Numerical and Computational Methods in Sciences & Engineering
An International Journal

http://dx.doi.org/10.18576/ncmse/020202

Analysis of boundary value problems for pantograph equations with ψ-type fractional derivative and nonlocal conditions

D. Vivek[1], *E. M. Elsayed*[2,*], *K. Kanagarajan*[3] *and J. Prasanth*[1]

[1]Department of Mathematics, PSG College of Arts & Science, Coimbatore-641014, India.
[2]Department of Mathematics, Faculty of Science,King Abdulaziz University, Jeddah 21589, Saudi Arabia.
[3]Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore-641020, India.

Received: 22 Jan. 2020, Revised: 23 May 2020, Accepted: 29 May 2020
Published online: 1 July 2020

Abstract: In this paper, we consider the existence and uniqueness of solutions to the nonlocal boundary value problem for pantograph equations involving ψ-type fractional derivative. With the help of properties of ψ-type fractional calculus, and fixed point methods, we derive existence and uniqueness results. Finally, an example is given to illustrate our theoretical results.

Keywords: ψ-fractional derivative; Boundary value problem; Existence; Fixed point; Nonlocal boundary conditions.

1 Introduction

In this paper, we go on intending to study the existence and uniqueness of solutions for the boundary value problems (BVP for short), for ψ-type fractional order pantograph equations and nonlocal boundary conditions of the form

$$^{c}\mathscr{D}^{\alpha;\psi}u(t) = f(t,u(t),u(\lambda t)), \quad \text{for each} \quad t \in J := [0,T], \quad 1 < \alpha \leq 2, \tag{1}$$

$$u(0) = g(u), \quad u(T) = u_T, \tag{2}$$

where $^{c}\mathscr{D}^{\alpha;\psi}$ is the ψ-type Caputo fractional derivative, $0 < \lambda < 1$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is a continuous function, $g : C(J,\mathbb{R}) \to \mathbb{R}$ a continuous function and $x_T \in \mathbb{R}$.

In the past decades, fractional differential equations (FDEs) have been widely used in the fields of physics, biology and engineering. There are many interesting results for qualitative analysis and applications. For more details on the development of this issue, one can refer to monographs [5,6,7,8] and the references therein.

As remarked by Byszewski [4], the nonlocal condition can be more useful than standard initial condition to describe some physical phenomena. For example, $g(u)$ may be given by

$$g(u) = \sum_{i=1}^{p} c_i u(t_i)$$

where c_i, $i = 1,...,p$, are given constants and $0 < t_1 < ... < t_p \leq T$.

Delay systems were widely applied to characterize propagation and transport development or population gestures. Also, since in economical system some choices such as investment plan and the evolution of commodity markets are divided over time intervals, delays appear in a natural way. In a mathematical area, such process is described by differential equations on functional spaces. It is shown that FDEs may be held as a possible choice to nonlinear differential equations, so recently fractional delay differential equations (DDEs) have been considered by some researchers. The application of fractional DDEs is seen in different technical systems, such as long communication lines, biology, automatic control and economy. Pantograph equation is one of the main types of DDEs arisen from the work for an electric locomotive. The

* Corresponding author e-mail: emmelsayed@yahoo.com

pantograph DDEs are found in large numbers of fields, namely electro-dynamic, so several numerical methods have been introduced for solving the integer pantograph DDEs, for more information, see [2,9,10,11].

2 Prerequisites

In this section we present some definitions and lemmas which will be used in our results later.

By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|u\|_{\infty} := \sup\{|u(t)| : t \in J\}.$$

Definition 1.[1] *The ψ-type fractional order integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$\mathscr{I}_a^{\alpha;\psi} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} h(s)ds,$$

where Γ is the Gamma function.

Definition 2.[1] *For a function h given on the interval $[a,b]$, the αth Riemann-Liouville fractional order derivative of h, is defined by*

$$\left(\mathscr{D}_{a^+}^{\alpha;\psi} h\right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1} h(s)ds.$$

Here $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Definition 3.[1] *For a function h given on the interval $[a,b]$, the ψ-type Caputo fractional order derivative of order α of h, is defined by*

$$\left({}^c\mathscr{D}_{a^+}^{\alpha;\psi} h\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha]+1$.

3 Existence of solutions

Let us start by defining what we mean by a solution of the problem (1)-(2).

Definition 4.*A function $u \in C^2([0,T],\mathbb{R})$ with its ψ-type α-derivative exists on $[0,T]$ is said to be a solution of (1)-(2) if u satisfies the equation ${}^c\mathscr{D}^{\alpha;\psi}u(t) = f(t,u(t),u(\lambda t))$ on J, and conditions $u(0) = g(u)$ and $u(T) = u_T$.*

For the existence of solutions for the problem (1)-(2), we need the following lemma. We adopt some ideas from [3].

Lemma 1.*Let $1 < \alpha \leq 2$ and let $h : [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be continuous. A function u is a solution of the following integral equation*

$$u(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s,u(s),u(\lambda s))ds \\ -\frac{\psi(t)}{\psi(T)\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s,u(s),u(\lambda s))ds \\ -\left(\frac{\psi(t)}{\psi(T)}-1\right) g(u) + \frac{\psi(t)}{\psi(T)} u_T \end{cases} \tag{3}$$

if and only if u is a solution of the ψ-fractional pantograph equation

$$ {}^c\mathscr{D}^{\alpha;\psi}u(t) = f(t,u(t),u(\lambda t)), \quad t \in [0,T], \quad 0<\lambda<1, \tag{4}$$
$$ u(0) = g(u), \quad u(T) = u_T. \tag{5}$$

At beginning, we give the following assumptions:

(A1)The function $f : [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous.

(A2)There exists a constant $k > 0$ such that

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \leq k(|x_1 - y_1| + |x_2 - y_2|)$$

for all $t \in J$, and all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

(A3)There exists a constant $k^* > 0$ such that

$$|g(x) - g(\bar{x})| \leq k^* |x - \bar{x}|,$$

for each $t \in J$, and all $x, \bar{x} \in C([0,T],\mathbb{R})$.

(A4)There exists a constant $M > 0$ such that

$$|f(t,x_1,x_2)| \leq M$$

for each $t \in J$ and all $x_1, x_2 \in \mathbb{R}$.

(A5)There exists a constant $M_1 > 0$ such that

$$|g(x)| \leq M_1 \quad \text{for all} \quad x \in C([0,T],\mathbb{R}).$$

Our first result based on Banach fixed point theorem.

Theorem 1. *Assume that (A2)-(A3) are satisfied. If*

$$\frac{4k(\psi(T))^{\alpha}}{\Gamma(\alpha+1)} + k^* < 1 \tag{6}$$

then the BVP (1)-(2) *has a unique solution on* $[0,T]$.

Proof. Transform the problem (1)-(2) into a fixed point problem. Consider the operator

$$N : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$$

defined by

$$N(u)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s,u(s),u(\lambda s))ds \\ -\frac{\psi(t)}{\psi(T)\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s,u(s),u(\lambda s))ds \\ -\left(\frac{\psi(t)}{\psi(T)} - 1\right) g(u) + \frac{\psi(t)}{\psi(T)} u_T. \end{cases}$$

Clearly, the fixed points of the operator N are solution of the problem (1)-(2). We shall use the Banach contraction principle to prove that N has fixed point. We shall show that N is a contraction.

Let $u, v \in C([0,T],\mathbb{R})$. Then, for each $t \in J$ we have

$$\begin{aligned} |N(u)(t) - N(v)(t)| \leq & \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |f(s,u(s),u(\lambda s)) - f(s,v(s),v(\lambda s))| ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} |f(s,u(s),u(\lambda s)) - f(s,v(s),v(\lambda s))| ds + |g(u) - g(v)| \\ \leq & \frac{2k\|u-v\|_\infty}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} ds \\ & + \frac{2k\|u-v\|_\infty}{\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} ds + k^* \|u-v\|_\infty \\ \leq & \frac{4k(\psi(T)^{\alpha})}{\alpha\Gamma(\alpha)} \|u-v\|_\infty + k^* \|u-v\|_\infty. \end{aligned}$$

Thus

$$|N(u) - N(v)| \leq \left[\frac{4k(\psi(T)^{\alpha})}{\Gamma(\alpha+1)} + k^*\right] \|u-v\|_\infty.$$

Consequently, N is a contraction. As a consequence of Banach fixed point theorem, we deduce that N has a fixed point which is a solution of the problem (1)-(2).

The second result is based on Schaefer's fixed point theorem.

Theorem 2. *Assume that (A1), (A4)-(A5) are satisfied. Then the BVP* (1)-(2) *has at least one solution on* $[0,T]$.

Proof. We shall use Schaefer's fixed point theorem to prove that N has a fixed point. The proof will be given in several steps.

Claim 1. N is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C([0,T],\mathbb{R})$. Then for each $t \in [0,T]$

$$\begin{aligned}|N(u_n)(t)-N(u)(t)| \leq& \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\,(\psi(t)-\psi(s))^{\alpha-1}\,|f(s,u_n(s),u_n(\lambda s))-f(s,u(s),u(\lambda s))|\,ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^T \psi'(s)\,(\psi(T)-\psi(s))^{\alpha-1}\,|f(s,u_n(s),u_n(\lambda s))-f(s,u(s),u(\lambda s))|\,ds+|g(u_n)-g(u)|\\ \leq& \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\,(\psi(t)-\psi(s))^{\alpha-1}\sup_{s\in[0,T]}|f(s,u_n(s),u_n(\lambda s))-f(s,u(s),u(\lambda s))|\,ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^T \psi'(s)\,(\psi(T)-\psi(s))^{\alpha-1}\sup_{s\in[0,T]}|f(s,u_n(s),u_n(\lambda s))-f(s,u(s),u(\lambda s))|\,ds\\ &+|g(u_n)-g(u)|.\end{aligned}$$

Since f and g are continuous functions, then we have

$$\|N(u_n)-N(u)\|_\infty \to 0 \quad \text{as} \quad n\to\infty.$$

Claim 2. N maps bounded sets into bounded sets in $C([0,T],\mathbb{R})$.

Indeed, it is enough to show that for any $\eta^*>0$, there exists a poitive constant l such that for each $u\in B_{\eta^*}=\{u\in C([0,T],\mathbb{R}):\|u\|_\infty\leq\eta^*\}$, we have $\|N(u)\|_\infty\leq l$. By (A4) and (A5) we have for each $t\in[0,T]$,

$$\begin{aligned}|N(u)(t)| \leq& \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\,(\psi(t)-\psi(s))^{\alpha-1}\,|f(s,u(s),u(\lambda s))|\,ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^T \psi'(s)\,(\psi(T)-\psi(s))^{\alpha-1}\,|f(s,u(s),u(\lambda s))|\,ds+2\,|g(u)|+|u_T|\\ \leq& \frac{M}{\Gamma(\alpha)}\int_0^t \psi'(s)\,(\psi(t)-\psi(s))^{\alpha-1}\,ds+\frac{M}{\Gamma(\alpha)}\int_0^T \psi'(s)\,(\psi(T)-\psi(s))^{\alpha-1}\,ds+2M_1+|u_T|\\ \leq& \frac{M}{\alpha\Gamma(\alpha)}(\psi(T))^\alpha+\frac{M}{\alpha\Gamma(\alpha)}(\psi(T))^\alpha+2M_1+|u_T|.\end{aligned}$$

Thus

$$\|N(u)\|_\infty \leq \frac{2M}{\alpha\Gamma(\alpha)}(\psi(T))^\alpha+2M_1+|u_T|:=l.$$

Claim 3. N maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$.

Let $t_1,t_2\in(0,T]$, $t_1<t_2$, B_{η^*} be a bounded set of $C([0,T],\mathbb{R})$ as in Claim 2, and $u\in B_{\eta^*}$. Then

$$\begin{aligned}|N(u)(t_2)-N(u)(t_1)| =& \Bigg|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}-\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}\right]f(s,u(s),u(\lambda s))ds\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} f(s,u(s),u(\lambda s))ds\Bigg|\\ &+\frac{(\psi(t_2)-(\psi(t_1)))}{(\psi(T))\Gamma(\alpha)}\int_0^T \psi'(s)\,(\psi(T)-\psi(s))^{\alpha-1}\,|f(s,u(s),u(\lambda s))|\,ds\\ &+\frac{(\psi(t_2)-(\psi(t_1)))}{(\psi(T))}|g(u)|+\frac{(\psi(t_2)-(\psi(t_1)))}{(\psi(T))}|u_T|\\ \leq& \frac{M}{\Gamma(\alpha+1)}\left[(\psi(t_2)-(\psi(t_1)))^\alpha+(\psi(t_1))^\alpha-(\psi(t_2))^\alpha\right]+\frac{M}{\Gamma(\alpha+1)}(\psi(t_2)-(\psi(t_1)))^\alpha\\ &+M\frac{(\psi(t_2)-\psi(t_1))}{(\psi(T))\Gamma(\alpha)}(\psi(T))^\alpha+\frac{(\psi(t_2)-\psi(t_1))}{(\psi(T))}M_1+\frac{(\psi(t_2)-\psi(t_1))}{(\psi(T))}|u_T|.\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Claim 1 to Claim 3 together with the Arzela-Ascoli theorem, we can conclude that $N : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ is completely continuous.

Claim 4. A priori bounds.

Now it remains to show that the set

$$\zeta = \{u \in C(J,\mathbb{R}) : u \in \delta N(u) \quad \text{for some} \quad 0 < \delta < 1\}$$

is bounded.

Let $u \in \zeta$, then $u \in \delta N(u)$ for some $0 < \delta < 1$. Thus, for each $t \in J$ we have

$$N(u)(t) = \begin{cases} \frac{\delta}{\Gamma(\alpha)} \int_0^t \psi'(s)\,(\psi(t)-\psi(s))^{\alpha-1} f(s,u(s),u(\lambda s))ds \\ -\frac{\delta\psi(t)}{\psi(T)\Gamma(\alpha)} \int_0^T \psi'(s)\,(\psi(t)-\psi(s))^{\alpha-1} f(s,u(s),u(\lambda s))ds \\ -\delta\left(\frac{\psi(t)}{\psi(T)}-1\right) g(u) + \delta\frac{\psi(t)}{\psi(T)} u_T. \end{cases}$$

This implies by (A4) and (A5) that for each $t \in J$ we have

$$\begin{aligned} |N(u)(t)| &\leq \frac{M}{\Gamma(\alpha)} \int_0^t \psi'(s)\,(\psi(t)-\psi(s))^{\alpha-1} ds + \frac{M}{\Gamma(\alpha)} \int_0^T \psi'(s)\,(\psi(T)-\psi(s))^{\alpha-1} ds + 2M_1 + |u_T| \\ &\leq \frac{2M(\psi(T))^{\alpha}}{\Gamma(\alpha+1)} + 2M_1 + |u_T|. \end{aligned}$$

Thus for every $t \in [0,T]$, we have

$$\|N(y)\|_\infty \leq \frac{2M(\psi(T))^{\alpha}}{\Gamma(\alpha+1)} + 2M_1 + |u_T| := R.$$

This shows that the set ζ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point which is a solution of the problem (1)-(2).

4 An example

In this section, we give an example to illustrate our main results derived in Sect.3.

$$^c\mathscr{D}^{\alpha;\psi}u(t) = \frac{1}{5} + \frac{1}{10}u(t) + \frac{1}{10}u(\lambda t), t \in J := [0,1], \quad 1 < \alpha \leq 2, \tag{7}$$

$$u(0) = \sum_{i=1}^{n} c_i u(t_i), \quad u(1) = 0, \tag{8}$$

where $\lambda \in (0,1)$, $0 < t_1 < t_2 < ... < t_n < 1$, c_i, $i = 1,...,n$ are given positive constants with $\sum_{i=1}^{n} c_i < \frac{4}{5}$. Set

$$f(t,x_1,x_2) = \frac{1}{5} + \frac{1}{10}x_1(t) + \frac{1}{10}x_2(\lambda t), \quad (t,x_1,x_2) \in J \times [0,\infty),$$

and

$$g(u) = \sum_{i=1}^{n} c_i u(t_i).$$

Let $u, v, \overline{u}, \overline{v} \in [0,\infty)$ and $t \in J$. Then we have

$$|f(t,u,v) - f(t,\overline{u},\overline{v})| \leq \frac{1}{10}(|u-\overline{u}| + |v-\overline{v}|).$$

Hence the condition (A2) holds with $k = \frac{1}{10}$. Also we have

$$|g(u) - g(v)| \leq \sum_{i=1}^{n} c_i |u-v|.$$

Hence (A3) is satified with $k^* = \sum_{i=1}^{n} c_i$. We shall check that condition (6) is satified with $\psi(T) = 1$. Indeed,

$$\frac{4k(\psi(T))^{\alpha}}{\Gamma(\alpha+1)} + k^* = \frac{2}{5\Gamma(\alpha+1)} + \sum_{i=1}^{n} c_i < 1,$$

which satisfies for $\alpha \in (1,2]$. Then by Theorem 1 the problem (7)-(8) has a unique solution on $[0,1]$.

Acknowledgment

The authors are thankful to the anonymous reviewers and the handling editor for the fruitful comments that made the presentation of the work more interested.

Authors contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

References

[1] R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Commun. Nonlinear Sci. Numer. Simul.*, 44 (2017) 460-481.
[2] K.Balachandran, S.Kiruthika, J.J.Trujillo, Existence of solutions of nonlinear fractional pantograph equations, *Acta Math. Sci. Ser. A*, 33 (2013) 712-720.
[3] M. Benchohra, S. Hamani, S.K. Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions, *Nonlinear Anal.*, 71 (2009) 2391-2396.
[4] L. Byszewski, V. Lakshmikantham, Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space, *Appl. Anal* 40 (1991) 11-19.
[5] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science B.V., Amsterdam (2006).
[6] V. Lakshmikantham, S. Leela, J.V. Devi, Theory of Fractional Dynamic Systems, Cambridge Scientific Publishers, Cambridge (2009).
[7] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Differential Equations, Wiley, New York (1993).
[8] I. Podlubny, Fractional Differential Equations, Academic Press, Boston (1999).
[9] D. Vivek, K. Kanagarajan, S. Sivasundaram, Dynamics and stability of pantograph equations via Hilfer fractional derivative, *Nonlinear Stud.*, 23(4) (2016) 685-698.
[10] D. Vivek, K. Kanagarajan, S. Sivasundaram, Theory and analysis of nonlinear neutral pantograph equations via Hilfer fractional derivative , *Nonlinear Stud.*, 24(3) (2017) 699-712.
[11] D. Vivek, K. Kanagarajan, S. Harikrishnan, Existence and uniqueness results for pantograph equations with generalized fractional derivative, *Journal of Nonlinear Analysis and Application*, 2 (2017) 105-112.