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On the Fractional Differential Equations Associated with Integral Operator Involving Aleph Function in the Kernel

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Abstract: In the present article, we introduced and explore an integral operator which consist Aleph function in the kernel with fractional calculus. In second section, we construct the characteristics of R-L fractional integral operator I^{β}_{a+} and derivative operator D^{β}_{a+} containing the Aleph-function and in third section, we develop the Sumudu transform of propose integral operator. In fourth section, we find the solutions of arbitrary order differential equations which consists the Hilfer derivative operator along with propose integral operator by applying Sumudu transform. We also established some fascinating corollaries and particular cases of our key results presented here in terms of a number of special functions particularly H-function, I-function, Mittag-Leffler, and generalized Bessel-Maitland function and exhibit to be their relation with certain known results. In the end of the article, we develop some graphical results to show the behavior of differential equation by assigning particular values to the parameters.

Keywords: Hilfer fractional operator, Sumudu transform, Aleph-function, Lebesgue measurable function.

1 Introduction, Motivation and Preliminaries

For the last few decades we have been seeing that the arbitrary order differential equations have experienced significant consideration to derive the solution of complex problems come to the light in the diverse field of Physical science, bio-science and engineering [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18]. If we talk about the literature fractional calculus, then there are many integrals and derivatives operators of arbitrary order are present. The names of few operators in this list are as follow: the R-L integral, R-L derivatives, Caputo derivative and Hilfer derivatives operator comprehensively analyzed and implemented in various problems occurring in mathematics science. From the research so far, we can say that Hilfer derivatives operator have too much potential to generalize different types of differential equations of arbitrary order. It would be fair to say that fractional differential equations (FDE's) can play a much larger role in solving a variety of problems problem which arises in Mathematics, physics, Statistics and engineering. Recently, Watugala [19] defined a novel integral transform and name as Sumudu transform. Watugala showed by his work the Sumudu transform play a vital role to solve fractional differential problems of engineering control [20]. Several authors used the results of Sumudu transform and implement to it to derive the solution of integral equations of convolution type [21] and discrete dynamic system [22]. Belgacem and Karaballi [23,24] investigate fundamental properties and applications that will assist us to derive the solution of complex problems in Bio-science, control engineering problems. In the meantime, Vashi and Timol [25] work showed the comparison between Laplace and Sumudu transform by finding their applications in physics and Electric circuit theory and confirmed that this transform is an effortless and very effective technique to explore the analytic results of intricate higher order ODE's. Wang and Liu [26] applied this transform to elucidate fractional differential equations and time fractional Cauchy reaction- diffusion equations. In the

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literature of fractional integration and differentiation there are several integral transform are studied such as Laplace, Fourier, Mellin etc. A new integral transform namely "Sumudu transform" and it is modified aspect of classical Laplace transform and has a good features. For more details concerning Sumudu transform (see [19,20,21,22,23,24,27,28,29] and many others. In this manuscript, we obtained the solution of FDE's based upon the Hilfer derivative operator by using a new integral transform technique 'Sumudu transform'. The motivation of this manuscript is to encourage further investigation of the integral operator by Sumudu transform and its application in the field of mathematics.

We will give some valuable definitions and properties related to Sumudu transform, Aleph-function and fractional calculus operator. Moreover, we also develop a new integral operator in this article. The Sumudu transform is used at large-scale to establish the solutions of several types of problems in manifold area of engineering and this well-known transform introduced by Watugala [19], and demonstrate over the set functions as
$A = \{y(\zeta)|\exists M, \sigma_1, \sigma_2 > 0, |y(\zeta)| < Me^{|\zeta|/\sigma_j}$, if $\zeta \in (-1)^j \times [0,\infty)\}$ by the following formula

$$Y(v) = S[y(\zeta);v] = \int_0^\infty y(v\zeta)e^{-\zeta}d\zeta, v \in (-\sigma_1, \sigma_2). \tag{1}$$

The existence and uniqueness was discussed in [24].
Now, the Sumudu transform of the function $\zeta y(\zeta)$ is given by

$$S[\zeta y(\zeta)] = \frac{vd(vy(v))}{dv} = v\left[Y(v) + v\frac{dY(v)}{dv}\right]. \tag{2}$$

Let $S[y(\zeta);v] = Y(v)$ and $S[l(\zeta);v] = L(v)$, then

$$S(y*l) = vS[y(\zeta)]S[l(\zeta)] = \int_0^\tau y(v)l(\tau - v)dv, \tag{3}$$

where $y*l$ is called convolution of y and l.
The R-L integral operator of order $\gamma > 0$ is express as (see [5,7])

$$I_{a+}^{\gamma} y(s) = \frac{1}{\Gamma(\gamma)}\int_a^s \frac{y(\tau)}{(s-\tau)^{(1-\gamma)}}d\tau, (\gamma \in C). \tag{4}$$

The R-L derivative of order $\gamma > 0$ is as follow (see [5,7])

$$D_{a+}^{\gamma} y(s) = \frac{1}{\Gamma(m-\gamma)}\frac{d^m}{ds^m}\int_a^s (s-\tau)^{(m-1-\gamma)}y(\tau)d\tau, \text{if} \quad (\gamma \in C, m = [\Re(\gamma)]+1). \tag{5}$$

The Caputo [2] fractional derivative of order $\gamma > 0$, which is defined as

$${}_a^cD_s^{\gamma}\gamma(s) = \begin{cases} I_{a+}^{m-\gamma}\frac{d^m}{ds^m}y(s) = \frac{1}{\Gamma(m-\gamma)}\int_a^s \frac{y^m(\tau)}{(s-\tau)^{(\gamma-m+1)}}d\tau, \text{if} \quad m-1<\gamma<m \\ = \frac{d^m y(s)}{ds^m}, \text{if} \quad \gamma = m, m \in N. \end{cases} \tag{6}$$

Hilfer operator is defined as follows (see [3,4,14]):

$${}_0D_{a+}^{\alpha,\beta}g(t) = \left(I_{a+}^{\beta(1-\alpha)}\frac{d}{dx}\left(I_{a+}^{(1-\beta)(1-\alpha)}g(t)\right)\right), (0<\alpha<1, 0\le\beta\le 1). \tag{7}$$

If we set $\beta = 1$, then eq. (7) reduces to (6).
Sumudu transform of Hilfer derivative [28,29] is express as follows:

$$S\left[{}_0D_{a+}^{\alpha,\beta}g(t);S\right] = S^{-\alpha}\bar{g}(s) - S^{-\beta(\alpha-1)-1}I_{0+}^{(1-\beta)(1-\alpha)}g(0+); (0<a\le 1). \tag{8}$$

For more details of this operator see [11].
The well known Aleph function $\aleph$ studied by Südland et al. [32] and express in terms of the Mellin-Barnes type integrals as

$$\begin{aligned}\aleph[t] = \aleph_{p_l,q_l,\sigma_l;r}^{m,n}(t) &= \aleph_{p_l,q_l,\sigma_l;r}^{m,n}\left[t \left| \begin{matrix} (\mathfrak{a}_j,\alpha_j)_{1,n}, [\sigma_l(\mathfrak{a}_{jl},\alpha_{jl})]_{n+1,p_l;r} \\ (\mathfrak{b}_j,\beta_j)_{1,m}, [\sigma_l(\mathfrak{b}_{jl},\beta_{jl})]_{m+1,q_l;r} \end{matrix}\right.\right] \\ &= \frac{1}{2\pi i}\int_L \Omega_{p_l,q_l,\sigma_l;r}^{m,n}(s)t^{-s}ds, \end{aligned} \tag{9}$$

for all $t \neq 0$, where $i = \sqrt{-1}$ and

$$\Omega^{m,n}_{p_l,q_l,\sigma_l;r}(s) = \frac{\prod_{j=1}^{m}\Gamma(\mathfrak{b}_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-\mathfrak{a}_j-\alpha_j s)}{\sum_{l=1}^{r}\sigma_l\prod_{j=n+1}^{p_l}\Gamma(\mathfrak{a}_{jl}+\alpha_{jl}s)\prod_{j=m+1}^{q_l}\Gamma(1-\mathfrak{b}_{jl}-\beta_{jl}s)}. \tag{10}$$

For details of existence conditions see [30,31,33].

*Remark.*If we set $\sigma_1 = \sigma_2 = \cdots = \sigma_r = 1$,then (9) reduces to the I-function [34]:

$$\begin{aligned} I[t] = \aleph^{m,n}_{p_l,q_l,1;r}(t) &= \aleph^{m,n}_{p_l,q_l,1;r}\left[t\left|\begin{matrix}(\mathfrak{a}_j,\alpha_j)_{1,n},[1(\mathfrak{a}_{jl},\alpha_{jl})]_{n+1,p_l;r}\\(\mathfrak{b}_j,\beta_j)_{1,m},[1(\mathfrak{b}_{jl},\beta_{jl})]_{m+1,q_l;r}\end{matrix}\right.\right]\\ &= \frac{1}{2\pi i}\oint \Omega^{m,n}_{p_l,q_l,1;r}(s)t^{-s}ds. \end{aligned} \tag{11}$$

*Remark.*If we assign particular value r=1 in (11), then we get the familiar Fox H-function introduced by Fox [35]:

$$H^{m,n}_{p,q}[t] = \aleph^{m,n}_{p_l,q_l,1;1}(t) = \aleph^{m,n}_{p_l,q_l,1;1}\left[t\left|\begin{matrix}(\mathfrak{a}_p,\alpha_p)\\(\mathfrak{b}_q,\beta_q)\end{matrix}\right.\right] = \frac{1}{2\pi i}\oint \Omega^{m,n}_{p_l,q_l,1;1}(s)t^{-s}ds. \tag{12}$$

Wright generalized hypergeometric function is also called the Fox - Wright function and which is express as (see[5,38]):

$$\begin{aligned} {}_p\Psi_q\left[\left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right|t\right] &= \sum_{k=0}^{\infty}\frac{\Gamma(a_1+kA_1)\ldots\Gamma(a_p+kA_p)}{\Gamma(b_1+kB_1)\ldots\Gamma(b_q+kB_q)}\frac{z^k}{k!}\\ &= H^{1,p}_{p,q+1}\left[-t\left|\begin{matrix}(1-a_1,A_1),\ldots,(1-a_p,A_p)\\(0,1),(1-b_1,B_1),\ldots,(1-b_q,B_q)\end{matrix}\right.\right], \end{aligned} \tag{13}$$

$\left(\Re(A_l) > 0 (l = 1,\ldots,p); \Re(B_j) > 0 (j = 1,\ldots,q); 1+\Re\left(\sum_{j=1}^{q}B_j - \sum_{l=1}^{p}A_l\right) \geq 0\right)$.
Several fractional integral operator were studied and investigated by various authors such as Kilbas et al. [6], Srivastava and Tomovski [9], Srivastava and Saxena [10] and Harjule and Jain [18].
Now, we introduced an integral operator as follows:

$$\left(\varepsilon^{\omega;\alpha,\beta,m,n}_{a+,p_l,q_l,\sigma_l;r}\vartheta\right)(z) = \int_a^z (z-\tau)^{\beta-1}\aleph^{m,n}_{p_l,q_l,\sigma_l;r}[\omega(z-\tau)^{\alpha}]\vartheta(\tau)d\tau, \tag{14}$$

$(z > a, \Re(\alpha) > 0, \Re(\beta) > 0, \omega \in \mathbb{C}, \Re(\beta) + \alpha\min_{1\leq j\leq m}\frac{\Re(b_j)}{B_j} > 0$ and $|arg(\omega)| < \pi\varphi_l/2, \varphi_l > 0, l = \overline{1,r})$.

2 Fractional Calculus Operator Involving the Aleph Function

In this portion, we established the relationship of the R-L fractional integral and derivative operators (4) and (5), I^{β}_{a+} and D^{β}_{a+} of order $\beta \in \mathbb{C}(\Re(\beta) > 0)$ with the Aleph function (9). The relationship is prescribed by following theorems:

Theorem 1.*Let $a \in \mathbb{R}^+, z > a, \mu > 0, \delta, \beta, \omega \in \mathbb{C}, (\Re(\delta) > 0, \Re(\beta), \Re(\mu) > 0)$, such that $\Re(\delta) + \mu\min_{1\leq j\leq m}\frac{\Re(b_j)}{B_j} > 0$ and $|arg(\omega)| < \pi\varphi_l/2, \varphi_l > 0, l = \overline{1,r}$. Then we have*

$$\begin{aligned} &\left(I^{\beta}_{a+}\left[(y-a)^{\delta-1}\aleph^{m,n}_{p_l,q_l,\sigma_l;r}(\omega(y-a)^{\mu})\right]\right)(z)\\ &= (z-a)^{\beta+\delta-1}\aleph^{m,n+1}_{p_l+1,q_l+1,\sigma_l;r}\left[\omega(z-a)^{\mu}\left|\begin{matrix}(1-\delta,\mu),(\mathfrak{a}_j,\alpha_j)_{1,n},[\sigma_l(\mathfrak{a}_{jl},\alpha_{jl})]_{n+1,p_l;r}\\(\mathfrak{b}_q,\beta_q)_{1,m},[\sigma_l(\mathfrak{b}_{jl},\beta_{jl})]_{m+1,q_l;r},(1-\beta-\delta,\mu)\end{matrix}\right.\right]. \end{aligned} \tag{15}$$

Proof. By using definition of the Aleph function and Riemann-Liouvilli integral operator given by (9) and (4), and also using the relation [5, p.71,eq.(2.1.16)]

$$\left(I^{\beta}_{a+}\left[(y-a)^{\delta-1}\right]\right)(z) = \frac{\Gamma(\delta)}{\Gamma(\delta+\beta)}(z-a)^{\delta+\beta-1}, \tag{16}$$

which will gives for $z > a$,

$$\left(I_{a+}^{\beta}\left[(y-a)^{\delta-1}\aleph_{p_l,q_l,\sigma_l;r}^{m,n}(\omega(y-a)^{\mu})\right]\right)(z)$$
$$=\left(I_{a+}^{\beta}\left[(y-a)^{\delta-1}\frac{1}{2\pi i}\oint\Omega_{p_l,q_l,\sigma_l;r}^{m,n}(s)(\omega^{-s}(y-a)^{-\mu s}ds\right]\right)(z),$$

demonstrating the $\aleph$-function in term of contour integral (9) and interchanging the order of integration and solve the inner integral by using eq. (16), then we have

$$=\frac{1}{2\pi i}\oint\Omega(s)\left[\omega^{-s}\frac{\Gamma(\delta-\mu s)}{\Gamma(\beta+\delta-\mu s)}(z-a)^{\beta+\delta-\mu s-1}\right]ds,$$
$$=(z-a)^{\beta+\delta-1}\frac{1}{2\pi i}\oint\Omega(s)\left[\omega^{-s}\frac{\Gamma(1-(1-\delta)-\mu s)}{\Gamma((1-(1-\beta-\delta)-\mu s)}(z-a)^{-\mu s}\right]ds.$$

Finally, we arrive at

$$\left(I_{a+}^{\beta}\left[(y-a)^{\delta-1}\aleph_{p_l,q_l,\sigma_l;r}^{m,n}(\omega(y-a)^{\mu})\right]\right)(z)$$
$$=(z-a)^{\beta+\delta-1}\aleph_{p_l+1,q_l+1,\sigma_l;r}^{m,n+1}\left[\omega(z-a)^{\mu}\left|\begin{matrix}(1-\delta,\mu),(\mathfrak{a}_j,\alpha_j)_{1,n},[\sigma_l(\mathfrak{a}_{jl},\alpha_{jl})]_{n+1,p_l;r}\\(\mathfrak{b}_q,\beta_q)_{1,m},[\sigma_l(\mathfrak{b}_{jl},\beta_{jl})]_{m+1,q_l;r},(1-\beta-\delta,\mu)\end{matrix}\right.\right].$$

This is the final result.

Theorem 2. *Under the same conditions declared already in Theorem 2.1, then we obtained the result:*

$$\left(D_{a+}^{\beta}\left[(y-a)^{\delta-1}\aleph_{p_l,q_l,\sigma_l;r}^{m,n}(\omega(y-a)^{\mu})\right]\right)(z)$$
$$=(z-a)^{\delta-\beta-1}\aleph_{p_l+1,q_l+1,\sigma_l;r}^{m,n+1}\left[\omega(z-a)^{\mu}\left|\begin{matrix}(1-\delta,\mu),(\mathfrak{a}_j,\alpha_j)_{1,n},[\sigma_l(\mathfrak{a}_{jl},\alpha_{jl})]_{n+1,p_l;r}\\(\mathfrak{b}_q,\beta_q)_{1,m},[\sigma_l(\mathfrak{b}_{jl},\beta_{jl})]_{m+1,q_l;r},(1+\beta-\delta,\mu)\end{matrix}\right.\right]. \quad (17)$$

Proof. To prove the result (17), we use (5) and also using the relation [5, p.71,eq.(2.1.17)]:

$$\left(D_{a+}^{\beta}\left[(y-a)^{\delta-1}\right]\right)(z)=\frac{\Gamma(\delta)}{\Gamma(\delta-\beta)}(z-a)^{\delta+\beta-1},(\Re(\delta)>0,\Re(\beta)\geq 0,\delta\in\mathbb{C}). \quad (18)$$

We find for $z > a$

$$\left(D_{a+}^{\beta}\left[(y-a)^{\delta-1}\aleph_{p_l,q_l,\sigma_l;r}^{m,n}(\omega(y-a)^{\mu})\right]\right)(z)$$
$$=\left(D_{a+}^{\beta}\left[(y-a)^{\delta-1}\frac{1}{2\pi i}\oint\Omega_{p_l,q_l,\sigma_l;r}^{m,n}(s)(\omega^{-s}(y-a)^{-\mu s}ds\right]\right)(z).$$

Solving in similar manner as we done in Theorem 2.1 and in the light of relation (18), yields

$$=\frac{1}{2\pi i}\oint\Omega(s)\left[\omega^{-s}\frac{\Gamma(\delta-\mu s)}{\Gamma(\beta-\delta-\mu s)}(z-a)^{\delta-\beta-\mu s-1}\right]ds,$$
$$=(z-a)^{\delta-\beta-1}\aleph_{p_l+1,q_l+1,\sigma_l;r}^{m,n+1}\left[\omega(z-a)^{\mu}\left|\begin{matrix}(1-\delta,\mu),(\mathfrak{a}_j,\alpha_j)_{1,n},[\sigma_l(\mathfrak{a}_{jl},\alpha_{jl})]_{n+1,p_l;r}\\(\mathfrak{b}_j,\beta_j)_{1,m},[\sigma_l(\mathfrak{b}_{jl},\beta_{jl})]_{m+1,q_l;r},(1+\beta-\delta,\mu)\end{matrix}\right.\right].$$

This is desired result

Theorem 3. *Let $a\in\mathbb{R}^{+},z>a$ and also let $\rho>0,0<\lambda<1,0\leq\nu\leq 1,\lambda,\mu,\omega\in\mathbb{C},(\Re(\mu)>0,\Re(\lambda)>0,\Re(\rho)>0)$, such that $\Re(\mu)+\rho\min_{1\leq j\leq m}\frac{\Re(b_j)}{B_j}>0$ and $|arg(\omega)|<\pi\varphi_l/2,\varphi_l>0,l=\overline{1,r}$. Then we arrive at the following result:*

$$\left(D_{a+}^{\lambda,\nu}\left[(y-a)^{\mu-1}\aleph_{p_l,q_l,\sigma_l;r}^{m,n}(\omega(y-a)^{\rho})\right]\right)(z)=(z-a)^{\mu-\lambda-1}$$
$$\aleph_{p_l+1,q_l+1,\sigma_l;r}^{m,n+1}\left[\omega(z-a)^{\rho}\left|\begin{matrix}(1-\mu,\rho),(\mathfrak{a}_j,\alpha_j)_{1,n},[\sigma_l(\mathfrak{a}_{jl},\alpha_{jl})]_{n+1,p_l;r}\\(\mathfrak{b}_j,\beta_j)_{1,m},[\sigma_l(\mathfrak{b}_{jl},\beta_{jl})]_{m+1,q_l;r},(1-\mu+\lambda,\rho)\end{matrix}\right.\right]. \quad (19)$$

Proof. In view of the relation [9, eq.(2.18)], we have to prove (19) and follow the same process as we done in (15) and (17), then we have

$$\begin{aligned}
&\left(D_{a+}^{\lambda,\nu}\left[(y-a)^{\mu-1}\aleph_{p_l,q_l,\sigma_l;r}^{m,n}(\omega(y-a)^{\rho})\right]\right)(z)\\
&=\left(D_{a+}^{\lambda,\nu}\left[(y-a)^{\mu-1}\frac{1}{2\pi i}\oint\Omega_{p_l,q_l,\sigma_l;r}^{m,n}(s)\omega^{-s}(y-a)^{-\rho s}ds\right]\right)(z),\\
&=\left(\frac{1}{2\pi i}\oint\Omega(s)\omega^{-s}\left[D_{a+}^{\lambda,\nu}\{(y-a)^{\mu-\rho s-1}\}\right]ds\right)(z),\\
&=\frac{1}{2\pi i}\oint\Omega(s)\omega^{-s}\frac{\Gamma(\mu-\rho s)}{\Gamma(\mu-\lambda-\rho s)}(z-a)^{\mu-\lambda-\rho s-1}ds.
\end{aligned}$$

Finally, we arrive at

$$\begin{aligned}
&\left(D_{a+}^{\lambda,\nu}\left[(y-a)^{\mu-1}\aleph_{p_l,q_l,\sigma_l;r}^{m,n}(\omega(y-a)^{\rho})\right]\right)(z)=(z-a)^{\mu-\lambda-1}\\
&\aleph_{p_l+1,q_l+1,\sigma_l;r}^{m,n+1}\left[\omega(z-a)^{\rho}\left|\begin{matrix}(1-\mu,\rho),(\mathfrak{a}_j,\alpha_j)_{1,n},[\sigma_l(\mathfrak{a}_{jl},\alpha_{jl}]_{n+1,p_l;r}\\(\mathfrak{b}_j,\beta_j)_{1,m},[\sigma_l(\mathfrak{b}_{jl},\beta_{jl})]_{m+1,q_l;r},(1-\mu+\lambda,\rho)\end{matrix}\right.\right]
\end{aligned}$$

This is desired result.
Thus, we obtained the compositional properties of the R-L fractional integral operator I_{a+}^{β} and derivative operator D_{a+}^{β} containing the Aleph-function.

*Remark.*Since we have already discussed in introduction when $\sigma_l=1, l=\overline{1,r}$ then the Aleph function turn down to the I-function (11) and hence, we get the results in similar manner for I-function [34].

*Remark.*Also if we set r=1 in eq.(15), (17) & (19), then we obtained a corollary containing the well-known fox's H-function [35].

Corollary 1.*Under the same parametric conditions stated already in Theorem 2.1, then the following result hold:*

$$\begin{aligned}
&\left(I_{a+}^{\beta}\left[(y-a)^{\delta-1}H_{p,q}^{m,n}(\omega(y-a)^{\mu})\right]\right)(z)\\
&=(z-a)^{\beta+\delta-1}H_{p_l,q_l}^{m,n}\left[\omega(z-a)^{\mu}\left|\begin{matrix}(1-\delta,\mu),(\mathfrak{a}_j,\alpha_j)1,p\\(\mathfrak{b}_j,\beta_j)_{1,q},(1-\beta-\delta,\mu)\end{matrix}\right.\right].
\end{aligned}\tag{20}$$

*Remark.*Let us take $m=n=p=q-1=1$ and use the identity [36, section(18.1)], then we arrive at the following relation

$$E_{\mu,\rho}(z)=H_{1,2}^{1,1}\left[-z\left|\begin{matrix}(0,1)\\(0,1),(1-\rho,\mu)\end{matrix}\right.\right],$$

where $E_{\mu,\rho}(z)$ stand for the Mittag-Leffler function [36], expressed as

$$E_{\mu,\rho}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu k+\rho)},\min\left(\Re(\mu),\Re(\rho)>0\right).$$

Thus, by using above result in (20), then we come at a known result obtained by Kilbas et al. [6].

3 An Integral Operator Consisting $\aleph$-Function and its Sumudu Transform

This section begins with the fractional integral operator connected with $\aleph$-function in its kernel and it is stated as follows:

$$\left(\varepsilon_{0+,p_l,q_l,\sigma_l;r}^{\omega;\alpha,\beta,m,n}\varphi\right)(z)=\int_a^z(z-\tau)^{\beta-1}\aleph_{p_l,q_l,\sigma_l;r}^{m,n}[\omega(z-\tau)^{\alpha}]\varphi(\tau)d\tau,(z>0).\tag{21}$$

$\Re(\alpha)>0,\Re(\beta)>0,\omega\in\mathbb{C},\Re(\beta)+\alpha\min_{1\le j\le m}\frac{\Re(b_j)}{B_j}>0$ and $|arg(\omega)|<\pi\varphi_l/2,\varphi_l>0,l=\overline{1,r}$.
Sumudu transform of Eq. (21) is given by

$$S\left(\varepsilon_{0+,p_l,q_l,\sigma_l;r}^{\omega;\alpha,\beta,m,n}\right)(z)=uS\left(z^{\beta-1}\aleph_{p_l,q_l,\sigma_l;r}^{m,n}[\omega z^{\alpha}]\right)S(1).$$

demonstrating the $\aleph$-function in term of contour integral (9), then we have

$$S\left(\varepsilon^{\omega;\alpha,\beta,m,n}_{0+,p_l,q_l,\sigma_l;r}\right)(z) = uS\left(z^{\beta-1}\frac{1}{2\pi i}\oint \Omega^{m,n}_{p_l,q_l,\sigma_l;r}(s)[(\omega z^{\alpha})^{-s}]ds\right).1$$
$$= \frac{u^{\beta}}{2\pi i}\oint \Omega^{m,n}_{p_l,q_l,\sigma_l;r}(s)\Gamma[1-(1-\beta)-\alpha s]w^{-s}u^{-\alpha s}ds.$$

Finally, we get the Sumudu transform of integral operator defined in (21), as

$$S\left(\varepsilon^{\omega;\alpha,\beta,m,n}_{0+,p_l,q_l,\sigma_l;r}\right)(z) = u^{\beta}\,\aleph^{m,n+1}_{p_l+1,q_l,\sigma_l;r}\left[\omega u^{\alpha}\left|\begin{matrix}(1-\beta,\alpha),(\mathfrak{a}_j,\alpha_j)_{1,n},[\sigma_l(\mathfrak{a}_{jl},\alpha_{jl})]_{n+1,p_l;r}\\(\mathfrak{b}_j,\beta_j)_{1,m},[\sigma_l(\mathfrak{b}_{jl},\beta_{jl})]_{m+1,q_l;r}\end{matrix}\right.\right]. \tag{22}$$

Theorem 4.*Let $a\in\mathbb{R}^+,\Re(\alpha)>0,\Re(\beta)>0,\omega\in\mathbb{C}$, such that $\Re(\alpha)+\mu\min_{1\le j\le m}\frac{\Re(b_j)}{B_j}>0$ and $|arg(\omega)|<\pi\varphi_l/2,\varphi_l>0,l=\overline{1,r}$. and the function ϑ defined on a finite interval $[a,b]\,(b>a)$ on R in the space of Lebesgue measurable function given in [8]*

$$L(a,b)=\{g:||g||_1=\int_a^b|g(z)|dz<\infty\}, \tag{23}$$

then the integral operator $\varepsilon^{\omega;\alpha,\beta,m,n}_{a+,p_l,q_l,\sigma_l;r}$ is bounded on L(a,b) and

$$\|\left(\varepsilon^{\omega;\alpha,\beta,m,n}_{a+,p_l,q_l,\sigma_l;r}\vartheta\right)(z)\|_1\le K\|\vartheta\|_1. \tag{24}$$

Where the constant $K\ (0<K<\infty)$ is given by

$$K=\frac{1}{2\pi i}\oint \Omega^{m,n}_{p_l,q_l,\sigma_l;r}(s)|\omega(b-a)^{\Re(\alpha)}|^{-s}\frac{(b-a)^{\Re(\beta)}}{\Re(\beta)-\Re(\alpha)s}ds. \tag{25}$$

Proof. In order to prove (24), using (23) and interchanged the order of integration by using Dirichlet formula [11, p.56], then

$$\begin{aligned}\|\left(\varepsilon^{\omega;\alpha,\beta,m,n)}_{a+,p_l,q_l,\sigma_l;r}\vartheta\right)(z)\|_1 &= \int_a^b|\int_a^z(z-t)^{\beta-1}\aleph^{m,n}_{p_l,q_l,\sigma_l;r}[\omega(z-t)^{\alpha}]\vartheta(t)dt|dz\\ &\le \int_a^b|\vartheta(t)|\left(\int_t^b(z-t)^{\Re(\beta)-1}|\aleph^{m,n}_{p_l,q_l,\sigma_l;r}[\omega(z-t)^{\alpha}]|dz\right)dt.\end{aligned} \tag{26}$$

Taking $(z-t)=\tau$ in inner integral, this yields

$$\begin{aligned}\|\left(\varepsilon^{\omega;\alpha,\beta,m,n)}_{a+,p_l,q_l,\sigma_l;r}\vartheta\right)(z)\|_1 &= \int_a^b|\int_a^z(z-t)^{\beta-1}\aleph^{m,n}_{p_l,q_l,\sigma_l;r}[\omega(z-t)^{\alpha}]\vartheta(t)dt|dz\\ &= \int_a^b|\vartheta(t)|\left(\int_t^b\tau^{\Re(\beta)-1}|\aleph^{m,n}_{p_l,q_l,\sigma_l;r}[\omega\tau^{\alpha}]|d\tau\right)dt.\end{aligned}$$

After little simplification, we get

$$\begin{aligned}&\|\left(\varepsilon^{\omega;\alpha,\beta,m,n)}_{a+,p_l,q_l,\sigma_l;r}\vartheta\right)(z)\|_1 = \int_a^b|\int_a^z(z-t)^{\beta-1}\aleph^{m,n}_{p_l,q_l,\sigma_l;r}[\omega(z-t)^{\alpha}]\vartheta(t)dt|dz\\ &\le\left(\frac{1}{2\pi i}\oint \Omega^{m,n}_{p_l,q_l,\sigma_l;r}(s)|\omega|^{-s}\left[\frac{\tau^{\Re(\beta)-\Re(\alpha)s}}{\Re(\beta)-\Re(\alpha)s}\right]_0^{b-a}ds\right)\|\vartheta\|_1=K\|\vartheta\|_1,\end{aligned} \tag{27}$$

where the constant K is finite and is given by (25). Finally, we get the desired result.

4 Differential Equations of Arbitrary Order Connected with Hilfer Derivative Operator

Theorem 5. *Let* $\Re(\alpha) > 0, \Re(\beta) > 0, 0 \le \nu \le 1, 0 < \lambda < 1, \omega \in \mathbb{C}$, *such that* $\Re(\beta) + \alpha \min_{1\le j\le m} \frac{\Re(b_j)}{B_j} > 0$ *and* $|arg(\omega)| < \pi\varphi_l/2, \varphi_l > 0, l = \overline{1,r}$, *then*

$$\left(D_{0+}^{\lambda,\nu} y\right) = \gamma\left(\varepsilon_{0+,p_l,q_l,\sigma_l;r}^{\omega;\alpha,\beta,m,n}\right)(x) + f(x), \tag{28}$$

under the condition: $\left(I_{0+}^{(1-\lambda),(1-\nu)} y\right)(0+) = k$, *is solvable in* $L(0,\infty)$, *then its solution is given by*

$$y(x) = k\frac{x^{\lambda+\nu(1-\lambda)-1}}{\Gamma[\lambda+\nu(1-\lambda)]} + \gamma x^{\beta+\lambda} \\ .\aleph_{p_l+1,q_l+1,\sigma_l;r}^{m,n+1}\left[\omega x^{\alpha}\left|\begin{matrix}(1-\beta,\alpha),(\mathfrak{a}_j,\alpha_j)_{1,n},[\sigma_l(\mathfrak{a}_{jl},\alpha_{jl})]_{n+1,p_l;r}\\(\mathfrak{b}_j,\beta_j)_{1,m},[\sigma_l(\mathfrak{b}_{jl},\beta_{jl})]_{m+1,q_l;r},(-\beta-\lambda,\alpha)\end{matrix}\right.\right] + \frac{1}{\Gamma(\lambda)}\int_0^x (x-t)^{\lambda-1} f(t)dt, \tag{29}$$

where k is an arbitrary constant.

Proof. Taking Sumudu transform on both sides of (28), and in view of the formula (8), (22) and by applying convolution property, we found that

$$u^{-\lambda}\bar{y}(u) - u^{-\nu(\lambda-1)-1}\left(I^{(1-\nu)(1-\lambda)} y\right)(0+) = \gamma u^{\beta} \\ .\aleph_{p_l+1,q_l+1,\sigma_l;r}^{m,n+1}\left[\omega u^{\alpha}\left|\begin{matrix}(1-\beta,\alpha),(\mathfrak{a}_j,\alpha_j)_{1,n},[\sigma_l(\mathfrak{a}_{jl},\alpha_{jl})]_{n+1,p_l;r}\\(\mathfrak{b}_j,\beta_j)_{1,m},[\sigma_l(\mathfrak{b}_{jl},\beta_{jl})]_{m+1,q_l;r}]\end{matrix}\right.\right] + \bar{f}(u),$$

which will gives

$$\bar{y}(u) = u^{\nu(\lambda-1)+\lambda-1}k + \gamma u^{\beta+\lambda} \\ .\aleph_{p_l+1,q_l+1,\sigma_l;r}^{m,n+1}\left[\omega u^{\alpha}\left|\begin{matrix}(1-\beta,\alpha),(\mathfrak{a}_j,\alpha_j)_{1,n},[\sigma_l(\mathfrak{a}_{jl},\alpha_{jl})_{n+1,p_l;r}]\\(\mathfrak{b}_j,\beta_j)_{1,m},[\sigma_l(\mathfrak{b}_{jl},\beta_{jl})]_{m+1,q_l;r}\end{matrix}\right.\right] + u^{\lambda}\bar{f}(u), \tag{30}$$

now, inverting Sumudu transform of (30), we find that

$$\begin{aligned} &y(x) \\ &= kS^{-1}\{u^{\nu+\lambda-\nu\lambda-1}\} + \gamma\frac{1}{2\pi i}\oint \Omega_{p_l,q_l,\sigma_l;r}^{m,n}(s)\Gamma(\beta-as)(\omega)^{-s}S^{-1}\{u^{\beta+\lambda-as}\}ds + S^{-1}\{u^{\lambda}\bar{f}(u)\}, \\ &= k\frac{x^{\lambda+\nu-\lambda\nu-1}}{\Gamma[\lambda+\nu(1-\lambda)]} + \gamma\oint \Omega_{p_l,q_l,\sigma_l;r}^{m,n}(s)\Gamma(1-(1-\beta)-as)(\omega)^{-s}\frac{x^{\beta+\lambda+\nu-as}}{\Gamma[\beta+\lambda-as+1)]}ds + \\ &S[S^{-1}\{\frac{x^{\lambda-1}}{\Gamma(\lambda)} * f(x)\}]. \end{aligned}$$

Finally, we get the solution in this form

$$y(x) = k\frac{x^{\lambda+\nu(1-\lambda)-1}}{\Gamma[\lambda+\nu(1-\lambda)]} + \gamma x^{\beta+\lambda} \\ .\aleph_{p_l+1,q_l+1,\sigma_l;r}^{m,n+1}\left[\omega x^{\alpha}\left|\begin{matrix}(1-\beta,\alpha),(\mathfrak{a}_j,\alpha_j)_{1,n},[\sigma_l(\mathfrak{a}_{jl},\alpha_{jl})]_{n+1,p_l;r}\\(\mathfrak{b}_j,\beta_j)_{1,m},[\sigma_l(\mathfrak{b}_{jl},\beta_{jl})]_{m+1,q_l;r},(-\beta-\lambda,\alpha)\end{matrix}\right.\right] + \frac{1}{\Gamma(\lambda)}\int_0^x (x-t)^{\lambda-1} f(t)dt. \tag{31}$$

This is desired result.

Theorem 6. *Under the same parametric conditions stated already in (28), then the following differential equation holds*

$$x\left(D_{0+}^{\lambda,\nu} y\right)(x) = \gamma\left(\varepsilon_{0+,p_l,q_l,\sigma_l;r}^{\omega;\alpha,\beta,m,n}\right)(x), \tag{32}$$

with the initial condition $\left(I_{0+}^{(1-\lambda),(1-\nu)}y\right)(0+) = k_1$, *Then its solution in the space* $L(0,\infty)$ *is given by*

$$y(x) = k_1 \frac{x^{\lambda-\nu(1-\lambda)-1}}{\Gamma[\lambda-\nu(1-\lambda)]} + k_2 \frac{x^{\lambda-1}}{\Gamma(\lambda)} + \frac{\gamma}{\Gamma(\lambda)} \int_0^x t^{\lambda-1}(x-t)^{\beta-1} \\ .\aleph_{p_l+1,q_l+1,\sigma_l;r}^{m,n+1}\left[\omega(x-t)^\alpha \middle| \begin{matrix} (1-\beta,\alpha),(\mathfrak{a}_j,\alpha_j)_{1,n},[\sigma_l(\mathfrak{a}_{jl},\alpha_{jl})]_{n+1,p_l;r} \\ (\mathfrak{b}_j,\beta_j)_{1,m},[\sigma_l(\mathfrak{b}_{jl},\beta_{jl})]_{m+1,q_l;r},(-\beta,\alpha) \end{matrix}\right] dt. \tag{33}$$

Where k_1 *and* k_2 *are an arbitrary constant.*

Proof. We can get the proof of Theorem 4.2 in the same way as we did in Theorem 4.1. In addition to the Sumudu transform and inverse Sumudu, we also use the following Sumudu derivative formula [24]:

$$S[t^n h(t)] = u^n \sum_{j=0}^{n} a_j^n u^j H_j(u), \tag{34}$$

where $a_0^n = !n, a_n^n = 1, a_1^n = n!n, a_{n-1}^n = n^2$ and for $j = 2,3,\dots,n-2$.
In particular when n=1, then eqs. (34) become

$$S[th(t)] = uH(u) + u^2 H_1(u) = uH(u) + u^2 \frac{dH(u)}{du} = u\frac{d}{du}(uS[h(t)]). \tag{35}$$

Now, by applying the formula given in (35) and using Sumudu transform method, the fractional differential equation (32) gives

$$S\left[x\left(D_{0+}^{\lambda,\nu}y\right)(x)\right] = \gamma S\left(\varepsilon_{0+,p_l,q_l,\sigma_l;r}^{\omega;\alpha,\beta,m,n}\right)(x).$$

The above Eq. can be presented as

$$u\frac{d}{dx}\left(u\left[u^{-\lambda}\bar{y}(u) - u^{-\nu(\lambda-1)-1}\left(I_{0+}^{(1-\nu)(1-\lambda)}y\right)(0+)\right]\right) \\ = \gamma u^\beta \aleph_{p_l+1,q_l,\sigma_l;r}^{m,n+1}\left[\omega u^\alpha \middle| \begin{matrix} (1-\beta,\alpha),(\mathfrak{a}_j,\alpha_j)_{1,n},[\sigma_l(\mathfrak{a}_{jl},\alpha_{jl})]_{n+1,p_l;r} \\ (\mathfrak{b}_j,\beta_j)_{1,m},[\sigma_l(\mathfrak{b}_{jl},\beta_{jl})]_{m+1,q_l;r} \end{matrix}\right].$$

It gives

$$(1-\lambda)u^{-\lambda}\bar{y}(u) + u^{1-\lambda}\bar{y}'(u) - k_1(-\nu[\lambda-1])u^{-\nu(\lambda-1)-1} \\ = \gamma u^{\beta-1} \aleph_{p_l+1,q_l,\sigma_l;r}^{m,n+1}\left[\omega u^\alpha \middle| \begin{matrix} (1-\beta,\alpha),(\mathfrak{a}_j,\alpha_j)_{1,n},[\sigma_l(\mathfrak{a}_{jl},\alpha_{jl})]_{n+1,p_l;r} \\ (\mathfrak{b}_j,\beta_j)_{1,m},[\sigma_l(\mathfrak{b}_{jl},\beta_{jl})]_{m+1,q_l;r} \end{matrix}\right].$$

This lead us to the following O.D.E. of first order and first degree for $\bar{y}(u)$:

$$\frac{d\bar{y}(u)}{du} + \frac{(1-\lambda)}{u}\bar{y}(u) - k_1(-\nu[\lambda-1])u^{-\nu(\lambda-1)+\lambda-2} - \gamma u^{\beta+\gamma-2} \\ .\aleph_{p_l+1,q_l,\sigma_l;r}^{m,n+1}\left[\omega u^\alpha \middle| \begin{matrix} (1-\beta,\alpha),(\mathfrak{a}_j,\alpha_j)_{1,n},[\sigma_l(\mathfrak{a}_{jl},\alpha_{jl})]_{n+1,p_l;r} \\ (\mathfrak{b}_j,\beta_j)_{1,m},[\sigma_l(\mathfrak{b}_{jl},\beta_{jl})]_{m+1,q_l;r} \end{matrix}\right] = 0 \tag{36}$$

After solving linear differential equation of first order (36), we find

$$\bar{y}(u) = k_1 u^{\lambda-\nu(\lambda-1)-1} + k_2 u^{\lambda-1} + \gamma \frac{1}{2\pi i} \oint \Omega_{p_l,q_l,\sigma_l;r}^{m,n}(s)(\omega)^{-s}\Gamma(\beta-\alpha s)\frac{u^{\beta-\alpha s}}{(\beta-\alpha s)} u^{\lambda-1} ds. \tag{37}$$

By taking inverse Sumudu transform of (37), we get

$$y(x) = k_1 \frac{u^{\lambda-\nu(\lambda-1)-1}}{\Gamma[\lambda-\nu(\lambda-1)]} + k_2 \frac{x^{\lambda-1}}{\Gamma(\lambda)} + \\ \gamma S^{-1}\left[uS\left(\frac{x^{\lambda-1}}{\Gamma(\lambda)}\right) \oint \Omega_{p_l,q_l,\sigma_l;r}^{m,n}(s)(\omega)^{-s}\frac{\Gamma(\beta-\alpha s)}{(\beta-\alpha s)} S\left(\frac{x^{\beta-as-1}}{\Gamma(\beta-as)}\right) ds\right].$$

On simplification, we get

$$y(x) = k_1 \frac{u^{\lambda-\nu(\lambda-1)-1}}{\Gamma[\lambda-\nu(\lambda-1)]} + k_2 \frac{x^{\lambda-1}}{\Gamma(\lambda)} + \frac{\gamma}{\Gamma(\lambda)} S^{-1}\left[uS\left(x^{\lambda-1}\right).S\{x^{\beta-1}\frac{1}{2\pi i}\oint \Omega^{m,n}_{p_l,q_l,\sigma_l;r}(s)\frac{(\omega x^{\alpha})^{-s}}{(\beta-\alpha s)}ds\}\right]. \tag{38}$$

By, means of sumudu convolution, yields the solution given by (33), then we come at our required result.

5 Particular Cases and Applications

In this portion, we discuss here some significant particular cases of Theorem 4.1, we can also comfortably obtained the results in identical manner of Theorem 4.2, but due to lack of space we don't considered them. Further, If we assign particular value to the parameters in Aleph-function then it reduce to Mittag-Leffler function [36] in the integral operator of (28), we come at the result possessed by Srivastava [9].

(i) In this case, we change the Aleph-function to the Saxena I-function [34] by assigning the value $\sigma_l = 1, l = \overline{1,r}$ in Theorem 4.1. Then, we found that

$$\left(D^{\lambda,\nu}_{0+}y\right)(x) = \gamma\left(\varepsilon^{\omega;\alpha,\beta,m,n}_{0+,p_l,q_l;r}\right)(x) + f(x), \tag{39}$$

(Under the same parametric conditions which stated already in Theorem 4.1)
then the following solution holds

$$y(x) = k\frac{x^{\lambda+\nu(1-\lambda)-1}}{\Gamma[\lambda+\nu(1-\lambda)]} + \gamma x^{\beta+\lambda} .I^{m,n+1}_{p_l+1,q_l;r}\left[\omega x^{\alpha}\left|\begin{matrix}(1-\beta,\alpha),(\mathfrak{a}_j,\alpha_j)_{1,n},(\mathfrak{a}_{jl},\alpha_{jl})_{n+1,p_l;r}\\(\mathfrak{b}_j,\beta_j)_{1,m},(\mathfrak{b}_{jl},\beta_{jl})_{m+1,q_l;r},(-\beta-\lambda,\alpha)\end{matrix}\right.\right] + \frac{1}{\Gamma(\lambda)}\int_0^x (x-t)^{\lambda-1}f(t)dt. \tag{40}$$

(ii) Now, in the above case if we set r=1 and also (12) taken into consideration, then the $\aleph$-function turn down to the Fox's H-function [35]. Thus, we reached the known result acquired by Harjule and Jain [18].

$$\left(D^{\lambda,\nu}_{0+}y\right)(x) = \gamma\left(\varepsilon^{\omega;\alpha,\beta,m,n}_{0+,p,q}\right)(x) + f(x), \tag{41}$$

then, its solution given by

$$y(x) = k\frac{x^{\lambda+\nu(1-\lambda)-1}}{\Gamma[\lambda+\nu(1-\lambda)]} + \gamma x^{\beta+\lambda} .H^{m,n+1}_{p+1,q+1}\left[\omega x^{\alpha}\left|\begin{matrix}(1-\beta,\alpha),(\mathfrak{a}_j,\alpha_j)_{1,p}\\(\mathfrak{b}_j,\beta_j)_{1,q},(-\beta-\lambda,\alpha)\end{matrix}\right.\right] + \frac{1}{\Gamma(\lambda)}\int_0^x (x-t)^{\lambda-1}f(t)dt. \tag{42}$$

(iii) The well-known Fox's H-function turn down to Wright generalized hypergeometric function ${}_p\psi_q$ by using the relation [37, p.25, eq.(1.140)] in the integral operator of R.H.S. of (41), then we obtained the result in the following manner

$$\left(D^{\lambda,\nu}_{0+}y\right)(x) = \gamma\left(\varepsilon^{\omega;\alpha,\beta,1,p}_{0+,p,q+1}\right)(x) + f(x), \tag{43}$$

then, we have

$$y(x) = k\frac{x^{\lambda+\nu(1-\lambda)-1}}{\Gamma[\lambda+\nu(1-\lambda)]} + \gamma x^{\beta+\lambda} .H^{1,p+1}_{p+1,q+2}\left[\omega x^{\alpha}\left|\begin{matrix}(1-\beta,\alpha),(1-\mathfrak{a}_p,\alpha_p)\\(0,1),(-\beta-\lambda,\alpha),(1-\mathfrak{b}_q,\beta_q)\end{matrix}\right.\right] + \frac{1}{\Gamma(\lambda)}\int_0^x (x-t)^{\lambda-1}f(t)dt. \tag{44}$$

(iv) If we assign particular value to the parameters in the Fox's H-function then it will change to Mittag-Leffler function [37, p.25, eq.(1.137)] in (41), given as

$$E^{\rho}_{\delta,\xi}(z) = \frac{1}{\Gamma(\rho)} H^{1,1}_{1,2}\left[-z\left|\begin{matrix}(1-\rho,1)\\(0,1),(1-\xi,\delta)\end{matrix}\right.\right],$$

then we obtained the result in the following manner

$$\left(D_{0+}^{\lambda,\nu}y\right)(x)=\gamma\left(\varepsilon_{0+,1,2}^{\omega;\alpha,\beta,1,1}\right)(x)+f(x), \tag{45}$$

then, we have

$$y(x)=c\frac{x^{\lambda+\nu(1-\lambda)-1}}{\Gamma[\lambda+\nu(1-\lambda)]}+\gamma x^{\beta+\lambda}.H_{2,3}^{1,2}\left[\omega x^{\alpha}\left|\begin{matrix}(1-\rho,1),(1-\beta,\alpha)\\(0,1),(1-\xi,\delta),(-\beta-\lambda,\alpha)\end{matrix}\right.\right]+\frac{1}{\Gamma(\lambda)}\int_0^x(x-t)^{\lambda-1}f(t)dt. \tag{46}$$

(v) The Fox's H-function reduces to generalized Bessel-Maitland function in the integral operator of (41), by using the following identity stated by Saxena et al. [37, p.25, eq.(1.139)],

$$J_{\delta,\eta}^{\rho}(z)=H_{1,3}^{1,1}\left[\frac{z^2}{4}\left|\begin{matrix}(\eta+\frac{\delta}{2},1)\\(\eta+\frac{\delta}{2},1),(\frac{\delta}{2},1),\left(\rho\left(\eta+\frac{\delta}{2}\right)-\eta-\delta,\rho\right)\end{matrix}\right.\right],$$

where $J_{\delta,\eta}^{\rho}(z)$ is the generalized Bessel-Maitland function, then we obtained the result in the following manner

$$\left(D_{0+}^{\lambda,\nu}y\right)(x)=\gamma\left(\varepsilon_{0+,1,3}^{\omega;\alpha,\beta,1,1}\right)(x)+f(x), \tag{47}$$

then, we have

$$y(x)=c\frac{x^{\lambda+\nu(1-\lambda)-1}}{\Gamma[\lambda+\nu(1-\lambda)]}+\gamma x^{\beta+\lambda} .H_{2,4}^{1,2}\left[\frac{\omega x^{2\alpha}}{4}\left|\begin{matrix}(\eta+\frac{\delta}{2},1),(1-\beta,\alpha)\\(\eta+\frac{\delta}{2},1),(\frac{\delta}{2},1),\left(\rho\left(\eta+\frac{\delta}{2}\right)-\eta-\delta,\rho\right)(-\beta-\lambda,\alpha)\end{matrix}\right.\right]+\frac{1}{\Gamma(\lambda)}\int_0^x(x-t)^{\lambda-1}f(t)dt. \tag{48}$$

Now, we compared the numerical results obtained from eq. (28), for different value of parameters to enhance the utility of fractional calculus.
(vi) If we assume $\nu=0$ and for different non-integer value of λ along with also consider $f(x)=1,\alpha=0,\beta=1/2$, then we arrive at the following differential equations:
When $\lambda=0.2$, then

$$\left(D_{0+}^{0.2,0}y\right)(x)=\gamma\left(\varepsilon_{0+,0,1,1,1}^{0;0,1/2,1,0}\right)(x)+1, \tag{49}$$

under the intial condition $\left(I_{0+}^{0.8}y\right)(0+)=k$,
Then, we have

$$y(x)=k\frac{x^{-0.8}}{\Gamma(0.2)}+\gamma x^{0.7}\frac{\Gamma(0.5)}{\Gamma(1.7)}+\frac{x^{0.2}}{\Gamma(1.2)}$$

When $\lambda=0.4$, then

$$\left(D_{0+}^{0.4,0}y\right)(x)=\gamma\left(\varepsilon_{0+,0,1,1,1}^{0;0,1/2,1,0}\right)(x)+1, \tag{50}$$

under the intial condition $\left(I_{0+}^{0.6}y\right)(0+)=k$,
Then, we get the solution of this form

$$y(x)=k\frac{x^{-0.6}}{\Gamma(0.4)}+\gamma x^{0.9}\frac{\Gamma(0.5)}{\Gamma(1.9)}+\frac{x^{0.4}}{\Gamma(1.4)}.$$

When $\lambda=0.6$, then

$$\left(D_{0+}^{0.6,0}y\right)(x)=\gamma\left(\varepsilon_{0+,0,1,1,1}^{0;0,1/2,1,0}\right)(x)+1, \tag{51}$$

under the initial condition $\left(I_{0+}^{0.4}y\right)(0+)=k$,
Then, we get the solution of this form

$$y(x)=k\frac{x^{-0.4}}{\Gamma(0.6)}+\gamma x^{1.1}\frac{\Gamma(0.5)}{\Gamma(2.1)}+\frac{x^{0.6}}{\Gamma(1.6)}.$$

When $\lambda=0.8$, then

$$\left(D_{0+}^{0.8,0}y\right)(x)=\gamma\left(\varepsilon_{0+,0,1,1,1}^{0;0,1/2,1,0}\right)(x)+1, \tag{52}$$

under the intial condition $\left(I_{0+}^{0.2}y\right)(0+)=k$,
Then, we get the solution of this form

$$y(x)=k\frac{x^{-0.2}}{\Gamma(0.8)}+\gamma x^{1.3}\frac{\Gamma(0.5)}{\Gamma(2.3)}+\frac{x^{0.8}}{\Gamma(1.8)}.$$

When $\lambda=1$, then

$$\left(D_{0+}^{1,0}y\right)(x)=\gamma\left(\varepsilon_{0+,0,1,1,1}^{0;0,1/2,1,0}\right)(x)+1, \tag{53}$$

under the intial condition $\left(I_{0+}^{0}y\right)(0+)=k$,
Then, we get the solution of this form

$$y(x)=k+\gamma x^{1.5}\frac{\Gamma(0.5)}{\Gamma(2.5)}+x.$$

(vii) If we set $\lambda=0.5$ and ν varies from 0 to 1 and also consider f(x)=x , in eq. (28), then we arrive at the following differential equations:
When $\nu=0.2$, then

$$\left(D_{0+}^{0.5,0.2}y\right)(x)=\gamma\left(\varepsilon_{0+,0,1,1,1}^{0;0,1/2,1,0}\right)(x)+x, \tag{54}$$

under the initial condition $\left(I_{0+}^{0.4}y\right)(0+)=k$,
Then, we get the solution of this form

$$y(x)=k\frac{x^{-0.4}}{\Gamma(0.6)}+\gamma x\frac{\Gamma(0.5)}{\Gamma(2)}+\frac{\Gamma(2)}{\Gamma(2.5)}x^{2.5}.$$

To find the last term of eq. (54), we use the following result:
$\left(I_{0+}^{\mu}f\right)(x)=\frac{1}{\Gamma(\mu)}\int_0^x(x-t)^{\mu-1}f(t)dt,(\Re(\mu)>0)$ and relation given in (16).
When $\nu=0.4$, then

$$\left(D_{0+}^{0.5,0.4}y\right)(x)=\gamma\left(\varepsilon_{0+,0,1,1,1}^{0;0,1/2,1,0}\right)(x)+x, \tag{55}$$

under the initial condition $\left(I_{0+}^{0.3}y\right)(0+)=k$,
Then, we get the solution of this form

$$y(x)=k\frac{x^{-0.3}}{\Gamma(0.7)}+\gamma x\frac{\Gamma(0.5)}{\Gamma(2)}+\frac{\Gamma(2)}{\Gamma(2.5)}x^{1.5}.$$

When $\nu=0.6$, then

$$\left(D_{0+}^{0.5,0.6}y\right)(x)=\gamma\left(\varepsilon_{0+,0,1,1,1}^{0;0,1/2,1,0}\right)(x)+x, \tag{56}$$

under the initial condition $\left(I_{0+}^{0.2}y\right)(0+)=k$,
Then, we get the solution of this form

$$y(x)=k\frac{x^{-0.2}}{\Gamma(0.8)}+\gamma x\frac{\Gamma(0.5)}{\Gamma(2)}+\frac{\Gamma(2)}{\Gamma(2.5)}x^{1.5}.$$

When $\nu=0.8$, then

$$\left(D_{0+}^{0.5,0.8}y\right)(x)=\gamma\left(\varepsilon_{0+,0,1,1,1}^{0;0,1/2,1,0}\right)(x)+x, \tag{57}$$

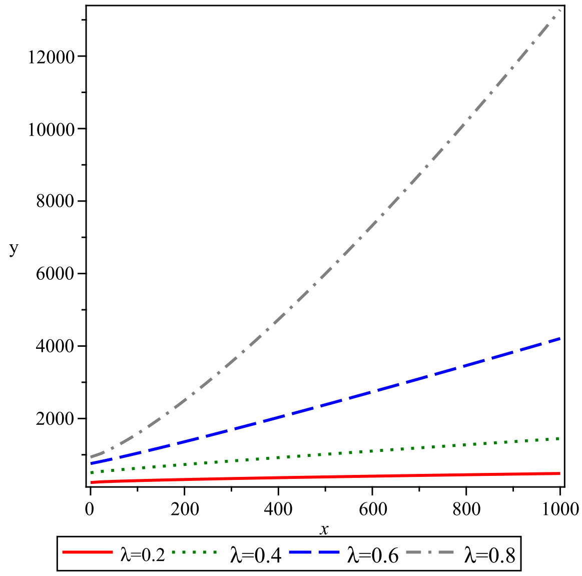

Fig. 1: The behavior of solutions of fractional differential Eqs. (49), (50), (51) and (52), when $y(0) = 10^3$ and λ varies from 0.2 to 0.8.

under the intial condition $\left(I_{0+}^{0.1}y\right)(0+) = k$,
Then, we get the solution of this form

$$y(x) = k\frac{x^{-0.1}}{\Gamma(0.9)} + \gamma x\frac{\Gamma(0.5)}{\Gamma(2)} + \frac{\Gamma(2)}{\Gamma(2.5)}x^{1.5}.$$

When $\nu = 1$, then

$$\left(D_{0+}^{0.5,1}y\right)(x) = \gamma\left(\varepsilon_{0+,0,1,1,1}^{0;0,1/2,1,0}\right)(x) + x, \tag{58}$$

under the initial condition $\left(I_{0+}^{0}y\right)(0+) = k$,
Then, we get the solution of this form

$$y(x) = k + \gamma x\frac{\Gamma(0.5)}{\Gamma(2)} + \frac{\Gamma(2)}{\Gamma(2.5)}x^{1.5}.$$

Where k and γ are considered as arbitrary constants.

6 Numerical Results and Discussions

In this part we established some numerical results for distinct order of fractional derivative. Figure 1 and 2 show the behavior of solutions of Eqs. (49), (50), (51),(52) when $\lambda = 0.2, 0.4, 0.6, 0.8, \nu = 0$ with $\gamma = 1$, for different initial conditions. It should be noted that for Riemann Liouville and Hilfer approaches, the integral type initial conditions are taken. Figure 3 and 4 show the behavior of solutions of Eqs. (54), (55), (56), (57), when $\nu = 0.2, 0.4, 0.6, 0.8, \lambda = 0.5$ with $\gamma = 1$ and $y(0) = 10^2$ and $y(0) = 1$ respectively.

7 Conclusion

We summarize this investigation by stating that the several compositional properties of the R-L integrals and derivatives operators of arbitrary order containing Aleph-function are derived. We established the boundedness of integral operator $\varepsilon_{a+,p_i,q_i,c_i;r}^{\omega;\alpha,\beta,m,n}$ and derived the outcomes of differential equations of arbitrary order comprising Hilfer derivative operator and integral operator $\varepsilon_{a+,p_i,q_i,c_i;r}^{\omega;\alpha,\beta,m,n}$ through Sumudu transform technique. In the last section of the paper we established

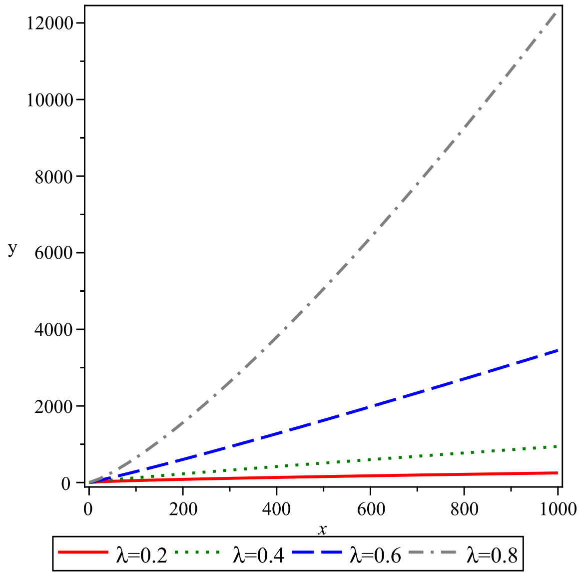

Fig. 2: The behavior of solutions of fractional differential Eqs. (49), (50), (51) and (52), when $y(0) = 1$ and λ varies from 0.2 to 0.8.

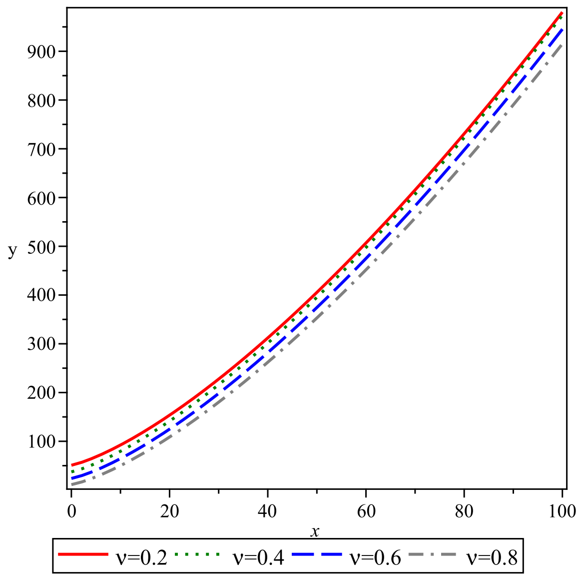

Fig. 3: The behavior of solution for fractional differential Eqs. (54), (55), (56) and (57), when $\nu = 0.2, 0.4, 0.6, 0.8$, $\lambda = 0.5$ with $\gamma = 1$ and $y(0) = 10^2$.

the particular cases and applications in term of well known Special function as well as by assigning particular values to the parameters are trying to show the comparative behavior of differential equations through graphically. Finally, the current investigation highlights the effectiveness of the proposed integral operator. We conclude that the results derived in this article are general in character and give some contributions in several areas of mathematics and engineering. Such a potential connection needs further investigation.

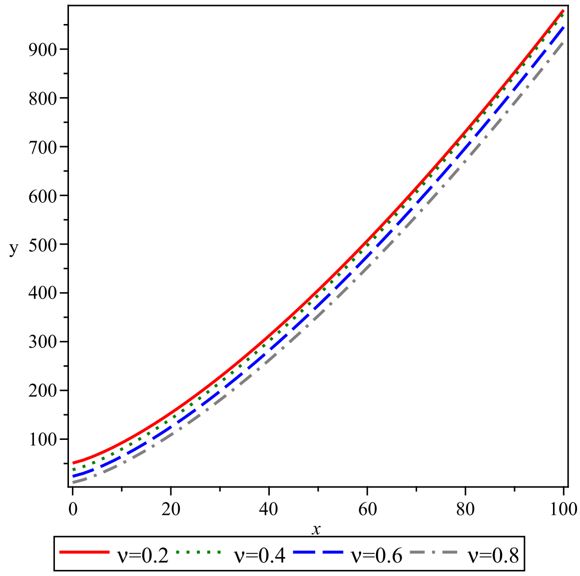

Fig. 4: The behavior of solution for fractional differential Eqs. (54), (55), (56) and (57), when $\nu = 0.2, 0.4, 0.6, 0.8$, $\lambda = 0.5$ with $\gamma = 1$ and $y(0) = 1$.

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