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A Fractional-Order Model of Dengue Fever with Awareness Effect : Numerical Solutions and Asymptotic Stability Analysis

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Abstract: In this paper, we study the effect of awareness of population on the spreading of a fractional-order dengue fever model. We calculate the equilibrium points (free-disease point and endemic point) and study local asymptotic stability by using Routh–Hurwitz conditions. The global asymptotic stability by using LaSalle's invariance principle has been studied. The stability analysis show the relation between reproductive ratio R_0 and the local and global stability. To support our theoretical results, we solve this model by using Adams-type predictor-corrector method. The numerical results show the effect of awareness (represented by σ parameter in model), also it show that the fractional order model has smaller peak than integer order model which mean better fitting of data.

Keywords: Dengue fever, fractional calculus, Adams-type predictor-corrector, stability analysis, numerical results.

1 Introduction

Mathematical models have a great importance in the natural sciences [1–13]. It can be used as an alternative to a biological system, and so much can be learned about the organism by devising an appropriate mathematical model. These models can be solved through analysis or numerical methods to know the behavior of the biological model. It is not limited to the study of the human body, but extends to the surrounding environment. It studied the spread of epidemics through environmental biology. Dengue fever is one of important bio-mathematical models which caused by four closely related viruses stereotypes [14–22]. There are many authors studied and developed outdated techniques to have exact, analytic, and approximate solutions for fractional differential equations arising from a variety of branches of various scientific fields [23–30]. Dengue fever is endemic in more than hundred nations. As of late, the quantity of dengue cases has been expanding significantly. Awareness of the infection can change the entire elements of the transmission of the disease. Because of awareness, individuals can take various types of safety measures towards the disease. Maintaining a strategic distance from mosquito's bite is the significant precautionary measure against dengue fever. A portion of the safety measures that can be taken are to keep home, environment and encompassing cleanliness, to evacuate all stale water and holders, to cover all compartments appropriately to forestall dengue mosquito reproducing there, to wrap all unused plastic tires, to utilize mosquito anti-agents to stay away from mosquito chomp, to utilize mist concentrates and mosquito loops to murder mosquitoes, to wear long sleeve and completely secured garments, to utilize mosquitoes net around bed while resting. To control the ailment adequately, one ought to comprehend the elements of the disease transmission and consider the entirety of the relating subtleties [16]. In [16], the authors study mathematical model of dengue fever with and without awareness in host population as for human population. In this paper, we will use Adams-type predictor-corrector method [31] to get an accurate solution to time fractional-order model of dengue fever with awareness effect

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which considered in [16] as:

$$\begin{aligned} D^{\alpha}u(t) &= \mu(1-u(t)) - \lambda u(t)w(t), \\ D^{\alpha}v(t) &= \lambda u(t)w(t) - \beta v(t), \\ D^{\alpha}w(t) &= \gamma(1-w(t))v(t) - \delta w(t), \\ D^{\alpha}N(t) &= \mu - \mu N(t) - \zeta v(t), \end{aligned} \tag{1}$$

where $u(t)$ is density of susceptible host individuals, $v(t)$ is density infected host, $w(t)$ is density infected vector, $N(t) = u(t)+v(t)$ is total population, μ is birth and death rate for host individuals, $\lambda = \dfrac{(1-\sigma)c\eta A}{\delta N}$, σ is density of susceptible host individuals aware of dengue transmission, c is the rate of biting by vector, A is recruitment rate(mosquitoes per time), η is the probability of transition from vector to human host, δ is death rate for vector, $\beta = \mu+\zeta$, ζ is recovery rate for human population, $\gamma = (1-\overline{\sigma})c\rho$, $\overline{\sigma}$ is density of infected host individuals aware of dengue transmission and ρ is the probability of transition from host human to vector. All previous parameters are calculated and found in [16]. The fractional order $0 < \alpha \leq 1$ and D^{α} is called Caputo fractional derivative sense [32], which defined, as:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t)\, dt,$$

$$\text{for } n-1 < \alpha \leq n\ ,\ n \in \mathrm{N}\ , x > 0.$$

2 Asymptotic stability

In this section, we calculate the equilibrium points of system (1) and then study asymptotic stability for each point. If we put $\dfrac{d^{\alpha}u}{dt} = 0, \quad \dfrac{d^{\alpha}v}{dt} = 0, \quad \dfrac{d^{\alpha}w}{dt} = 0$, the following equilibrium points exist for system (1) as:

1.Free-disease equilibrium point is

$$E_0(1,0,0).$$

2.The epidemic equilibrium point is

$$E_1\left(\frac{\gamma\mu_h+\beta\delta}{\lambda\gamma+\gamma\mu_h}, \frac{\lambda\gamma\mu_h-\beta\mu_h\delta}{\lambda\beta\gamma+\beta\gamma\mu_h}, \frac{\lambda\gamma\mu_h-\beta\mu_h\delta}{\lambda\gamma\mu_h+\alpha\beta\delta}\right).$$

We evaluate the Jacobian matrix J of equilibrium point to study asymptotic stability for each of them. The jacobian matrix J is computed as $J(E) = j_{ij}$
where

$$j_{11} = -(\mu+\lambda w),\ \ j_{12}=0,\ \ j_{13} = -\lambda u,\ \ j_{21} = \lambda w,\ \ j_{22} = -\beta,\ \ j_{23} = \lambda u,\ \ j_{31} = 0, j_{32} = \gamma - \gamma w \text{ and } \ j_{33} = -(\delta+\gamma v).$$

Theorem 2.1. The equilibrium point E_0 is locally asymptotically stable if $R_0^2 < 1$, *where* $R_0^2 = \frac{\alpha\gamma}{\beta\delta}$.
Proof:
The Jacobian matrix of system (1) at equilibrium point $E_0(1,0,0)$:

$$J(E_0) = \begin{pmatrix} -\mu & 0 & -\lambda \\ 0 & -\beta & \lambda \\ 0 & \gamma & -\delta \end{pmatrix}. \tag{2}$$

The eigenvalues corresponding to the equilibrium E_0 are:

$$\rho_1 = -\frac{\beta}{2} - \frac{\delta}{2} - \frac{\sqrt{\left((\beta-\delta)^2+4\lambda\gamma\right)}}{2}, \quad \rho_2 = -\frac{\beta}{2} - \frac{\delta}{2} + \frac{\sqrt{\left((\beta-\delta)^2+4\lambda\gamma\right)}}{2}, and \quad \rho_3 = -\mu.$$

If we put $\rho_2 < 0$ then

$$-\frac{\beta}{2} - \frac{\delta}{2} + \frac{\sqrt{\left((\beta-\delta)^2+4\lambda\gamma\right)}}{2} < 0,$$

$$\left(\beta-\delta)^2+4\lambda\gamma\right)<(\beta+\delta)^2.$$

Then we get $R_0^2=\dfrac{\lambda\gamma}{\beta\delta}<1$. See [16, 18].

It's shown thatρ_2 negative, if and only if $R_0^2<1$ and ρ_1,ρ_3 are negative where all parameters are positive. We conolude that E_0 is locally asymptotic stable if and only if $R_0^2<1$. Proof complete

Theorem 2.2. The equilibrium point E_1 is locally asymptotically stable if and only if $R_0^2>1$.

Proof:

The Jacobian matrix of system (1) at equilibrium point $E_1(u^*,v^*,w^*)$, where $u^*=\dfrac{\gamma\mu_h+\beta\delta}{\lambda\gamma+\gamma\mu_h}, v^*=\dfrac{\lambda\gamma\mu_h-\beta\mu_h\delta}{\lambda\beta\gamma+\beta\gamma\mu_h}$ and $w^*=\dfrac{\lambda\gamma\mu_h-\beta\mu_h\delta}{\lambda\gamma\mu_h+\alpha\beta\delta}$ is

$$J(E_1)=\begin{pmatrix} -\mu_h-\lambda(\frac{\lambda\gamma\mu_h-\beta\mu_h\delta}{\lambda\gamma\mu_h+\lambda\beta\delta}) & 0 & -\lambda(\frac{\gamma\mu_h+\beta\delta}{\lambda\gamma+\gamma\mu_h}) \\ \lambda(\frac{\lambda\gamma\mu_h-\beta\mu_h\delta}{\lambda\gamma\mu_h+\lambda\beta\delta}) & -\beta & \lambda(\frac{\gamma\mu_h+\beta\delta}{\lambda\gamma+\gamma\mu_h}) \\ 0 & \gamma-\gamma(\frac{\lambda\gamma\mu_h-\beta\mu_h\delta}{\lambda\gamma\mu_h+\alpha\beta\delta}) & -\delta-\gamma(\frac{\lambda\gamma\mu_h-\beta\mu_h\delta}{\lambda\beta\gamma+\beta\gamma\mu_h}) \end{pmatrix}. \tag{3}$$

Consider the polynomial of eigenvalues :

$$\rho^3+a_1\rho^2+a_2\rho+a_3 \quad =0, \tag{4}$$

where

$$a_1=\beta+\mu+\delta+\lambda w^*+\gamma v^*,$$
$$a_2=\beta\mu+\beta\delta+\mu\delta+\lambda\beta w^*-\lambda\gamma u^*+\beta\gamma v^*+\gamma\mu v^*+\lambda\delta w^*+\lambda\gamma\mu w^*+\lambda\delta u^*w^*,$$
$$a_3=\beta\mu\delta-\lambda\gamma\mu u^*+\mu\delta+\beta\gamma\mu v^*+\lambda\beta\delta w^*+\beta\gamma v^*+\gamma\mu v^*+\lambda\delta w^*+\lambda\beta\delta u^*w^*+\lambda\beta\gamma v^*w^*+\alpha\delta\mu u^*w^*.$$

It is obvious that as a_1,a_2 and $a_3>0$ if $R_0^2>1$. Further, we have $a_1a_2-a_3\ >\ 0$, therefore, the Routh-Hurwitz conditions are satisfied. See [12]. We complete proof

3 Global asymptotic stability

Theorem 3.1. If $R_0^2<1$, then the disease-free equilibrium E_0 is globally asymptotically stable.

Proof.

For system (1), we get the following Lyapunov function:

$$L_0(t)=v+\frac{\beta}{\gamma}w,$$

$$D^\alpha L_0(t)=\lambda uw-\beta\, v+\beta\,(1-w)\,v-\frac{\delta\beta}{\gamma}w,$$

$$D^\alpha L_0(t)<w\left(\lambda u-\frac{\delta\beta}{\gamma}\right)<w\frac{\gamma}{\delta\beta}\left(R_0^2-1\right)<0\ ,\ if\ R_0^2<1.$$

We have $L_0(t)>0$, $D^\alpha\ L_0(t)=0$ at E_0 and $D^\alpha L_0<0$, then E_0 is globally asymptotically stable according to [30] when $R_0^2<1$, which implies the disease will disappear regardless the initial infected individuals.

Theorem 3.2. If $R_0^2>1$, $a>0$ where a is positive constant and $N=av$, then the endemic equilibrium E_1 is globally asymptotically stable.

Proof.

For system (1), we get the following Lyapunov function:

$$L_1(t)=\frac{1}{2}(N-N^*)^2,$$

$$D^{\alpha}L_1(t) = \frac{1}{2}(N-N^*)D^{\alpha}N = \frac{1}{2}(N-N^*)(\mu-\mu N-\zeta v) = \frac{1}{2}(N-N^*)\left(\mu-\mu N-\frac{\zeta}{a}N\right) = -(\mu+\frac{\zeta}{a})(N-N^*)^2 < 0,$$

where

$$N = av, \quad \mu = \left(\mu+\frac{\zeta}{a}\right)N^*.$$

We have $L_1(t) > 0$, $D^{\alpha}\,L_1(t) = 0$ at E_1 and $D^{\alpha}L_1 < 0$,t then the unique positive epidemic equilibrium E_1 is globally asymptotically stable if $R_0^2 > 1$.

4 Numerical simulation

To support our theoretical results, we use Adams-type predictor-corrector method which explained in details in [31] to solve fractional dengue fever model (1) . The initial conditions are: $u(0) = 0.9,\ v(0) = 0.1,\ w(0) = 0$ .The parameters in figures 1,2 and 3 are $\lambda = 0.8750,\ \beta = 0.1429,\ \gamma = 0.25,\ \delta = 0.1429,\ \sigma = 0.8,\ \overline{\sigma} = 0.9$ and $R_0^2 = 0.46$.The parameters in figures 4,5 and 6 are $\lambda = 0.1750,\ \beta = 0.1429,\ \gamma = 0.025,\ \delta = 0.1429,\ \sigma = \overline{\sigma} = 0$ and $R_0^2 = 3.27$

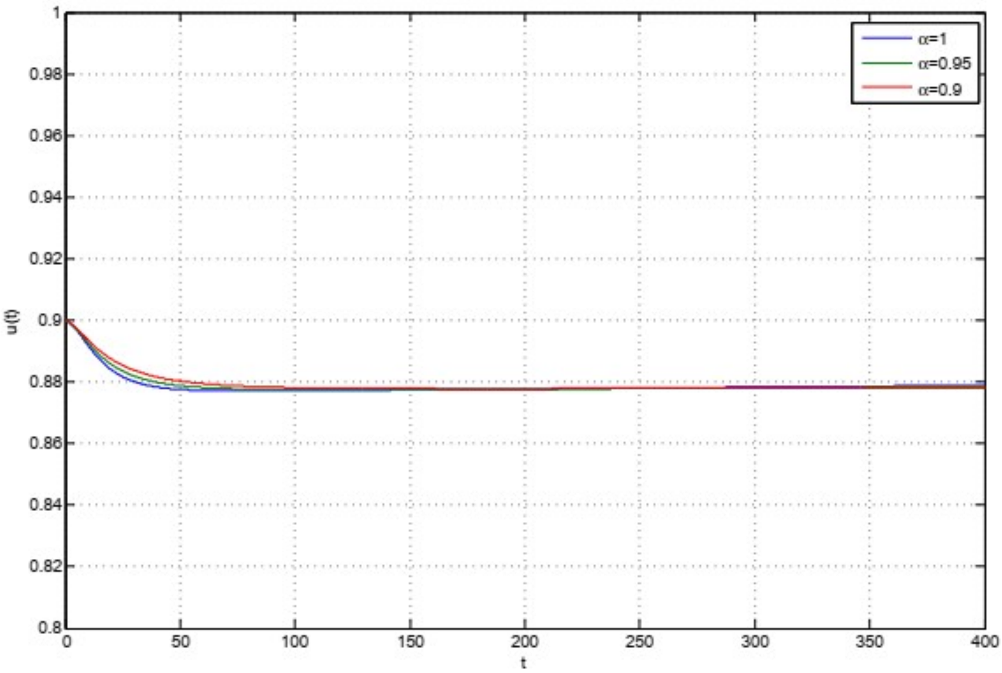

Fig. 1: The relation between susceptible host individuals *u(t)* and time *t* with different values of fractional orders α.

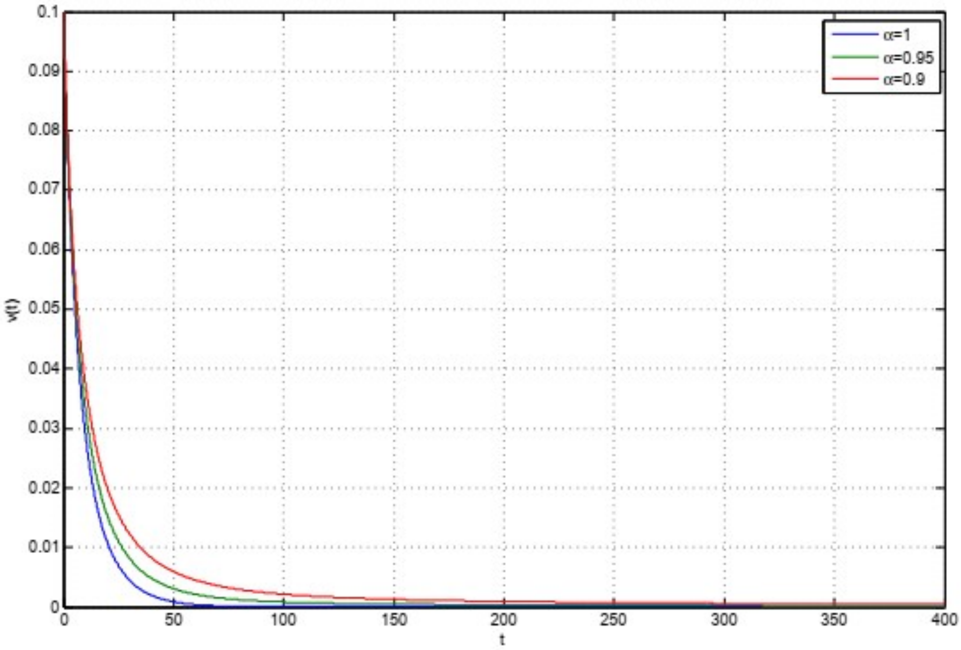

Fig. 2: The relation between infected host individuals *v(t)* and time *t* with different values of fractional orders α.

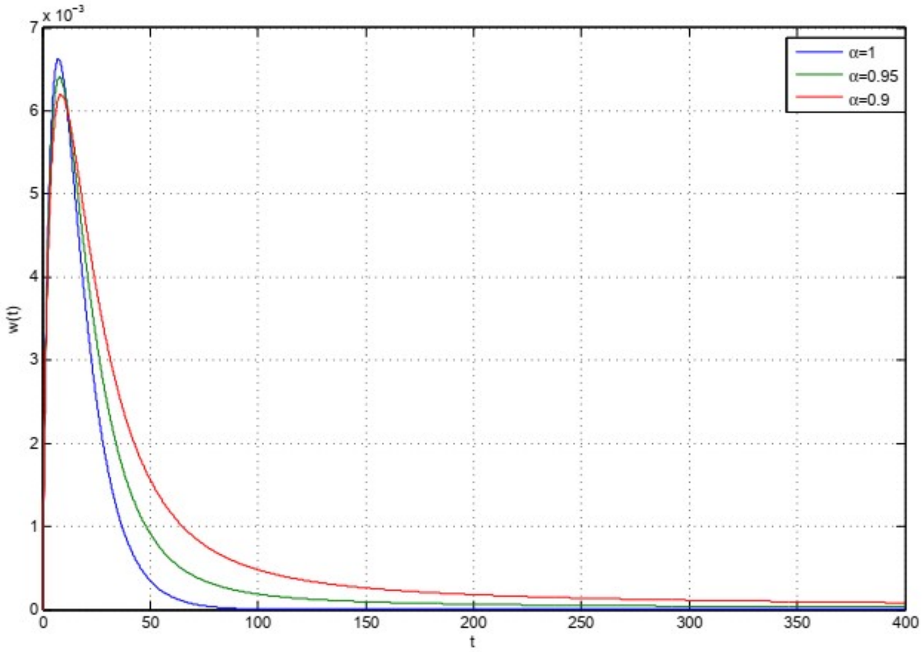

Fig. 3: The relation between infected vector *w(t)* and time *t* with different values of fractional orders α.

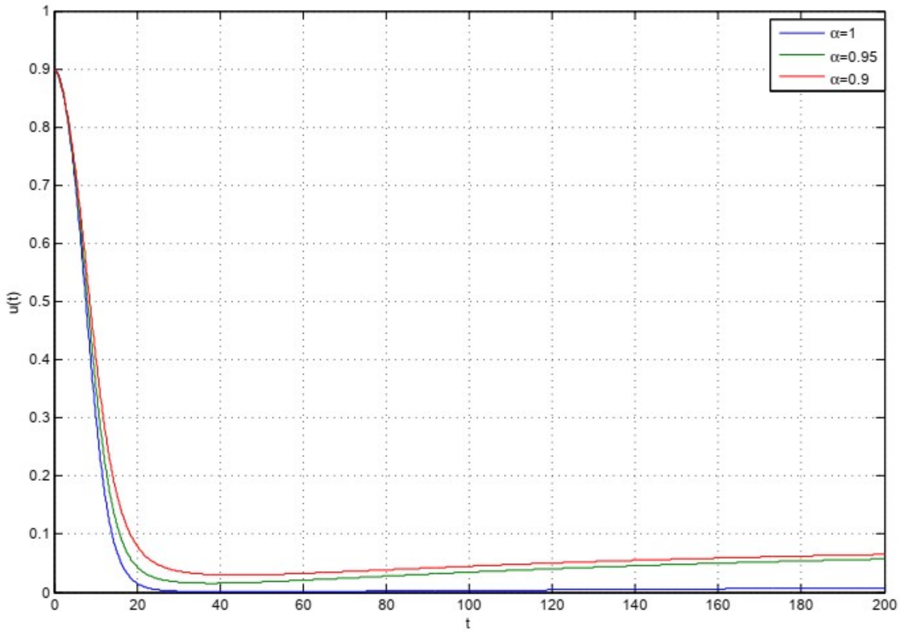

Fig. 4: The relation between susceptible host individuals *u(t)* and time *t* with different values of fractional orders α.

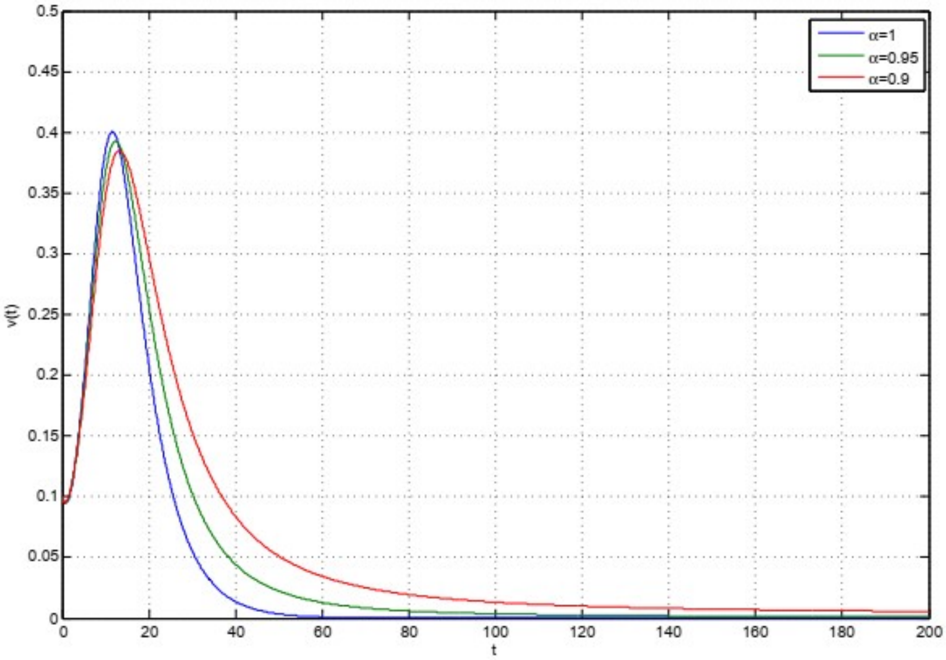

Fig. 5: The relation between infected host individuals *v(t)* and time *t* with different values of fractional orders α.

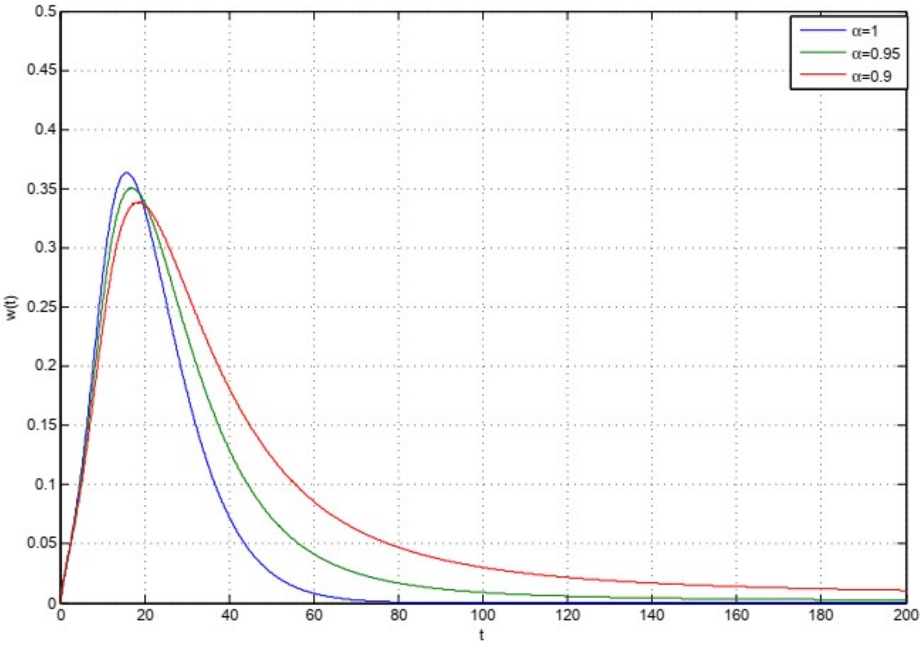

Fig. 6: The relation between infected vector *w(t)* and time *t* with different values of fractional orders α.

5 Conclusion

We applied Adams-type predictor-corrector method to study fractional dengue fever model. Figures 1,2 and 3 show that the free disease point E_0 is locally asymptotic stable, but epidemic equilibrium point is not. Where $R_0^2 < 1$. Figure 2 shows that infected host individuals $v(t)$ tend to 0 at $t \to \infty$, the effect of fractional order appears in how much time curves tends to 0. Actually, we can easily see that fractional order takes much time to reach 0; this in turn helps us to compare with clinical data. Figures 3 and 6 show that the infected vector $w(t)$, fractional order curve has smaller peak than integer order which show cases better data fitting. Obviously, figures 4,5 and 6 show that endemic point is local and global asymptotic is stable with $R_0^2 > 1$. Further, figures 4 and 5 show that integer order tends to 0, showing that we haven't any data of susceptible host individuals or infected host individuals. Fractional order, however, does not tend to 0, exposing more data for comparison with clinical data. Figure 2 shows the effect of awareness towards disease if we take value of awareness of susceptible and infected human as ,$\sigma = 0.8, \overline{\sigma} = 0.9$ that mean highly awareness of disease effect of reproductive ratio, we can easily show that the value of $R_0^2 < 1$ and $u(t)$ infected individuals tends to zero. Meanly, the awareness plays important rule for transition of infection. In Figure 5, the awareness parameter $\sigma = \overline{\sigma} = 0$, we can see the effect of disappearing the effect of awareness in reproductive ratio $R_0^2 > 1$ and the curve of solution go up to peak that mean highly infection comparing the curve of figure 2. we Finally, the results show that mathematical modeling based on fractional order has more advantages than classical integer-order. It could also be concluded that appropriate fractional order can be taken for the best comparison with clinical data.

Conflicts of Interests
The authors declare that they have no conflicts of interests.

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