$	0
α= 2 and $\beta=2$	2.29477
α= 2 and $\beta=0.2$	-0.531786
α= 2 and $\beta=0.5$	-0.252485
α= 2 and $\beta=1$	-1.13027
α= 0.2 and $\beta=0.5$	-2.14808
α= 0.2 and $\beta=1$	-2.12409
α= 0.2 and $\beta=2$	-2.10991
α= 0.5 and $\beta=2$	-0.666869
α= 1 and $\beta=2$	0.838473
α= 0.5 and $\beta=0.2$	-1.5249
α= 1 and $\beta=0.2$	-1.06015

4.3 Moments

Theorem 2. *If X follows the OIP-Mo distribution given by the pdf (18), then the* $\mathrm{n^{th}}$ *moment of X is given by*

$$\mu_{\mathrm{n}}^{'} = \sum_{k,j,m=0}^{\infty}\sum_{r=0}^{n} v_{r,m,n,k,j}\left(\alpha,\beta\right)\Gamma_r(m+\frac{k+j+1}{2}), \tag{20}$$

where

$$\left.\begin{array}{l} v_{r,m,n,k,j}\left(\alpha,\beta\right)=\alpha\binom{-\alpha-1}{k}\binom{-k-2}{j}\binom{n}{r}\left(-1\right)^{j+n}\frac{\beta^{k+1}\left(\ln(2)\right)^{n-r}}{\left(\sqrt{\pi}\right)^{k+j+1}}c_{m,\mathrm{k+j}}, \\ \text{and } c_{m,k+j}=m^{-1}\sum_{l=1}^{m}\frac{(-1)^l((k+j)l-m+l)}{(2l+1)l!}c_{m-l,k+j} \; for\; m = 1,2,\ldots \; and\; c_{0,k+j}=2^{k+j}. \end{array}\right\} \tag{21}$$

Proof. The $\mathrm{n^{th}}$ moment of the OIP-Mo distribution is given as follows

$$\mu_{\mathrm{n}}^{\prime}=E\left(x^{n}\right)=\int_{-\infty}^{\infty}x^{n}f\left(x\right)dx\,.$$

Therefore

$$\mu_{\mathrm{n}}^{\prime}=\alpha\sum_{k,j=0}^{\infty}\binom{-\alpha-1}{k}\binom{-k-2}{j}\frac{\beta^{k+1}(-1)^{j}}{\left(\sqrt{\pi}\right)^{k+j+1}}\int_{-\infty}^{\infty}x^{n}\,\frac{\mathrm{e}^{-\frac{1}{2}\left(\mathrm{e}^{-x}+x\right)}}{\sqrt{2}}\gamma^{k+j}\left(\frac{1}{2},\frac{e^{-x}}{2}\right)dx.$$

Setting $u=\frac{e^{-x}}{2}$, $\mu_{\mathrm{n}}^{\prime}$ reduces to

$$\mu_{\mathrm{n}}^{\prime}=\alpha\sum_{k,j=0}^{\infty}\binom{-\alpha-1}{k}\binom{-k-2}{j}\frac{\beta^{k+1}(-1)^{j+n}}{\left(\sqrt{\pi}\right)^{k+j+1}}\int_{0}^{\infty}u^{\frac{-1}{2}}\left(\ln\left(2u\right)\right)^{n}e^{-u}\gamma^{k+j}\left(\frac{1}{2},u\right)du.$$

Using the binomial expansion in the last equation, we obtain

$$\mu_{\mathrm{n}}^{\prime}=\alpha\sum_{k,j=0}^{\infty}\sum_{r=0}^{n}\binom{-\alpha-1}{k}\binom{-k-2}{j}\binom{n}{r}\frac{\beta^{k+1}(-1)^{j+n}}{\left(\sqrt{\pi}\right)^{k+j+1}}\left(\ln(2)\right)^{n-r}\int_{0}^{\infty}u^{\frac{-1}{2}}\left(\ln\left(u\right)\right)^{r}e^{-u}\gamma^{k+j}\left(\frac{1}{2},u\right)du. \tag{22}$$

Let $I_{r,k,j}=\int_0^{\infty}u^{\frac{-1}{2}}\left(\ln\left(u\right)\right)^{r}e^{-u}\gamma^{k+j}\left(\frac{1}{2},u\right)du$.

Using the series expansion $\gamma(\alpha,x) = x^{\alpha}\sum_{m=0}^{\infty}\frac{(-x)^m}{(\alpha+m)m!}$,in the last equation ,we have

$$I_{r,k,j}=\int_{0}^{\infty}u^{\frac{-1}{2}}\left(\ln\left(u\right)\right)^{r}e^{-u}\left[u^{\frac{1}{2}}\sum_{m=0}^{\infty}\frac{(-u)^{m}}{\left(\frac{1}{2}+m\right)m!}\right]^{k+j}du.$$

Using the identity of a power series raised to an integer, namely $\left(\sum_{k=0}^{\infty}a_kx^k\right)^n=\sum_{k=0}^{\infty}c_{k,n}x^k$ (see Gradshteyn and Ryzhik, [22], p.14 Section 0.314), where $c_{0,n} = a_0^n$ and $c_{k,n}=\left(ka_0\right)^{-1}\sum_{l=1}^{k}\left(nl-k+l\right)a_lc_{k-l,n}$.

Hence, $I_{r,k,j}=\int_0^{\infty}u^{\frac{k+j-1}{2}}\left(\ln\left(u\right)\right)^{r}e^{-u}\sum_{m=0}^{\infty}c_{m,k+j}u^{m}du$, where $c_{m,k+j}=m^{-1}\sum_{l=1}^{m}\frac{(-1)^l((k+j)l-m+l)}{(2l+1)l!}c_{m-l,k+j}$ $for\; m = 1,2,\ldots\; and\; c_{0,k+j}=2^{k+j}$.

Then Equation (22) can be written as

$$\mu_{\mathrm{n}}^{\prime}=\alpha\sum_{k,j,m=0}^{\infty}\sum_{r=0}^{n}\binom{-\alpha-1}{k}\binom{-k-2}{j}\binom{n}{r}\frac{\beta^{k+1}(-1)^{j+n}}{\left(\sqrt{\pi}\right)^{k+j+1}}\left(\ln(2)\right)^{n-r}c_{m,k+j}\,J\left(r\right) \tag{23}$$

where $J\left(r\right)=\int_0^{\infty}u^{m+\frac{k+j+1}{2}-1}\left(\ln\left(u\right)\right)^{r}e^{-u}du$.

This integral $J(r)$ in (23) can be calculated from the result given by Gradshteyn and Ryzhik, [22], (p. 578 Section 4.358 integral 5). From the definition of $\Gamma_r\left(p\right)=\frac{\partial^r\Gamma(\mathrm{p})}{\partial p^r}$, we have

$$J\left(r\right)=\int_{0}^{\infty}u^{m+\frac{k+j+1}{2}-1}\left(\ln\left(u\right)\right)^{r}e^{-u}du=\Gamma_r\left(m+\frac{k+j+1}{2}\right).$$

This yields the $\mathrm{n^{th}}$ moment given in Equation (20).

Putting n=1 in Equation (20), we easily obtain the mean of OIP-Mo distribution.

4.4 Moment generating function

Theorem 3. *If X follows the OIP-Mo distribution given by the pdf (18), the moment generating function (mgf) of X is given by*

$$\left.\begin{array}{c} M_x(t) = 2^{-t}\sum_{k,j,m=0}^{\infty} w_{k,j,m}(\alpha,\beta)\,\Gamma\left(m+\frac{k+j+1}{2}-t\right), \\ \text{where} \quad w_{k,j,m}(\alpha,\beta)=\alpha\binom{-\alpha-1}{k}\binom{-k-2}{j}\frac{\beta^{k+1}}{(\sqrt{\pi})^{k+j+1}}(-1)^j c_{m,k+j}\ , \\ \text{and } c_{m,k+j} \text{ is defined in Equation (21)} \end{array}\right\} \tag{24}$$

Proof. The mgf of OIP-Mo distribution is defined by

$$M_x(t) = E\left(e^{tx}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

Using Equation (18), the mgf of OIP-Mo distribution reduces to

$$M_x(t) = \alpha \sum_{k,j=0}^{\infty}\binom{-\alpha-1}{k}\binom{-k-2}{j}\frac{\beta^{k+1}(-1)^j}{\left(\sqrt{\pi}\right)^{k+j+1}}\int_{-\infty}^{\infty} e^{tx}\frac{e^{\frac{-e^{-x}}{2}}e^{\frac{-x}{2}}}{\sqrt{2}}\gamma^{k+j}\left(\frac{1}{2},\frac{e^{-x}}{2}\right)dx$$

Substituting $u = \frac{e^{-x}}{2}$,we have

$$M_x(t) = \alpha 2^{-t}\sum_{k,j=0}^{\infty}\binom{-\alpha-1}{k}\binom{-k-2}{j}\frac{\beta^{k+1}(-1)^j}{\left(\sqrt{\pi}\right)^{k+j+1}}\int_0^{\infty} u^{-t-\frac{1}{2}}e^{-u}\gamma^{k+j}\left(\frac{1}{2},u\right)du$$

Let $A_{t,k,j} = \int_0^{\infty} u^{-t-\frac{1}{2}}e^{-u}\gamma^{k+j}\left(\frac{1}{2},u\right)du$
Following similar steps of Theorem 2, $M_x(t)$ takes the form

$$M_x(t) = \alpha 2^{-t}\sum_{k,j,m=0}^{\infty}\binom{-\alpha-1}{k}\binom{-k-2}{j}\frac{\beta^{k+1}(-1)^j}{\left(\sqrt{\pi}\right)^{k+j+1}}c_{m,k+j}\int_0^{\infty} u^{m+\frac{k+j+1}{2}-t-1}e^{-u}du$$

which yields the result (24).

4.5 Mean deviations

Theorem 4. *Let X follow OIP-Mo* distribution *given by the pdf (18). The mean deviation of X about the mean μ_1' and the median M are defined, respectively, by*

$$\delta_1(x) = 2\mu_1' F\left(\mu_1'\right) - 2T\left(\mu_1'\right)$$

and $\delta_2(x) = \mu_1' - 2T(M)$ where $T(q)$ is given by

$$T(q) = \sum_{k,j,m=0}^{\infty}\sum_{r=0}^{1} v_{r,m,1,k,j}(\alpha,\beta)\,\Gamma_r\left(m+\frac{k+j+1}{2},\frac{e^{-q}}{2}\right), \tag{25}$$

and $v_{r,m,1,k,j}(\alpha,\beta)$ is defined in Equation (21) for $n=1$.

Proof. The mean deviations of X about the mean and the median are given by

$$\delta_1(x) = \int_{-\infty}^{\infty}\left|x-\mu_1'\right| f(x)dx = 2\mu_1' F\left(\mu_1'\right) - 2T\left(\mu_1'\right)$$

and

$$\delta_2(x) = \int_{-\infty}^{\infty}|x-M| f(x)dx = \mu_1' - 2T(M)\ ,$$

where $T(q) = \int_{-\infty}^{q} x f(x)\ dx,\ F(M) = \frac{1}{2}$ and $F\left(\mu_1'\right)$ can be easily calculated from Equation (11)

Using Equation (18), we write $T(q)$ in the following form

$$T(q)=\alpha\sum_{k,j=0}^{\infty}\binom{-\alpha-1}{k}\binom{-k-2}{j}\frac{\beta^{k+1}(-1)^{j}}{(\sqrt{\pi})^{k+j+1}}\int_{-\infty}^{q}x\frac{\mathrm{e}^{-\frac{1}{2}(\mathrm{e}^{-x}+x)}}{\sqrt{2}}\gamma^{k+j}\left(\frac{1}{2},\frac{e^{-x}}{2}\right)dx.$$

Again setting $u=\frac{e^{-x}}{2}$, we have

$$T(q)=\alpha\sum_{k,j,m=0}^{\infty}\binom{-\alpha-1}{k}\binom{-k-2}{j}\frac{\beta^{k+1}(-1)^{j+1}}{(\sqrt{\pi})^{k+j+1}}\int_{\frac{e^{-q}}{2}}^{\infty}u^{\frac{-1}{2}}\ln(2u)\,e^{-u}\gamma^{k+j}\left(\frac{1}{2},u\right)du.$$

Following similar steps of Theorem 2, $T(q)$ takes the form

$$T(q)=\alpha\sum_{k,j,m=0}^{\infty}\sum_{r=0}^{1}\binom{-\alpha-1}{k}\binom{-k-2}{j}\binom{1}{r}\frac{\beta^{k+1}(-1)^{j+1}}{(\sqrt{\pi})^{k+j+1}}(\ln(2))^{1-r}c_{m,k+j}\int_{\frac{e^{-q}}{2}}^{\infty}u^{m+\frac{k+j-1}{2}}(\ln(u))^{r}e^{-u}du.$$

From the last equation

$$\int_{\frac{e^{-q}}{2}}^{\infty}u^{m+\frac{k+j-1}{2}}(\ln(u))^{r}e^{-u}du=\frac{\partial^{r}}{\partial\left(m+\frac{k+j+1}{2}\right)^{r}}\int_{\frac{e^{-q}}{2}}^{\infty}u^{m+\frac{k+j+1}{2}-1}e^{-u}du.$$

$$=\frac{\partial^{r}}{\partial\left(m+\frac{k+j+1}{2}\right)^{r}}\Gamma\left(m+\frac{k+j+1}{2},\frac{e^{-q}}{2}\right)=\Gamma_{r}(m+\frac{k+j+1}{2},\frac{e^{-q}}{2}).$$

This yields the $T(q)$ given in Equation (25).
Therefore the measures $\delta_1(x)$ *and* $\delta_2(x)$ are determined from Equation (25).

4.6 Incomplete moments

Theorem 5. *If X follows the OIP-Mo distribution defined in Equation*(18)*, then the n^{th} incomplete moment is given by*

$$m_n(z)=\sum_{k,j,m=0}^{\infty}\sum_{r=0}^{n}v_{r,m,n,k,j}(\alpha,\beta)\Gamma_r\left(m+\frac{k+j+1}{2},\frac{e^{-z}}{2}\right),\tag{26}$$

where $v_{r,m,n,k,j}(\alpha,\beta)$ is defined in Equation (21).

Proof. The n^{th} incomplete moments denoted as $m_n(z)$ can be obtained as follows:

$$m_n(z)=\int_{-\infty}^{z}x^n f(x)dx$$

From Equation (18), we have

$$m_n(z)=\alpha\sum_{k,j=0}^{\infty}\binom{-\alpha-1}{k}\binom{-k-2}{j}\frac{\beta^{k+1}(-1)^{j}}{(\sqrt{\pi})^{k+j+1}}\int_{-\infty}^{z}x^{n}\frac{\mathrm{e}^{-\frac{1}{2}(\mathrm{e}^{-x}+x)}}{\sqrt{2}}\gamma^{k+j}\left(\frac{1}{2},\frac{e^{-x}}{2}\right)dx.$$

Using the transformation $u=\frac{e^{-x}}{2}$, we have

$$m_n(z)=\alpha\sum_{k,j,m=0}^{\infty}\binom{-\alpha-1}{k}\binom{-k-2}{j}\frac{\beta^{k+1}(-1)^{j+n}}{(\sqrt{\pi})^{k+j+1}}\int_{\frac{e^{-z}}{2}}^{\infty}u^{\frac{-1}{2}}(\ln(2u))^{n}e^{-u}\gamma^{k+j}\left(\frac{1}{2},u\right)du.$$

Following similar steps of Theorem 2, $m_n(z)$ takes the form

$$m_n(z)=\alpha\sum_{k,j,m=0}^{\infty}\sum_{r=0}^{1}\binom{-\alpha-1}{k}\binom{-k-2}{j}\binom{n}{r}\frac{\beta^{k+1}(-1)^{j+n}}{(\sqrt{\pi})^{k+j+1}}(\ln(2))^{n-r}c_{m,k+j}\int_{\frac{e^{-z}}{2}}^{\infty}u^{m+\frac{k+j-1}{2}}(\ln(u))^{r}e^{-u}du.$$

which is similar to the integral in Section 4.5
Therefore, this reduces to the result in Equation (26).

4.7 Lorenz, Bonferroni and Zenga curves

These curves have different applications in many fields such as medicine, insurance, reliability, demography and economics.

The Lorenz, Bonferroni and Zenga curves are defined, respectively, as follows:

$$L(F(z)) = \frac{1}{E(z)} \int_{-\infty}^{z} x f(x)\, dx = \frac{m_1(z)}{\mu_1'},$$

$$B(F(z)) = \frac{L(F(z))}{F(z)}$$

and

$$A(z) = 1 - \frac{M^-(z)}{M^+(z)}$$

where

$$M^-(z) = \frac{1}{F(z)} \int_{-\infty}^{z} x f(x)\, dx\,, \quad M^+(z) = \frac{1}{1-F(z)} \int_{z}^{\infty} x f(x)\, dx$$

Therefore, using Equations (20) and (26), we obtain the Lorenz curve as follows

$$L(F(z)) = \frac{\sum_{k,j,m=0}^{\infty} \sum_{r=0}^{1} v_{r,m,1,k,j}(\alpha,\beta)\, \Gamma_r\left(m + \frac{k+j+1}{2}, \frac{e^{-z}}{2}\right)}{\sum_{k,j,m=0}^{\infty} \sum_{r=0}^{1} v_{r,m,1,k,j}(\alpha,\beta)\, \Gamma_r(m + \frac{k+j+1}{2})}, \tag{27}$$

From Equations (27) and (11), we find the Bonferroni curve as

$$B(F(z)) = \frac{\sum_{k,j,m=0}^{\infty} \sum_{r=0}^{1} v_{r,m,1,k,j}(\alpha,\beta)\, \Gamma_r\left(m + \frac{k+j+1}{2}, \frac{e^{-z}}{2}\right)}{\left[1 + \frac{\beta \operatorname{erf}\left(\frac{\sqrt{e^{-z}}}{\sqrt{2}}\right)}{1-\operatorname{erf}\left(\frac{\sqrt{e^{-z}}}{\sqrt{2}}\right)}\right]^{-\alpha} \sum_{k,j,m=0}^{\infty} \sum_{r=0}^{1} v_{r,m,1,k,j}(\alpha,\beta)\, \Gamma_r(m + \frac{k+j+1}{2})}, \tag{28}$$

Hence, the Zenga curve can be defined as follows

$$A(z) = 1 - \left\{ \left(\left[1 + \frac{\beta \operatorname{erf}\left(\frac{\sqrt{e^{-z}}}{\sqrt{2}}\right)}{1 - \operatorname{erf}\left(\frac{\sqrt{e^{-z}}}{\sqrt{2}}\right)}\right]^{\alpha} - 1 \right) \frac{\sum_{k,j,m=0}^{\infty} \sum_{r=0}^{1} v_{r,m,1,k,j}(\alpha,\beta)\, \Gamma_r\left(m + \frac{k+j+1}{2}, \frac{e^{-z}}{2}\right)}{\sum_{k,j,m=0}^{\infty} \sum_{r=0}^{1} v_{r,m,1,k,j}(\alpha,\beta)\, \gamma_r(m + \frac{k+j+1}{2}, \frac{e^{-z}}{2})} \right\}, \tag{29}$$

where

$$M^-(z) = \frac{\sum_{k,j,m=0}^{\infty} \sum_{r=0}^{1} v_{r,m,1,k,j}(\alpha,\beta)\, \Gamma_r\left(m + \frac{k+j+1}{2}, \frac{e^{-z}}{2}\right)}{\left[1 + \frac{\beta \operatorname{erf}\left(\frac{\sqrt{e^{-z}}}{\sqrt{2}}\right)}{1-\operatorname{erf}\left(\frac{\sqrt{e^{-z}}}{\sqrt{2}}\right)}\right]^{-\alpha}},$$

$$M^+(z) = \frac{\sum_{k,j,m=0}^{\infty} \sum_{r=0}^{1} v_{r,m,1,k,j}(\alpha,\beta)\, \gamma_r\left(m + \frac{k+j+1}{2}, \frac{e^{-z}}{2}\right)}{\left\{1 - \left[1 + \frac{\beta \operatorname{erf}\left(\frac{\sqrt{e^{-z}}}{\sqrt{2}}\right)}{1-\operatorname{erf}\left(\frac{\sqrt{e^{-z}}}{\sqrt{2}}\right)}\right]^{-\alpha}\right\}},$$

and

$$\gamma_r\left(m + \frac{k+j+1}{2}, \frac{e^{-z}}{2}\right) = \frac{\partial^r}{\partial\left(m + \frac{k+j+1}{2}\right)^r} \gamma\left(m + \frac{k+j+1}{2}, \frac{e^{-z}}{2}\right) = \frac{\partial^r}{\partial\left(m + \frac{k+j+1}{2}\right)^r} \int_0^{\frac{e^{-z}}{2}} u^{m+\frac{k+j+1}{2}-1} e^{-u} du$$

$$= \int_0^{\frac{e^{-z}}{2}} u^{m+\frac{k+j+1}{2}-1} (\ln(u))^r e^{-u} du\,.$$

4.8 Rényi entropy

The entropy of a random variable represents the amount of variation of the uncertainty. The Rényi entropy has broad applications in different areas such as statistics, physics and ecology as the index of diversity. The Rényi entropy is defined as

$$J_R(\xi) = \frac{1}{1-\xi}\log\left(I(\xi)\right),$$

where $I(\xi) = \int f^{\xi}(x)\,dx$, $\xi \geq 0$, *and* $\xi \neq 1$.

Using this notion, we deduce the Rényi entropy of a random variable following the OIP-Mo pdf (15), in Theorem 6.

Theorem 6. *Let X be a continuous random variable following the OIP-Mo distribution given by Equation* (15). *The Rényi entropy of X is given by*

$$J_R(\xi) = (1-\xi)^{-1}\left\{\xi\log(\alpha) + \log\left[\sum_{k,j,m=0}^{\infty}\binom{\xi(-\alpha-1)}{k}\binom{-k-2\xi}{j}c_{m,k+j}\frac{(-1)^j\beta^{k+\xi}}{(\pi)^{\frac{k+j+\xi}{2}}}\xi^{-\left(m+\frac{\xi+k+j}{2}\right)}\Gamma\left(m+\frac{\xi+k+j}{2}\right)\right]\right\} \tag{30}$$

Proof. Setting the pdf of OIP-Mo (15) in the definition of Rényi entropy given above, we have

$$I(\xi) = \frac{\alpha^{\xi}\beta^{\xi}}{(2\pi)^{\frac{\xi}{2}}}\int_{-\infty}^{\infty}\frac{e^{-\frac{\xi}{2}(e^{-x}+x)}}{\left(1-\mathrm{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right)^{2\xi}}\left\{1+\frac{\beta\,\mathrm{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}{1-\mathrm{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}\right\}^{\xi(-\alpha-1)}dx$$

Using the series expansion (16) in the last equation, we obtain

$$I(\xi) = \frac{\alpha^{\xi}}{(2\pi)^{\frac{\xi}{2}}}\sum_{k=0}^{\infty}\binom{\xi(-\alpha-1)}{k}\beta^{\xi+k}\int_{-\infty}^{\infty}e^{-\frac{\xi}{2}(e^{-x}+x)}\left[\mathrm{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^{k}\left(1-\mathrm{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right)^{-k-2\xi}dx$$

Again using the expansion (16) and $erf(\sqrt{x}) = \frac{\gamma\left(\frac{1}{2},x\right)}{\sqrt{\pi}}$, we obtain

$$I(\xi) = \frac{\alpha^{\xi}}{(2)^{\frac{\xi}{2}}}\sum_{k,j=0}^{\infty}\binom{\xi(-\alpha-1)}{k}\binom{-k-2\xi}{j}\frac{\beta^{\xi+k}(-1)^j}{(\pi)^{\frac{k+j+\xi}{2}}}\int_{-\infty}^{\infty}e^{-\frac{\xi}{2}(e^{-x}+x)}\gamma^{k+j}\left(\frac{1}{2},\frac{e^{-x}}{2}\right)dx$$

Substituting $u = \frac{e^{-x}}{2}$, we have

$$I(\xi) = \alpha^{\xi}\sum_{k,j=0}^{\infty}\binom{\xi(-\alpha-1)}{k}\binom{-k-2\xi}{j}\frac{\beta^{\xi+k}(-1)^j}{(\pi)^{\frac{k+j+\xi}{2}}}\int_{0}^{\infty}u^{\frac{\xi}{2}-1}e^{-\xi u}\gamma^{k+j}\left(\frac{1}{2},u\right)du$$

Following similar steps of Theorem 2, we have

$$I(\xi) = \alpha^{\xi}\sum_{k,j,m=0}^{\infty}\binom{\xi(-\alpha-1)}{k}\binom{-k-2\xi}{j}c_{m,k+j}\frac{\beta^{\xi+k}(-1)^j}{(\pi)^{\frac{k+j+\xi}{2}}}\int_{0}^{\infty}u^{m+\frac{\xi+k+j}{2}-1}e^{-\xi u}du$$

where $c_{m,k+j}$ is defined in Section 4.3. The integral in the last equation can be easily calculated. Hence,

$$I(\xi) = \alpha^{\xi}\sum_{k,j,m=0}^{\infty}\binom{\xi(-\alpha-1)}{k}\binom{-k-2\xi}{j}c_{m,k+j}\frac{\beta^{\xi+k}(-1)^j}{(\pi)^{\frac{k+j+\xi}{2}}}\xi^{-\left(m+\frac{\xi+k+j}{2}\right)}\Gamma\left(m+\frac{\xi+k+j}{2}\right)$$

Finally, the Rényi entropy can be expressed as in Equation (30).

4.9 Continuous entropy

The Continuous entropy defined by Marsh [23] is as follows

$$h(x) = E\left[-\ln(f(x))\right] = \int -\ln(f(x))\, f(x)\, dx$$

Theorem 7. *Let X be a continuous random variable following the OIP-Mo distribution given by Equation (15). The Continuous entropy of X is given by*

$$h(x) = -\ln(\alpha) - \ln(\beta) + \frac{1}{2}\left[\ln(2\pi) + M_x(-1) + \mu_1'\right] - 2\alpha \sum_{k,r=0}^{\infty} \frac{\binom{-\alpha-1}{k}\binom{k}{r}\beta^{k+1}}{(r-k-1)^2}(-1)^{r+1} + \frac{\alpha+1}{\alpha} \tag{31}$$

Proof. Substituting the pdf of OIP-Mo (15) in the definition of the Continuous entropy given above, we have

$$h(x) = \left[-\ln(\alpha) - \ln(\beta) + \frac{1}{2}\ln(2\pi)\right] \mathrm{I}_1 + \frac{1}{2}\,\mathrm{I}_2 + \frac{1}{2}\mathrm{I}_3 + 2\,\mathrm{I}_4 + (\alpha+1)\,\mathrm{I}_5 \tag{32}$$

$\mathrm{I}_1 = \int_{-\infty}^{\infty} f(x)dx = 1$ (i)
$\mathrm{I}_2 = \int_{-\infty}^{\infty} e^{-x} f(x)dx = M_x(-1)$ (ii)
$\mathrm{I}_3 = \int_{-\infty}^{\infty} x f(x)dx = \mu_1'$ (iii)
Also,

$$\mathrm{I}_4 = \int_{-\infty}^{\infty} \ln\left[1 - \operatorname{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right] f(x)\ dx$$

$$= \int_{-\infty}^{\infty} \ln\left[1 - \operatorname{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right] \frac{\alpha\beta\, \mathrm{e}^{-\frac{1}{2}\left(\mathrm{e}^{-x}+x\right)}}{\sqrt{2\pi}\left[1 - erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^2} \left[1 + \beta \frac{erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}{1 - erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}\right]^{-\alpha-1} dx$$

Let $v = 1 - \operatorname{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)$, then we have

$$\int_{-\infty}^{\infty} \ln\left[1 - \operatorname{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right] f(x)\ dx\ = \alpha\beta \int_0^1 \ln(v)\ v^{-2} \left[(1 + \beta\frac{1-v}{v})\right]^{\alpha-1} dv$$

$$= \alpha \sum_{k=0}^{\infty} \binom{-\alpha-1}{k} \beta^{k+1} \int_0^1 \ln(v)\ v^{-k-2}\ (1-v)^k dv$$

$$= \alpha \sum_{k,r=0}^{\infty} \binom{-\alpha-1}{k}\binom{k}{r} \beta^{k+1}(-1)^r \int_0^1 \ln(v)\ v^{r-k-2} dv$$

Integration by parts yields
$\mathrm{I}_4 = \int_{-\infty}^{\infty} \ln\left[1 - \operatorname{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right] f(x)\ dx\ = \alpha \sum_{k,r=0}^{\infty} \frac{\binom{-\alpha-1}{k}\binom{k}{r}\beta^{k+1}}{(r-k-1)^2}(-1)^{r+1}$ (iv)
Now the last integral will give

$$\mathrm{I}_5 = \int_{-\infty}^{\infty} \ln\left[1 + \beta \frac{erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}{1 - erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}\right] f(x)\, dx$$

$$= \int_{-\infty}^{\infty} \ln\left[1 + \beta \frac{erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}{1 - erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}\right] \frac{\alpha\beta\, \mathrm{e}^{-\frac{1}{2}\left(\mathrm{e}^{-x}+x\right)}}{\sqrt{2\pi}\left[1 - erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^2} \left[1 + \beta \frac{erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}{1 - erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}\right]^{-\alpha-1} dx$$

Now, we have

$$\int_{-\infty}^{\infty} \ln \left[1 + \beta \frac{erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}{1 - erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)} \right] f(x)\,dx = \alpha\beta \int_{0}^{1} \ln \left[\frac{v + \beta - \beta v}{v} \right] \frac{v^{\alpha-1}}{(v + \beta - \beta v)^{\alpha+1}} dv$$

Let $z = v + \beta - \beta v$, in the last integration

$$\int_{-\infty}^{\infty} \ln \left[1 + \beta \frac{erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}{1 - erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)} \right] f(x)\,dx = \frac{\alpha\beta}{(1-\beta)^{\alpha}} \int_{\beta}^{1} \ln \left[\frac{1-\beta}{1-\frac{\beta}{z}} \right] z^{-2} \left[1 - \frac{\beta}{z} \right]^{\alpha-1} dz$$

And let $w = 1 - \frac{\beta}{z}$, in the last integration

$$\int_{-\infty}^{\infty} \ln \left[1 + \beta \frac{erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}{1 - erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)} \right] f(x)\,dx = \frac{\alpha}{(1-\beta)^{\alpha}} \int_{0}^{1-\beta} \ln \left[\frac{1-\beta}{w} \right] w^{\alpha-1} dw$$

Therefore, $I_5 = \int_{-\infty}^{\infty} \ln \left[1 + \beta \frac{erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}{1-erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)} \right] f(x)\,dx = \frac{1}{\alpha}$ (v)

Substituting (i), (ii), (iii), (iv) and (v) in Equation (32), the Continuous entropy can be expressed as the result (31).

5 Order Statistics

Order statistics play an important role in probability and statistics. Let $X_{1:m} \leq X_{2:m} \leq \cdots \leq X_{m:m}$be the order sample from a continuous population with pdf $f(x)$ and cdf $F(x)$. The pdf of $X_{k:m}$, the k^{th} order statistic is given by

$$f_{X_{k:m}}(x) = \frac{m!}{(k-1)!\,(m-k)!} f(x)\,[F(x)]^{k-1}[1-F(x)]^{m-k} \quad ;k = 1,2,\ldots,m.$$

Then , the pdf of the k^{th}order OIP-Mo random variable $X_{k:m}$can be obtained using Equations (11) and (15) in the last equation to give

$$f_{X_{k:m}}(x) = \frac{m!\,\alpha\beta\, \mathrm{e}^{-\frac{1}{2}(\mathrm{e}^{-x}+x)}}{(k-1)!\,(m-k)!\sqrt{2\pi}\left[1-\mathrm{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^{2}} \left(1 + \beta \frac{erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}{1 - erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}\right)^{-\alpha k-1} \left(1 - \left[1 + \beta \frac{erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}{1 - erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}\right]^{-\alpha}\right)^{m-k}$$

Using the binomial expansion, we obtain

$$f_{X_{k:m}}(x) = \frac{m!\,\alpha\beta\, \mathrm{e}^{-\frac{1}{2}(\mathrm{e}^{-x}+x)}}{(k-1)!\,(m-k)!\sqrt{2\pi}} \left[1-\mathrm{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^{-2} \sum_{i=0}^{\infty} \binom{m-k}{i} (-1)^{i} \left(1 + \beta \frac{erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}{1 - erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)}\right)^{-\alpha(k+i)-1}$$

From the series expansion (16), we have

$$f_{X_{k:m}}(x) = \frac{m!\,\alpha\, \mathrm{e}^{-\frac{1}{2}(\mathrm{e}^{-x}+x)}}{(k-1)!\,(m-k)!\sqrt{2\pi}} \sum_{i,j=0}^{\infty} \binom{m-k}{i} \binom{-\alpha(k+i)-1}{j} \beta^{j+1}(-1)^{i} \left[\mathrm{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^{j} \left[1-\mathrm{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^{-j-2}$$

Again, using the binomial expansion,

$$f_{X_{k:m}}(x) = \frac{m!\,\alpha\, \mathrm{e}^{-\frac{1}{2}(\mathrm{e}^{-x}+x)}}{(k-1)!\,(m-k)!\sqrt{2\pi}} \sum_{i,j,r=0}^{\infty} \binom{m-k}{i} \binom{-\alpha(k+i)-1}{j} \binom{-j-2}{r} \beta^{j+1}(-1)^{i+r} \left[\mathrm{erf}\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^{j+r}$$

Therefore ,the pdf of the k^{th} order OIP-Mo random variable $X_{k:m}$ is as follows

$$f_{X_{k:m}}(x) = \frac{m!\,\alpha\, \mathrm{e}^{-\frac{1}{2}(\mathrm{e}^{-x}+x)}}{(k-1)!\,(m-k)!\sqrt{2}} \sum_{i,j,r=0}^{\infty} \binom{m-k}{i} \binom{-\alpha(k+i)-1}{j} \binom{-j-2}{r} \frac{\beta^{j+1}(-1)^{i+r}}{\pi^{\frac{j+r+1}{2}}} \gamma^{j+r}\left(\frac{1}{2}, \frac{e^{-x}}{2}\right) \quad (33)$$

Also, the n^{th}moment for the k^{th}order statistic with pdf $f_{X_{k:m}}(x)$ is given by

$$\mu_{k:m}^{(n)}=\int_{-\infty}^{\infty}x^{n}\ f_{k:m}(x)dx$$

Then , the n^{th}moment for the k^{th}order OIP-Mo random variable $X_{k:m}$can be obtained using Equation (33)

$$\mu_{k:m}^{(n)}=\frac{m!\ \alpha}{(k-1)!\,(m-k)!\sqrt{2}}\sum_{i,j,r=0}^{\infty}\binom{m-k}{i}\binom{-\alpha\,(k+i)-1}{j}\binom{-j-2}{r}\frac{\beta^{j+1}(-1)^{i+r}}{\pi^{\frac{j+r+1}{2}}}\int_{-\infty}^{\infty}x^{n}\ \mathrm{e}^{-\ \frac{1}{2}(\mathrm{e}^{-x}+x)}\gamma^{j+r}\left(\frac{1}{2}\ ,\ \frac{e^{-x}}{2}\right)\ dx$$

Again, using $u=\frac{e^{-x}}{2}$, we have

$$\mu_{k:m}^{(n)}=\frac{m!\ \alpha}{(k-1)!\,(m-k)!}\sum_{i,j,r=0}^{\infty}\binom{m-k}{i}\binom{-\alpha\,(k+i)-1}{j}\binom{-j-2}{r}\frac{\beta^{j+1}(-1)^{i+r+n}}{\pi^{\frac{j+r+1}{2}}}\int_{0}^{\infty}[\ln(2u)]^{n}u^{\frac{-1}{2}}e^{-u}\gamma^{j+r}(\frac{1}{2},u)du$$

which yields

$$\mu_{k:m}^{(n)}=\frac{m!\ \alpha}{(k-1)!\,(m-k)!}$$
$$\times\sum_{i,j,r=0}^{\infty}\sum_{z=0}^{n}\binom{m-k}{i}\binom{-\alpha\,(k+i)-1}{j}\binom{-j-2}{r}\binom{n}{z}\frac{\beta^{j+1}(-1)^{i+r+n}}{\pi^{\frac{j+r+1}{2}}}[\ln(2)\,]^{n-z}\int_{0}^{\infty}u^{\frac{-1}{2}}e^{-u}[\ln(u)\,]^{z}\gamma^{j+r}\left(\frac{1}{2},u\right)du$$

Set $B_{r,j,z}=\int_{0}^{\infty}u^{\frac{-1}{2}}e^{-u}[\ln(u)\,]^{z}\gamma^{j+r}\left(\frac{1}{2},u\right)du$

Using the series expansion $\gamma(\alpha,x)\ =x^{\alpha}\sum_{m=0}^{\infty}\frac{(-x)^{m}}{(\alpha+m)m!}$ and the identity of a power series raised to an integer, namely $\left(\sum_{k=0}^{\infty}a_{k}x^{k}\right)^{n}=\sum_{k=0}^{\infty}c_{k,n}x^{k}$in the last equation, we have

$$B_{r,j,z}=\sum_{s=0}^{\infty}c_{s,j+r}\int_{0}^{\infty}u^{s+\frac{j+r-1}{2}}e^{-u}[\ln(u)\,]^{z}du$$

where $c_{s,j+r}$ is defined in Section 4.3.

The integral in $B_{r,j,z}$ can be calculated from the result given by Gradshteyn and Ryzhik. [22], (p. 578, Section 4.358, integral 5). From the definition of $\Gamma_{r}(p)=\frac{\partial^{r}\Gamma(\mathrm{p})}{\partial p^{r}}$,we have

$$B_{r,j,z}=\sum_{s=0}^{\infty}c_{s,j+r}\Gamma_{z}(s+\frac{j+r+1}{2})$$

This yields the n^{th} moment of $X_{k:m}$ given by

$$\mu_{k:m}^{(n)}=\frac{m!\ \alpha}{(k-1)!\,(m-k)!}$$

$$\sum_{i,j,r,s=0}^{\infty}\sum_{z=0}^{n}\binom{m-k}{i}\binom{-\alpha\,(k+i)-1}{j}\binom{-j-2}{r}\binom{n}{z}\frac{\beta^{j+1}(-1)^{i+r+n}}{\pi^{\frac{j+r+1}{2}}}[\ln(2)\,]^{n-z}c_{s,j+r}\Gamma_{z}(s+\frac{j+r+1}{2})\qquad(34)$$

6 Estimation of Parameters

In this section, we describe the maximum likelihood estimators (MLEs) and the observed information matrix of the OIP-Mo distribution. Let $X_1,X_2,\ldots,X_n$ be an independent random sample from the OIP-Mo distribution, then the log-likelihood function is given by

$$l=n\left[\ln(\alpha)+\ln(\beta)-\ln(\sigma)-\frac{1}{2}\ln(2\pi)\quad\right]-\frac{1}{2}\sum_{i=1}^{n}\mathrm{e}^{-\left(\frac{x_i-\mu}{\sigma}\right)}-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)+$$

$$(\alpha-1)\sum_{i=1}^{n}\ln\left[1-erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)\right]-(\alpha+1)\sum_{i=1}^{n}\ln\left[1-erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)+\beta\, erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)\right] \tag{35}$$

Then

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\ln\left(1-erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)\right)-\sum_{i=1}^{n}\ln\left[1-erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)+\beta\, erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)\right] \tag{36}$$

$$\frac{\partial l}{\partial\mu}=\frac{n}{2\sigma}-\frac{1}{2}\sum_{i=1}^{n}\frac{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}{\sigma}-(\alpha-1)\sum_{i=1}^{n}\frac{\mathrm{e}^{-\frac{1}{2}e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2\pi}\sigma\left[1-erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)\right]}$$

$$+(\alpha+1)\sum_{i=1}^{n}\frac{\mathrm{e}^{-\frac{1}{2}e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\,(1-\beta\ )}{\sqrt{2\pi}\sigma\left[1-erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)+\beta\, erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)\right]} \tag{37}$$

$$\frac{\partial l}{\partial\sigma}=-\frac{n}{\sigma}+\frac{1}{2}\sum_{i=1}^{n}\frac{x_i-\mu}{\sigma^2}-\frac{1}{2}\sum_{i=1}^{n}\frac{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}(x_i-\mu)}{\sigma^2}-(\alpha-1)\sum_{i=1}^{n}\frac{\mathrm{e}^{-\frac{1}{2}e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\,(x_i-\mu)}{\sqrt{2\pi}\sigma^2\left[1-\mathrm{erf}\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)\right]}$$

$$+(\alpha+1)\sum_{i=1}^{n}\frac{\mathrm{e}^{-\frac{1}{2}e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\,(x_i-\mu)(1-\beta)}{\sqrt{2\pi}\sigma^2\left[1-erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)+\beta\, erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)\right]} \tag{38}$$

and

$$\frac{\partial l}{\partial\beta}=\frac{n}{\beta}-(\alpha+1)\sum_{i=1}^{n}\frac{erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)}{1-erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)+\beta\,\mathrm{erf}\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)} \tag{39}$$

The MLEs $\left(\widehat{\alpha},\widehat{\beta},\widehat{\mu},\widehat{\sigma}\right)$ of the parameters $(\alpha,\beta,\mu,\sigma)$ are obtained by solving the system of nonlinear equations (36), (37), (38) and (39). These equations cannot be solved analytically, but can be solved using numerical techniques such as Newton-Raphson method.

For interval estimation and testing of hypotheses of the parameters$(\alpha,\beta,\mu,\sigma)$, we require the 4×4 unit observed information matrix.

$$J(\psi)=\begin{pmatrix}\frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\partial\beta} & \frac{\partial^2 l}{\partial\alpha\partial\mu} & \frac{\partial^2 l}{\partial\alpha\partial\sigma}\\ \frac{\partial^2 l}{\partial\beta\partial\alpha} & \frac{\partial^2 l}{\partial\beta^2} & \frac{\partial^2 l}{\partial\beta\partial\mu} & \frac{\partial^2 l}{\partial\beta\partial\sigma}\\ \frac{\partial^2 l}{\partial\mu\partial\alpha} & \frac{\partial^2 l}{\partial\mu\partial\beta} & \frac{\partial^2 l}{\partial\mu^2} & \frac{\partial^2 l}{\partial\mu\partial\sigma}\\ \frac{\partial^2 l}{\partial\sigma\partial\alpha} & \frac{\partial^2 l}{\partial\sigma\partial\beta} & \frac{\partial^2 l}{\partial\sigma\partial\mu} & \frac{\partial^2 l}{\partial\sigma^2}\end{pmatrix}$$

whose elements are given by the following.

$$\frac{\partial^2 l}{\partial\alpha^2}=-\frac{n}{\alpha^2}$$

$$\frac{\partial^2 l}{\partial\alpha\partial\beta}=\frac{\partial^2 l}{\partial\beta\partial\alpha}=-\sum_{i=1}^{n}\frac{B}{D}$$

$$\frac{\partial^2 l}{\partial\alpha\partial\mu}=\frac{\partial^2 l}{\partial\mu\partial\alpha}=-\sum_{i=1}^{n}\frac{e^{-\frac{1}{2}A}\sqrt{A}}{\sqrt{2\pi}\sigma(1-B)}-\sum_{i=1}^{n}\frac{e^{-\frac{1}{2}A}\sqrt{A}\,(\beta-1)}{\sqrt{2\pi}\sigma D}$$

$$\frac{\partial^2 l}{\partial\alpha\partial\sigma}=\frac{\partial^2 l}{\partial\sigma\partial\alpha}=-\sum_{i=1}^{n}\frac{e^{-\frac{1}{2}A}\sqrt{A}(x_i-\mu)}{\sqrt{2\pi}\sigma^2(1-B)}-\sum_{i=1}^{n}\frac{e^{-\frac{1}{2}A}\sqrt{A}\,(x_i-\mu)(\beta-1)}{\sqrt{2\pi}\sigma^2 D}$$

$$\frac{\partial^2 l}{\partial\beta^2}=-\frac{n}{\beta^2}+(\alpha+1)\sum_{i=1}^{n}\frac{B^2}{D^2}$$

$$\frac{\partial^2 l}{\partial\beta\partial\mu}=\frac{\partial^2 l}{\partial\mu\partial\beta}=-(\alpha+1)\sum_{i=1}^{n}\frac{e^{-\frac{1}{2}A}\sqrt{A}}{\sqrt{2\pi}\sigma D}\left(1-\frac{(\beta-1)B}{D}\right)$$

$$\frac{\partial^2 l}{\partial\beta\partial\sigma}=\frac{\partial^2 l}{\partial\sigma\partial\beta}=-(\alpha+1)\sum_{i=1}^{n}\frac{e^{-\frac{1}{2}A}\sqrt{A}(x_i-\mu)}{\sqrt{2\pi}\sigma^2 D}\left(1-\frac{(\beta-1)B}{D}\right)$$

$$\frac{\partial^2 l}{\partial\mu^2}=-\frac{1}{2\sigma^2}\sum_{i=1}^{n}A+(\alpha-1)\sum_{i=1}^{n}\frac{e^{-\frac{1}{2}A}A}{2\sqrt{\pi}\sigma^2(1-B)}\left(\frac{-e^{-\frac{1}{2}A}}{\sqrt{\pi}(1-B)}-\frac{1}{\sqrt{2A}}+\sqrt{\frac{A}{2}}\right)$$
$$+(\alpha+1)\sum_{i=1}^{n}\left(-\frac{\left[e^{-\frac{1}{2}A}\sqrt{A}\,(\beta-1)\right]^2}{2\pi\sigma^2 D^2}+\frac{e^{-\frac{1}{2}A}\sqrt{A}\,(\beta-1)(1-A)}{2\sqrt{2\pi}\sigma^2 D}\right)$$

$$\frac{\partial^2 l}{\partial\mu\partial\sigma}=\frac{\partial^2 l}{\partial\sigma\partial\mu}=\frac{-n}{2\sigma^2}+\frac{1}{2\sigma^2}\sum_{i=1}^{n}A\left(1-\frac{x_i-\mu}{\sigma}\right)+(\alpha-1)\sum_{i=1}^{n}\frac{e^{-\frac{1}{2}A}\sqrt{A}}{\sqrt{2\pi}\sigma^2(1-B)}\left(1-\frac{e^{-\frac{1}{2}A}\sqrt{A}(x_i-\mu)}{\sqrt{2\pi}\sigma(1-B)}-\frac{(x_i-\mu)(1-A)}{2\sigma}\right)$$
$$-(\alpha+1)\sum_{i=1}^{n}\left(\frac{e^{-\frac{1}{2}A}\sqrt{A}(x_i-\mu)}{2\sqrt{2\pi}\sigma^3 D}(1-\beta)\left(\frac{2\sigma}{x_i-\mu}-(1-A)\right)-\frac{\sqrt{2}e^{-\frac{1}{2}A}\sqrt{A}\,(\beta-1)^2}{\sqrt{\pi}\,D}\right)$$

$$\frac{\partial^2 l}{\partial\sigma^2}=\frac{n}{\sigma^2}-\sum_{i=1}^{n}\frac{x_i-\mu}{\sigma^3}-\frac{1}{2}\sum_{i=1}^{n}\frac{A(x_i-\mu)}{\sigma^3}\left(\frac{x_i-\mu}{\sigma}-2\right)+(\alpha-1)\sum_{i=1}^{n}\frac{e^{-\frac{1}{2}A}\sqrt{A}(x_i-\mu)^2}{2\sqrt{2\pi}\sigma^4(1-B)}\left(\frac{4\sigma}{x_i-\mu}-(1-A)-\frac{\sqrt{2}e^{-\frac{1}{2}A}\sqrt{A}}{\sqrt{\pi}\,(1-B)}\right)$$
$$-(\alpha+1)\sum_{i=1}^{n}\left(\frac{-\left(e^{-\frac{1}{2}A}\sqrt{A}(x_i-\mu)(\beta-1)\right)^2}{2\pi\sigma^4 D^2}+\frac{e^{-\frac{1}{2}A}\sqrt{A}(x_i-\mu)^2}{2\sqrt{2\pi}\sigma^4\,D}(1-\beta)\left(\frac{4\sigma}{x_i-\mu}-(1-A)\right)\right)$$

where $A=e^{-\left(\frac{x_i-\mu}{\sigma}\right)}$, $B=erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)$ and $D=1-erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)+\beta\, erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)$

7 Applications

In this section, we introduce two applications of OIP-Mo distribution to two real data sets. The first data set is given by Linhart and Zucchini [24], which represents the failure times of the air conditioning system of an airplane: 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95.

The second data are vinyl chloride data obtained from clean up gradient monitoring wells in mg/L. The data are: 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8, 0.8, 0.4, 0.6, 0.9, 0.4, 2, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1, 0.2, 0.1, 0.1, 1.8, 0.9, 2, 4, 6.8, 1.2, 0.4, 0.2. This data was used by Bhaumik et al. [25].

We use these two data sets to compare the fit of the new model OIP-Mo with Beta Moyal (BMo) and Moyal (Mo). First, we obtain the maximum likelihood estimates (MLEs) for the unknown parameters of each model and then compare the results via goodness -of- fit statistics AIC (Akaike information criterion), AICC (corrected Akaike information criterion), CAIC (consistent Akaike information criterion) and BIC (Bayesian information criterion).

Table 2: MLEs for OIP-Mo, BMo, Mo models and the statistics AIC, AICC, CAIC, BIC for the first data set

Model	$\widehat{\alpha}$	$\widehat{\beta}$	$\hat{a}$	$\hat{b}$	$\widehat{\mu}$	$\widehat{\sigma}$	$-\ell$	AIC	AICC	CAIC	BIC
OIP-Mo	4.20888	0.18603	—-	—-	15.0458	22.6501	139.87	287.74	289.34	289.34	285.648
BMo	—-	—-	3.38285	0.760179	-14.425	13.9304	144.927	297.854	299.454	299.454	295.76
Mo	—-	—-	—-	—-	24.645	22.9871	159.426	322.852	323.296	323.296	321.8

Table 3: MLEs for OIP-Mo, BMo, Mo models and the statistics AIC, AICC, CAIC, BIC for the second data set

Model	$\widehat{\alpha}$	$\widehat{\beta}$	$\hat{a}$	$\hat{b}$	$\widehat{\mu}$	$\widehat{\sigma}$	$-\ell$	AIC	AICC	CAIC	BIC
OIP-Mo	0.494299	0.921582	—-	—-	1.63958	0.742177	59.6542	127.3084	128.688	128.688	125.434
BMo	—-	—-	0.486451	0.0242931	-26.235	0.353301	167.795	343.59	344.969	344.969	341.715
Mo	—-	—-	—-	—-	0.886804	0.678708	1322.56	2649.12	2649.51	2649.51	2648.18

The best model corresponds to the smallest AIC, AICC, CAIC and BIC values where,

$$AIC = 2K - 2l$$

$$\text{AICC} = AIC + \frac{2k(k+1)}{n-k-1},$$

$$CAIC = \frac{2kn}{n-k-1} - 2l,$$

$$BIC = k\log n \; - 2l,$$

l denotes the log – likelihood function evaluated at MLEs, k is the number of parameters and n is the sample size.

8 Conclusion

In this article, we proposed a new distribution namely the odd inverse pareto - Moyal (OIP-Mo) distribution which is considered a new extension of the Moyal distribution. We provide a mathematical treatment of the new distribution including the density, hazard rate functions, quantile function, mode, n^{th} moment, moment generating functions, mean deviation, incomplete moment, Lorenz, Bonferroni and Zenga curves, Rényi entropy and continuous entropy and the moments of order statistics. The parameters of the new distribution are estimated by using the method of maximum likelihood and the information matrix is derived. Two real data sets are applied to demonstrate that the odd inverse pareto - Moyal (OIP-Mo) distribution can provide a better fit than the Moyal (Mo) and beta Moyal (BMo) distributions.

Conflict of Interest

The authors declare that they have no conflict of interest.

References

[1] N. Eugene, C. Lee and F. Famoye, Beta –normal distribution and its applications, Communications in Statistics – Theory and Methods, 31(4), 497-512, (2002).

[2] M. C. Jones, Families of distributions arising from distributions of order statistics, Test, 13(1), 1-43, (2004).

[3] S. Nadarajah and S. Kotz, The beta Gumbel distribution, Mathematical problems in Engineering, 10, 323-332, (2004).

[4] S. Nadarajah and S. Kotz, The beta exponential distribution, Reliability Engineering and system Safety, 91, 689-697, (2005).
[5] G. M. Cordeiro and A. J. Lemonte, The beta Laplace distribution, Statistics and Probability Letters, 81(8), 973-982, (2011).
[6] G. M. Cordeiro, J. S. Nobre, R. R. Pescim and E. M. M. Ortega, The beta Moyal: A useful-skew distribution, International Journal of Research and in Applied Sciences, 10, 171-192, (2012).
[7] M. M. Nassar and N. K. Nada, The beta generalized Pareto distribution, Journal of Statistics-Advances in Theory and Applications, 6, 1-17, (2011).
[8] M. M. Nassar and N. K. Nada, A new generalization of the exponential geometric distribution, Journal of Statistics-Advances in Theory and Applications, 7, 25-48, (2012).
[9] M. M. Nassar and N. K. Nada, A new generalization of the Pareto geometric distribution, Journal of Egyptian Mathematical Society, 21(2), 148-155, (2013).
[10] M. M. Nassar and A. Elmasry, A study of generalized Logistic distributions, Journal of Egyptian Mathematical Society, 20(2), 126-133, (2012).
[11] M. Mahmoud, M. M. Nassar and M. A. Aefa, A new generalization of Kumaraswamy Lindley distribution, Journal of Statistics – Advances in Theory and Applications, 14, 69-105, (2015).
[12] M. C. Jones, Kumaraswamy distribution: A beta – type Distribution with some tractability advantages, Statistical Methodology, 6(1), 70-81, (2009).
[13] G. M. Cordeiro, S. Nadarajah and E. M. M. Ortega, The Kumaraswamy Gumbel distribution, Statistical Methods & Applications, 21(2), 139-168, (2012).
[14] G. M. Cordeiro, E. M. M. Ortega and S. Nadarajah, The Kumaraswamy Weibull distribution with application to failure data, Journal of the Franklin Institute, 347(8), 1399-1429, (2010).
[15] G. M. Cordeiro and M. de Castro, A new family of generalized distribution, Journal of Statistical Computation and Simulation, 81(7), 883-898, (2011).
[16] Elbatal and M. Elgarhy, Transmuted Quasi Lindley Distributions: A generalization of the Quasi Lindley Distribution, International Journal of Pure and Applied Sciences and Technology, 18(2), 59-69, (2013).
[17] M. M. Nassar, The Kumaraswamy Laplace Distribution, Pakistan Journal of Statistics and Operation Research, 12(4), 609-624, (2016).
[18] K. Zografos and N. Balakrishnan, On families of beta-and generalized gamma-generateddistributions and associated inference, Statistical Methodology, 6, 344-362, (2009).
[19] J. E. Moyal, Theory of Ionization Fluctuation, The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science, 46, 263-280, (1955).
[20] M. M. Nassar and M. Ibrahem, A New Generalization of the Moyal Distribution, Journal of Statistics Applications and Probability, 9(2), 361-378, (2020).
[21] A. Alzaatreh, C. Lee, F. Famoye, A new method for generating families of distributions, Metron, 71, 63–79, (2013).
[22] I. S. Gradshteyn and I. M. Ryzhik, Table of integrals, series, and products, Academic Press, San Francisco, 10-580, (1980).
[23] C. Marsh, Introduction to Continuous Entropy, crmarsh@princeton.edu, (2013).
[24] H. Linhart and W. Zucchini, Model Selection, Wiley, New York, 20-295, (1986).
[25] D. K. Bhaumik, K. Kapur, R. D. Gibbons, Testing Parameters of a Gamma Distribution for Small Samples, Technometrics, 51, 326-334, (2009).