Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/100322

A New Analysis of a Fluid Queue Driven By *M/M/*1 Queue

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Received: 16 May 2020, Revised: 20 Sep. 2020, Accepted: 26 Oct. 2020.
Published online: 1 Nov. 2021.

Abstract: In this paper, we consider a fluid queue with an infinite buffer capacity which is both filled and depleted by a fluid at constant rates. These rates are uniquely determined by the number of customers in an *M/M/*1 queue with constant arrival and service rates. A simple series form is obtained for the joint stationary distribution of the buffer occupancy. This method is explicit where the coefficients of the series are obtained in closed form.

Keywords: Fluid queue; *M/M/*1 queue; Stationary state; Buffer content distribution; Series approach.

1 Introduction

In the last years, many researchers in queueing theory pay attention to study stochastic fluid flow models and their applications. The necessity of studying fluid queues appear when the system has continuous stream of customers. In fluid queues, some quantity accumulates or is depleted and it is proved to be efficient for studying performance analysis of telecommunication and manufacturing models, for example see among others [1-4].

Many authors analyze fluid models which have infinite state space of the Markov process that modulates the input rate of fluid in the buffer such as [5-8]. The technique of series approach is used by [9,10] to manipulate classical queueing systems [11]. For the best of our knowledge, series approach has not been used before to manipulate fluid queueing systems. In this study, we obtain the stationary distributions of the buffer occupancy and the buffer content for a fluid queue driven by an *M/M/*1 queue using a series approach. We give the formula of the stationary distribution of the buffer occupancy with unknown constant coefficients which need to be determined, then we derive a general form for these coefficients by extrapolation.

The rest of the paper is organized as follows: In Section 2, the preliminaries of the proposed fluid queue model are provided. The fluid queue model driven by an *M/M/*1 queue is analyzed in Section 3 using power series technique. The buffer content distribution is obtained for the fluid model in Section 4. In Section 5, the conclusion is stated.

2 Preliminaries

We consider a fluid queue driven by an *M/M/*1 queue which can be represented by two-dimensional Markov process $\{(Y(t), C(t)), \quad t \geq 0\}$. The background queue can be represented by a continuous time Markov chain with arrival rate λ and service rate μ, where $Y(t)$ is a random variable denoting the number of customers in the system at time t and let the generator of the process $\{Y(t)\}$ be denoted by Q that is

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$$Q=\begin{pmatrix} -\lambda & \mu & & \\ \lambda & -(\lambda+\mu) & \mu & \\ & \ddots & \ddots & \ddots \\ & & & \end{pmatrix}.$$

In addition, we consider a fluid queue with an infinite buffer capacity which has input rate q_j and service rate c_j such that $q_j > c_j$, $q_j > 0$ to avoid the trivial case where the queue remains always empty and $C(t)$ denote the content of the buffer at time t with $C(t) \ge 0$ where the content of the buffer cannot decrease whenever the reservoir is empty.

Let the following stability condition for this process is satisfied

$$\sum_{j=0}^{\infty} p_j (q_j - c_j) < 0 ,$$

where p_j be the stationary state probabilities of the infinite birth-death queue. The buffer occupancy distribution $F_j(t,u)$ given by

$$F_j(t,u) = prob\{X(t) = j, C(t) \le u\},\ j \in S\ ,\ u \ge 0 .$$

In steady state case, we obtain

$$F_j(u) = \underset{t\to\infty}{prob}\{X(t) = j, C(t) \le u\},$$

and the boundary conditions are $F_0(0) = 1$ and $F_j(0) = 0$.

3 The Proposed Approach

The basic system of differential equations of the proposed model is given by

$$\theta \dot{F}_0(u) = -\lambda F_0(u) + \mu F_1(u),\ j = 0,$$

$$\theta \dot{F}_j(u) = \lambda F_{j-1}(u) - (\lambda+\mu) F_j(u) + \mu F_{j+1}(u),\ j \ge 1,$$

where $\theta = q - c$, $\dot{F}_j(u) = \dfrac{dF_j(u)}{du}$.

Let $\tau = u\mu$, $\rho = \dfrac{\lambda}{\mu}$, $b = \dfrac{\rho}{1+\rho}$ and then we can write $\rho = \dfrac{b}{1-b}$,the new system will be in the following form

$$\theta \dot{F}_0(\tau) = -(\frac{b}{1-b}) F_0(\tau) + F_1(\tau),\ j = 0, \tag{1}$$

$$\theta \dot{F}_j(\tau) = (\frac{b}{1-b}) F_{j-1}(\tau) - (\frac{1}{1-b}) F_j(\tau) + F_{j+1}(\tau),\ j \ge 1, \tag{2}$$

the proposed solution for the previous system can be written

$$F_j(\tau) = e^{\frac{-\tau}{f}} \rho^j \sum_{n=0}^{\infty} a(n,j) [\frac{\left(\frac{\tau}{f}\right)^n}{n!}],\ j \ge 0,$$

(3)

where

$f = \theta(1-b)$, and $a(n,j)$ are constants need to be determined.

Differentiate equation (3) w.r.t. τ , we have

$$\dot{F}_j(\tau) = \left(\frac{1}{f}\right) e^{\frac{-\tau}{f}} \rho^j \left[\sum_{n=0}^{\infty} a(n+1,j)\left[\frac{\left(\frac{\tau}{f}\right)^n}{n!}\right] - \sum_{n=0}^{\infty} a(n,j)\left[\frac{\left(\frac{\tau}{f}\right)^n}{n!}\right]\right]. \quad (4)$$

Substituting equations (3)-(4) in equations (1)-(2) we get

$$a(n+1,0) = (1-b)a(n,0) + ba(n,1)\,,\ j = 0, \quad (5)$$

$$a(n+1,j) = (1-b)a(n,j-1) + ba(n,j+1)\,,\ j \ge 1, \quad (6)$$

where $a(0,j) = \delta_{0,j}$, such that $\delta_{0,j}$ is the well-known Kronecker delta [12].

We obtain the formula of $a(n,j)$ by using extrapolation principle, as the following table

Table 1: Some values of the coefficient *a* (*n*, *j*).

n/j	0	1	2	3	4	5
0	1	0	0	0	0	0
1	$1-b$	$1-b$	0	0	0	0
2	$1-b$	$1-2b+b^2$	$1-2b+b^2$	0	0	0
3	$1-b-b^2+b^3$	$1-b-b^2+b^3$	$1-3b+3b^2-b^3$	$1-3b+3b^2-b^3$	0	0
4	$1-b-b^2+b^3$	$1-b-3b^2+5b^3-2b^4$	$1-b-3b^2+5b^3-2b^4$	$1-4b+6b^2-4b^3+b^4$	$1-4b+6b^2-4b^3+b^4$	0
5	$1-b-b^2-b^3+4b^4-2b^5$	$1-b-b^2-b^3+4b^4-2b^5$	$1-b-6b^2+14b^3-11b^4+3b^5$	$1-b-6b^2+14b^3-11b^4+3b^5$	$1-5b+10b^2-10b^3+5b^4-b^5$	$1-5b+10b^2-10b^3+5b^4-b^5$

Theorem 1. For any non-negative integer n, j , $a(n,j)$ can be given as:

$$a(n,j) = \begin{cases} 0, & n < j, \\ \sum_{r=0}^{\left\lfloor \frac{n-j}{2} \right\rfloor} \left[\binom{n}{r} - \binom{n}{r-1}\right](b)^r (1-b)^{n-r}, & n \ge j, \end{cases} \quad (7)$$

where $\lfloor . \rfloor$ is the floor function, *r* and *n* are coefficients.

Proof.

It is easy to see that equation (7) gives $a(n,j) = 0$ when $n < j$, also for $n, j = 0,1,2$, we find equation (7) satisfies equations (5)-(6).

Now, for $n > 0, j \ge 1$ we have

$$(1-b)a(n,j-1)+ba(n,j+1)=\sum_{r=0}^{\left\lfloor\frac{n+1-j}{2}\right\rfloor}[\binom{n}{r}-\binom{n}{r-1}](b)^{r}(1-b)^{n+1-r}+$$

$$\sum_{r=0}^{\left\lfloor\frac{n-j-1}{2}\right\rfloor}[\binom{n}{r}-\binom{n}{r-1}](b)^{r+1}(1-b)^{n-r}.$$

Replace r by $r-1$ in the last two terms, and by using the following identity

$$\binom{n}{r}-\binom{n}{r-1}=\binom{n+1}{r},$$

we have

$$R.H.S=\sum_{r=0}^{\left\lfloor\frac{n+1-j}{2}\right\rfloor}[\binom{n+1}{r}-\binom{n+1}{r-1}](b)^{r}(1-b)^{n+1-r}$$

$$=a(n+1,j).$$

Hence, the formula of $a(n,j)$ satisfies the system of the difference equations (5)-(6) which complete the proof.

Lemma 1. For any non-negative integer j , we have

$$\sum_{j=0}^{\infty}a(n,j)(\frac{b}{1-b})^{j}=1.$$

Proof. It is clear that for $n=0,1,2$ the identity is true.

For $n\geq 3$, we have

$$\sum_{j=0}^{\infty}a(n,j)(\frac{b}{1-b})^{j}=\sum_{j=0}^{n}a(n,j)(\frac{b}{1-b})^{j}$$

$$=\sum_{j=0}^{n}\sum_{r=0}^{\left\lfloor\frac{n-j}{2}\right\rfloor}[\binom{n}{r}-\binom{n}{r-1}](b)^{r+j}(1-b)^{n-r-j}.$$

$$R.H.S=(1-b)^{n}[\sum_{k=0}^{n}\binom{n}{k}(\frac{b}{1-b})^{k}+\sum_{k=\left\lfloor\frac{n}{2}\right\rfloor+1}^{n}\binom{n}{k}(\frac{b}{1-b})^{k}]$$

$$=(1-b)^{n}[\frac{1}{(1-b)^{n}}]=1,$$

which complete the proof.

4 Other Results

Equation (3) can be written as

$$F_{j}(\tau)=e^{\frac{-\tau}{f}}\rho^{j}\sum_{n=0}^{\infty}[(1-\rho)+(\rho-1)+a(n,j)][\frac{\left(\frac{\tau}{f}\right)^{n}}{n!}]$$

$$=(1-\rho)\rho^{j}+e^{\frac{-\tau}{f}}\rho^{j}\sum_{n=0}^{\infty}[(\rho-1)+a(n,j)][\frac{\left(\frac{\tau}{f}\right)^{n}}{n!}].$$

Hence the final form of the buffer occupancy distribution is given by

$$F_{j}(u)=(1-\rho)\rho^{j}+e^{\frac{-(\lambda+\mu)}{q-c}}\rho^{j}\sum_{n=0}^{\infty}[(\rho-1)+a(n,j)][\frac{(\frac{\lambda+\mu}{q-c})^{n}}{n!}],$$

where

$$\tau=u\mu,\ \rho=\frac{\lambda}{\mu},b=\frac{\rho}{1+\rho},f=\theta(1-b),\ \rho=(\frac{b}{1-b})\ \text{ and }\ \theta=q-c\ .$$

Now, we can derive the formula of the marginal distribution of the buffer occupancy $P(C\leq u)$ as the following:

$$P(C\leq u)=\sum_{j=0}^{\infty}F_{j}(u).$$

$$P(C\leq u)=1+e^{\frac{-(\lambda+\mu)}{q-c}}\sum_{n=0}^{\infty}[\sum_{j=0}^{\infty}\rho^{j}[(\rho-1)+a(n,j)][\frac{(\frac{\lambda+\mu}{q-c})^{n}}{n!}]],$$

by derivation the both sides we easily get the formula of distribution of the buffer content.

5 Conclusions

In this paper, a fluid queue model driven by *M/M/*1 queue is studied. We used a series approach to obtain the stationary distribution of the buffer occupancy. Further, the buffer content can be obtained directly. All the obtained formulas are in closed form. In the future, we will discuss the transient state for this model by using series approach.

Conflict of Interest: The authors declare that they have no conflict of interest.

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