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Fixed Point Results for Some Contractions in Non-Triangular Non-Archimedean Extended Fuzzy b-Metric Spaces

V. Parvaneh[1,*], *R. J. Shahkoohi*[2], *N. Hussain*[3] *and Z. Bagheri*[4]

[1] Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran
[2] Department of Mathematics, Aliabad Katoul Branch, Islamic Azad University, Aliabad katoul, Iran.
[3] Department of Mathematics, King Abdulaziz University P.O. Box 80203, Jeddah 21589, Saudi Arabia
[4] Department of Mathematics, Azadshahr Branch, Islamic Azad University, Azadshahr, Iran.

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Abstract: In this paper, we investigate the existence of fixed points under various contractive conditions such as Geraghty contractions, comparison contractions and JS-contractions in the setup of extended fuzzy b-metric spaces. Moreover, some examples are provided here to illustrate the usability of the obtained results. Our results generalize and extend comparable results in fuzzy metric and fuzzy b-metric spaces.

Keywords: fixed point, fuzzy metric space, fuzzy b-metric space, extended fuzzy b-metric space.

1 Introduction

In 1988, Grabiec [1] defined contractive mappings on a fuzzy metric space and extended fixed point theorems of Banach and Edelstein in such spaces. Successively, George and Veeramani [2] slightly modified the notion of a fuzzy metric space introduced by Kramosil and Michálek and then defined a Hausdorff and first countable topology on it.

Definition 1.*(Schweizer and Sklar [3]) A binary operation $\star:[0,1]\times[0,1]\to[0,1]$ is called a continuous t-norm if it satisfies the following assertions:*

(T1)$\star$ is commutative and associative;
(T2)$\star$ is continuous;
(T3)$a\star 1=a$ for all $a\in[0,1]$;
(T4)$a\star b\leq c\star d$ when $a\leq c$ and $b\leq d$, with $a,b,c,d\in[0,1]$.

Four basic examples of the continuous t-norm are $a\star_1 b=min\{a,b\}$, $a\star_2 b=\frac{ab}{max\{a,b,\lambda\}}$ for $\lambda\in(0,1)$, $a\star_3 b=ab$ and $a\star_4 b=\max\{a+b-1,0\}$ for all $a,b\in[0,1]$.

Definition 2.*A 3-tuple $(X,M,*)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2\times(0,\infty)$ satisfying the following conditions, for all $x,y,z\in X$ and $t,s>0$,*

(i)$M(x,y,t)>0$;
(ii)$M(x,y,t)=1$ for all $t>0$ if and only if $x=y$;
(iii)$M(x,y,t)=M(y,x,t)$;
(iv)$M(x,y,t)*M(y,z,s)\leq M(x,z,t+s)$;
(v)$M(x,y,.):(0,\infty)\to[0,1]$ is continuous;

The function $M(x,y,t)$ denotes the degree of nearness between x and y with respect to t.

If, in the above definition, the triangular inequality (iv) is replaced by the following condition:

$$(NA)\qquad M(x,y,t)*M(y,z,s)\leq M(x,z,\max\{t,s\})$$

for all $x,y,z\in X$ and $t,s>0$ then the triple $(X,M,*)$ is called a non-Archimedean fuzzy metric space. It is easy to check that (NA) implies (iv), that is, every non-Archimedean fuzzy metric space is itself a fuzzy metric space.

*Example 1.*Let (X,d) be a metric space. Then $(X,M,*)$ is a fuzzy metric space, where $a*b=ab$ for all $a,b\in[0,1]$

* Corresponding author e-mail: zam.dalahoo@gmail.com

and $M(x,y,t) = \frac{t}{t+d(x,y)}$ for all $x,y \in X$ and for all $t > 0$. We call this M as the standard fuzzy metric induced by the metric d. Even if we define $a \star b = min\{a,b\}$ for all $a,b \in [0,1]$, then the triple $(X,M,*)$ will be a fuzzy metric space.

Definition 3.[4] *A fuzzy b-metric space is an ordered triple $(X,B,\star)$ such that X is a nonempty set, $\star$ is a continuous t-norm and B is a fuzzy set on $X \times X \times (0,+\infty)$ satisfying the following conditions, for all $x,y,z \in X$ and $t,s > 0$:*

(F1)$B(x,y,t) > 0$;
(F2)$B(x,y,t) = 1$ *if and only if* $x = y$;
(F3)$B(x,y,t) = B(y,x,t)$;
(F4)$B(x,y,t) \star B(y,z,s) \leq B(x,z,\lambda(t+s))$ *where* $\lambda \geq 1$;
(F5)$B(x,y,\cdot) : (0,+\infty) \to (0,1]$ *is left continuous.*

Definition 4.*Let $(X,B,\star)$ be a fuzzy b-metric space. Then*

(i)a sequence $\{x_n\}$ converges to $x \in X$, if and only if $\lim_{n\to+\infty} B(x_n,x,t) = 1$ for all $t > 0$;
(ii)a sequence $\{x_n\}$ in X is a Cauchy sequence if and only if for all $\varepsilon \in (0,1)$ and for all $t > 0$, there exists n_0 such that $B(x_n,x_m,t) > 1-\varepsilon$ for all $m,n \geq n_0$;
(iii)the fuzzy b-metric space $(X,B,\star)$ is called complete if every Cauchy sequence in it converges to some $x \in X$.

Parvaneh and Ghoncheh introduced in [5] the following generalization of the concept of b-metric space.

Definition 5.[5] *Let X be a (nonempty) set. A function $\widetilde{d} : X \times X \to R^+$ is an extended b-metric if there exists a strictly increasing continuous function $\Omega : [0,\infty) \to [0,\infty)$ with $\Omega^{-1}(t) \leq t \leq \Omega(t)$ such that for all $x,y,z \in X$, the following conditions hold:*

$(p_1)\widetilde{d}(x,y) = 0$ *iff* $x = y$,
$(p_2)\widetilde{d}(x,y) = \widetilde{d}(y,x)$,
$(p_3)\widetilde{d}(x,z) \leq \Omega(\widetilde{d}(x,y) + \widetilde{d}(y,z))$.

In this case, the pair $(X,\widetilde{d})$ is called an extended b-metric space.

It should be noted that the class of extended b-metric spaces is considerably larger than the class of b-metric spaces, since a b-metric is an extended b-metric space when $\Omega(t) = st$ for fixed $s \geq 1$, while a metric is an extended b-metric space when $\Omega(t) = t$. For more details on extended b-metric spaces see [6].

Several examples of extended b-metric spaces can be constructed using the following easy proposition.

Proposition 1.[6] *Let (X,d) be a b-metric space with coefficient $s \geq 1$ and let $\rho(x,y) = \xi(d(x,y))$ where $\xi : [0,\infty) \to [0,\infty)$ is a strictly increasing continuous function with $t \leq \xi(t)$ for $t \geq 0$ and $\xi(0) = 0$. Then ρ is an extended b-metric space with $\Omega(t) = \xi(st)$.*

The following concept is the concept of extended fuzzy b-metric space introduced in [7]. Parvaneh and Hussain presented some fixed point results in this framework using the notion of extended parametric b-metric space.

Definition 6.*An extended fuzzy b-metric space is an ordered quadruple $(X,B,\star,\Omega)$ such that X is a nonempty set, $\star$ is a continuous t-norm, B is a fuzzy set on $X \times X \times (0,+\infty)$ and Ω is a strictly increasing continuous function from $[0,\infty) \to [0,\infty)$ with $\Omega^{-1}(x) \leq x \leq \Omega(x)$ and $\Omega^{-1}(0) = 0 = \Omega(0)$ such that the following conditions hold for all $x,y,z \in X$ and $t,s > 0$:*

(F1)$B(x,y,t) > 0$;
(F2)$B(x,y,t) = 1$ *if and only if* $x = y$;
(F3)$B(x,y,t) = B(y,x,t)$;
(F4)$B(x,y,t) \star B(y,z,s) \leq B(x,z,\Omega(t+s))$;
(F5)$B(x,y,\cdot) : (0,+\infty) \to (0,1]$ *is left continuous.*

Proposition 2.*Let (X,d) be a b-metric space (with parameter $\lambda \geq 1$) and there exists a strictly increasing continuous function $\Omega : [0,\infty) \to [0,\infty)$ with $\lambda\Omega^{-1}(t) \leq t \leq \Omega(\frac{t}{\lambda})$ for all $t > 0$ and $\Omega^{-1}(0) = 0 = \Omega(0)$. Let $M_\Omega : X \times X \times (0,\infty) \to [0,1]$ be defined by $M_\Omega(x,y,t) = \frac{\lambda\Omega^{-1}(t)}{\lambda\Omega^{-1}(t)+d(x,y)}$, for all $t > 0$. Then $(X,M_\Omega,\wedge,\Omega)$ is an extended fuzzy b-metric space. M_Ω will be called the standard extended fuzzy b-metric.*

Taking $\Omega(t) = \lambda t$ in the above proposition, our result is the extension of Example 8 of [8].

Taking various functions Ω in the previous proposition, we can obtain a lot of examples of extended fuzzy b-metric. We state just a few of them which we will use later in the text.

*Example 2.*Let (X,d) be a metric space.

1.If $\Omega(t) = e^t - 1$, then $\Omega^{-1}(u) = \ln(1+u)$. So, $M_\Omega : X \times X \times (0,\infty) \to [0,1]$ defined by $M_\Omega(x,y,t) = \frac{\ln(1+t)}{\ln(1+t)+d(x,y)}$ is an extended fuzzy b-metric.
2.If $\Omega(t) = te^t$, then $\Omega^{-1}(u) = W(u)$, for $u \geq 0$, where W is the Lambert W-function (see, e.g., [9]). Let $M_\Omega : X \times X \times (0,\infty) \to [0,1]$ be defined by $M_\Omega(x,y,t) = \frac{W(t)}{W(t)+d(x,y)}$. Then $(X,M_\Omega,\wedge,\Omega)$ is an extended fuzzy b-metric space.
3.If $\Omega(t) = t+\ln(1+t)$, then $\Omega^{-1}(u) = (W(e^{u+1})-1)$ for $u \geq 0$. Assume that $M_\Omega : X \times X \times (0,\infty) \to [0,1]$ be defined by $M_\Omega(x,y,t) = \frac{W(e^{t+1})-1}{W(e^{t+1})-1+d(x,y)}$. Then $(X,M_\Omega,\wedge,\Omega)$ is an extended fuzzy b-metric space.

Definition 7.*Let X be a nonempty set. If $(X,M,\star)$ is a fuzzy metric space and $(X,\preceq)$ is a partially-ordered set, then $(X,M,\star,\preceq)$ is called an ordered fuzzy metric space. Also, $x,y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. Let $(X,\preceq)$ be a partially-ordered set and $T : X \to X$ be a mapping. T is called a non-decreasing if $Tx \preceq Ty$, whenever $x \preceq y$ for all $x,y \in X$.*

Motivated by [8] we present the following Lemma.

Lemma 1. *For all $x,y \in X$ the mapping $M(x,y,.) : [0,\infty) \to [0,1]$ is nondecreasing.*

Proof. For $0 < t \leq s$ we have

$$M(x,y,s) \geq M(x,x,\Omega^{-1}[s]-t)*M(x,y,t) = 1*M(x,y,t) = M(x,y,t).$$

In general, an extended fuzzy b-metric function for nontrivial function Ω is not jointly continuous in all its variables. So, we need the following simple lemma about the convergent sequences.

Lemma 2. *Let $(X,B,\star,\Omega)$ be an extended fuzzy b-metric space.*

1. Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent to x, and y, respectively. Then we have

$$B(x,y,t) \geq \limsup_{n\longrightarrow\infty} B(x_n,y_n,\Omega^{-2}(t)) \geq \liminf_{n\longrightarrow\infty} B(x_n,y_n,\Omega^{-2}(t)) \geq B(x,y,\Omega^{-4}(t)).$$

2. Suppose that $\{x_n\}$ is convergent to x. Then we have

$$B(x,y,t) \geq \limsup_{n\longrightarrow\infty} B(x_n,y,\Omega^{-1}(t)) \geq \liminf_{n\longrightarrow\infty} B(x_n,y,\Omega^{-1}(t)) \geq B(x,y,\Omega^{-2}(t)).$$

Proof. 1. Using the triangle inequality in an extended fuzzy b-metric space it is easy to see that,

$$\begin{aligned} B(x,y,t) &\geq B(x,x_n,\Omega^{-1}(t))*B(y,x_n,\Omega^{-1}(t)) \\ &\geq B(x,x_n,\Omega^{-1}(t))*B(x_n,y_n,\Omega^{-2}(t))*B(y_n,y,\Omega^{-2}(t)) \end{aligned}$$

and

$$\begin{aligned} B(x_n,y_n,t) &\geq B(x_n,x,\Omega^{-1}(t))*B(y_n,x,\Omega^{-1}(t)) \\ &\geq B(x_n,x,\Omega^{-1}(t))*B(x,y,\Omega^{-2}(t))*B(y,y_n,\Omega^{-2}(t)). \end{aligned}$$

Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality we obtain the desired result.

2. Using the rectangle inequality we see that,

$$B(x,y,t) \geq B(x,x_n,\Omega^{-1}(t))*B(y,x_n,\Omega^{-1}(t))$$

and

$$B(x_n,y,t) \geq B(x_n,x,\Omega^{-1}(t))*B(y,x,\Omega^{-1}(t)).$$

Recall that an extended fuzzy b-metric space $(X,\preceq)$ is said to have the s.l.c. property, if whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$

One of the methods to obtain fixed point results in the framework of fuzzy metric spaces is the use of the concept of parametric spaces. In this method, the discussed fuzzy metric should be triangular which is a strong assumption. For more details, we refer the reader to [10] and [4]. In this paper, without using the triangularity condition we present some fixed point theorems in an extended fuzzy b-metric space.

2 Main results

2.1 Results under Geraghty-type conditions

Fixed point theorems for monotone operators in ordered metric spaces have had various applications in differential and integral equations (see [11,12,13] and references therein). In 1973, M. Geraghty [14] proved a fixed point result, generalizing the Banach contraction principle. Several authors proved later various results using Geraghty-type conditions. Fixed point results of this kind in b-metric spaces were obtained by Đukić et al. in [15].

Following [15], let $\mathscr{F}_\Omega$ denote the class of all functions $\beta : [0,\infty) \to [0,\Omega^{-1}(1))$ satisfying the following condition:

$$\beta(t_n) \to \Omega^{-1}(1) \text{ as } n \to \infty \text{ implies } t_n \to 0 \text{ as } n \to \infty.$$

Definition 8. *Let $(X,B,\star,\Omega)$ be an extended fuzzy b-metric space. We say that $T : X \to X$ is an extended fuzzy Geraghty contractive type mapping if there exist two functions β and Ω such that:*

$$\frac{1}{B(Tx,Ty,\Omega^{-2}(t))} - 1 \leq \beta\left(\frac{1}{B(x,y,t)} - 1\right)\left(\frac{1}{B(x,y,\Omega(t))} - 1\right) \quad (1)$$

for all $t > 0$ and for all comparable elements $x,y \in X$.

Theorem 1. *Let $(X,\preceq)$ be a partially-ordered set and $(X,B,\star,\Omega)$ be a complete non-Archimedean extended fuzzy b-metric space. Suppose that $T : X \to X$ is a self-mapping satisfying the following assertions:*

(i) T is an extended fuzzy Geraghty contractive type mapping;
(ii) There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$,
(iii) T is continuous, or, let $(X,\preceq)$ has the s.l.c. property.

Then T has a fixed point, that is, there exists $x_0 \in X$ such that $Tx_0 = x_0$.

Proof. Put $x_n = T^n(x_0)$.

Step I: We will show that $\lim_{n\to\infty} B(x_n,x_{n+1},t) = 1$. We assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by 1 we have:

$$\begin{aligned} \frac{1}{B(Tx_{n-1},Tx_n,t)} - 1 &\leq \frac{1}{B(Tx_{n-1},Tx_n,\Omega^{-2}(t))} - 1 \\ &\leq \beta\left(\frac{1}{B(x_{n-1},x_n,t)} - 1\right)\left(\frac{1}{B(x_{n-1},x_n,\Omega(t))} - 1\right) \\ &< \Omega^{-1}(1)\left(\frac{1}{B(x_{n-1},x_n,\Omega(t))} - 1\right) \\ &\leq \left(\frac{1}{B(x_{n-1},x_n,\Omega(t))} - 1\right), \end{aligned}$$

i.e.,

$$B(x_n,x_{n+1},t) \geq B(x_{n-1},x_n,\Omega(t)) \geq B(x_{n-1},x_n,t). \quad (2)$$

Therefore, the sequence $\{B(x_n,x_{n+1},t)\}$ is an increasing sequence of positive real numbers in $[0,1]$. Let $B(t) = \lim_{n\to\infty} B(x_{n-1},x_n,t)$. We show that $B(t) = 1$ for all

$t > 0$. Suppose that there is $t_0 > 0$ such that $B(t_0) < 1$, then

$$\frac{1}{(B(Tx_{n-1},Tx_n,t_0))} - 1 = \frac{1}{(B(x_n,x_{n+1},t_0))} - 1 \tag{3}$$

$$< \Omega^{-1}(1)(\frac{1}{B(x_{n-1},x_n,t_0)} - 1). \tag{4}$$

As $n \to \infty$ we have,

$$\frac{1}{B(t_0)} - 1 \leq \Omega^{-1}(1)[\frac{1}{B(t_0)} - 1], \tag{5}$$

which is a contradiction, as $\Omega^{-1}(1) < 1$ and so, we get $\{B(x_n,x_{n+1},t)\} \to 1$ for all $t > 0$.

Step II: Now, we prove that the sequence $\{x_n\}$ is a Cauchy sequence. If the sequence $\{x_n\}$ is not Cauchy, then there are $\varepsilon \in (0,1)$, $t_0 > 0$ and $k_0 \in \mathbb{N}$ such that, for each $k \in \mathbb{N}$ with $k \geq k_0$, there exist $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ and $B(x_{m(k)},x_{n(k)},t_0) \leq 1-\varepsilon$.

Let, for each k, $m(k)$ be the least positive integer exceeding $n(k)$ satisfying the above property, that is, $B(x_{m(k)-1},x_{n(k)},t_0) > 1-\varepsilon$ and $B(x_{m(k)},x_{n(k)},t_0) \leq 1-\varepsilon$.

Then, for each positive integer $k \geq k_0$, by condition (NA) we have:

$$1-\varepsilon \geq B(x_{m(k)},x_{n(k)},\Omega(t_0)) \geq B(x_{m(k)-1},x_{m(k)},t_0) \star B(x_{m(k)-1},x_{n(k)},t_0). \tag{6}$$

Taking the limit as $k \to \infty$ in the above relation, we deduce that:

$$\lim_{k\to\infty} B(x_{m(k)},x_{n(k)},\Omega(t_0)) \geq 1-\varepsilon. \tag{7}$$

Also,

$$\begin{aligned} &B(x_{m(k)},x_{n(k)},t_0)\\ &\geq B(x_{m(k)},x_{m(k)+1},\Omega^{-1}[t_0]) \star B(x_{m(k)+1},x_{n(k)},\Omega^{-1}[t_0])\\ &\geq B(x_{m(k)},x_{m(k)+1},\Omega^{-1}[t_0]) \star B(x_{m(k)+1},x_{n(k)+1},\Omega^{-2}[t_0]) \star B(x_{n(k)+1},x_{n(k)},\Omega^{-2}[t_0]), \end{aligned} \tag{8}$$

we get:

$$\lim_{k\to\infty} B(x_{m(k)+1},x_{n(k)+1},\Omega^{-2}[t_0]) \leq 1-\varepsilon. \tag{9}$$

According to the contractive condition 1

$$\frac{1}{B(x_{m(k)+1},x_{n(k)+1},\Omega^{-2}[t_0])} - 1 \leq \beta(\frac{1}{B(x_{m(k)},x_{n(k)},t_0)} - 1)(\frac{1}{B(x_{m(k)},x_{n(k)},\Omega[t_0])} - 1). \tag{10}$$

Taking the limit as $k \to \infty$ in the above inequality and using (7) and (9), we get:

$$\frac{1}{1-\varepsilon} - 1 \leq \lim_{k\to\infty} \beta(\frac{1}{B(x_{m(k)},x_{n(k)},t_0)} - 1)[\frac{1}{1-\varepsilon} - 1]$$

which yields that $\lim_{k\to\infty} \beta(\frac{1}{B(x_{m(k)},x_{n(k)},t_0)} - 1) = \Omega^{-1}(1)$. Hence,

$$\lim_{k\to\infty} B(x_{m(k)},x_{n(k)},t_0)) = 1,$$

which implies that $1-\varepsilon \geq \lim_{k\to\infty} B(x_{m(k)},x_{n(k)},t_0) = 1$, a contradiction. Therefore, $\{x_n\}$ is a cauchy sequence. Since $(X,B,\star,\Omega)$ is a complete extended fuzzy b-metric space, it follows that the sequence $\{x_n\}$ converges to some $x^\star \in X$. Now, the continuity of T implies that $Tx_n \to Tx^\star$ as $n \to +\infty$ and so $\lim_{m,n\to+\infty} B(Tx_n,Tx^\star,t) = 1$ for all $t > 0$. It follows that:

$$\lim_{m,n\to+\infty} B(x_{n+1},Tx^\star,t) = \lim_{m,n\to+\infty} B(Tx_n,Tx^\star,t) = 1$$

for all $t > 0$, that is, $x_n \to Tx^\star$ as $n \to +\infty$. By the uniqueness of the limit, we get $x^\star = Tx^\star$, i.e., $x^\star$ which is a fixed point of T.

Now, let $(X,\preceq)$ enjoys the s.l.c. property. Using this assumption we have $x_n \preceq x^\star$. Now, by Lemma 2,

$$\begin{aligned} \frac{1}{B(x^\star,Tx^\star,t)} - 1 &\leq \limsup_{n\to\infty}[\frac{1}{B(x_{n+1},Tx^\star,\Omega^{-1}(t))} - 1]\\ &\leq \limsup_{n\to\infty}[\frac{1}{B(x_{n+1},Tx^\star,\Omega^{-2}(\Omega(t)))} - 1]\\ &\leq \limsup_{n\to\infty}\beta(\frac{1}{B(x_n,x^\star,\Omega(t))} - 1)\limsup_{n\to\infty}[\frac{1}{B(x_n,x^\star,\Omega^2(t))} - 1]\\ &\leq \limsup_{n\to\infty}\beta(\frac{1}{B(x_n,x^\star,\Omega(t))} - 1)[\frac{1}{B(x^\star,x^\star,\Omega(t))} - 1] = 0. \end{aligned}$$

Therefore, we deduce that $B(x^\star,Tx^\star,t) = 1$, so, $x^\star = Tx^\star$.

2.2 Fixed point results via comparison functions

Let Φ be the family of all nondecreasing functions $\phi : [0,\infty) \to [0,\infty)$ such that

$$\lim_{n\to\infty} \phi^n(t) = 0$$

for all $t > 0$.

Lemma 3. *If $\phi \in \Phi$, then the followings are satisfied.*
(a) $\phi(t) < t$ for all $t > 0$;
(b) $\phi(0) = 0$.

Definition 9. *Let $(X,B,\star,\Omega)$ be a non-Archimedean extended fuzzy b-metric space. We say that $T : X \to X$ is an extended-fuzzy comparison contractive mapping if there exist two functions ϕ and Ω such that:*

$$\frac{1}{B(Tx,Ty,\Omega^{-2}(t))} - 1 \leq \phi(\frac{1}{B(x,y,\Omega(t))} - 1)$$

for all $t > 0$ and for all comparable elements $x, y \in X$.

Theorem 2. *Let $(X,\preceq)$ be a partially-ordered set and $(X,B,\star,\Omega)$ be a complete extended fuzzy b-metric space. Suppose that $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) T is an extended fuzzy comparison contractive mapping;
(ii) There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$,
(iii) T is continuous, or, let $(X,\preceq)$ have the s.l.c. property.

Then T has a fixed point, that is, there exists $x_0 \in X$ such that $Tx_0 = x_0$.

Proof. The proof is similar to the above theorem and the fact that $\phi(r) < r$.

2.3 Fixed point results related to JS-contractions

Jleli et al. [16] have introduced the class Θ_0 consisting of all functions $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

(θ_1) θ is non-decreasing;
(θ_2) for each sequence $\{t_n\} \subseteq (0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;
(θ_3) there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = \ell$;
(θ_4) θ is continuous.

They proved the following result:

Theorem 3.[16, Corollary 2.1] *Let (X,d) be a complete metric space and let $T : X \to X$ be a given mapping. Suppose that there exist $\theta \in \Theta_0$ and $k \in (0,1)$ such that*

$$x,y \in X, \quad d(Tx,Ty) \neq 0 \implies \theta\big(d(Tx,Ty)\big) \leq \theta\big(d(x,y)\big)^k. \tag{11}$$

Then T has a unique fixed point.

From now on, we denote by Θ the set of all functions $\theta : [0,\infty) \to [1,\infty)$ satisfying the following conditions:
θ_1. θ is a continuous strictly increasing function;
θ_2. for each sequence $\{t_n\} \subseteq (0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;

Remark. It is clear that $f(t) = e^t$ does not belong to Θ_0, but it belongs to Θ. Other examples are $f(t) = \cosh t$, $T(t) = \frac{2\cosh t}{1+\cosh t}$, $T(t) = 1+\ln(1+t)$, $T(t) = \frac{2+2\ln(1+t)}{2+\ln(1+t)}$, $T(t) = e^{te^t}$ and $T(t) = \frac{2e^{te^t}}{1+e^{te^t}}$, for all $t \geq 0$.

Definition 10. *Let $(X,B,\star,\Omega)$ be a M-complete extended fuzzy b-metric space. We say that $T : X \to X$ is an extended-fuzzy JS-contraction mapping if there exist two functions θ and Ω such that:*

$$\theta\Big(\frac{1}{B(Tx,Ty,\Omega^{-2}[t])} - 1\Big) \leq \theta\Big(\frac{1}{B(x,y,\Omega(t))} - 1\Big)^k \tag{12}$$

for all comparable elements $x,y \in X$ and for all $t > 0$ where $\theta \in \Theta$ and $k \in [0,1)$.

Theorem 4. *Let $(X,\preceq)$ be a partially-ordered set and $(X,B,\star,\Omega)$ be a complete non-Archimedean extended fuzzy b-metric space. Suppose that $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) T is an extended-fuzzy JS-contraction type mapping;
(ii) There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$,
(iii) T is continuous, or, let $(X,\preceq)$ has the s.l.c. property.

Then f has a fixed point.

Proof. Put $x_n = T^n(x_0)$.
Step 1. We will show that $\lim_{n\to\infty} B(x_n,x_{n+1},t) = 1$. Without any loss of generality, we may assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by 12 we have

$$\begin{aligned}\theta\Big(\frac{1}{B(x_n,x_{n+1},t)} - 1\Big) &\leq \theta\Big(\frac{1}{B(x_n,x_{n+1},\Omega^{-2}[t])} - 1\Big)\\ &= \theta\Big(\frac{1}{B(Tx_{n-1},Tx_n,\Omega^{-2}[t])} - 1\Big)\\ &\leq \theta\Big(\frac{1}{B(x_{n-1},x_n,t)} - 1\Big)^k.\end{aligned} \tag{13}$$

From (17) we deduce that,

$$\Theta\Big(\frac{1}{B(x_n,x_{n+1},t)} - 1\Big) \leq \Theta\Big(\frac{1}{B(x_{n-1},x_n,t)} - 1\Big)^k.$$

Therefore,

$$1 \leq \Theta\Big(\frac{1}{B(x_n,x_{n+1},t)} - 1\Big) \leq \Theta\Big(\frac{1}{B(x_{n-1},x_n,t)} - 1\Big)^k \leq \ldots \leq \Theta\Big(\frac{1}{B(x_0,x_1,t)} - 1\Big)^{k^n}. \tag{14}$$

Taking the limit as $n \to \infty$ in (14) we have,

$$\lim_{n\to\infty} \Theta\Big(\frac{1}{B(x_n,x_{n+1},t)} - 1\Big) = 1$$

and since $\Theta \in \Delta_\Theta$ we obtain,

$$\lim_{n\to\infty} \frac{1}{B(x_n,x_{n+1},t)} - 1 = 0, \tag{15}$$

i.e, $\lim_{n\to\infty} B(x_n,x_{n+1},t) = 1$.
Step 2. Now, we prove that the sequence $\{x_n\}$ is a M-Cauchy sequence. If the sequence $\{x_n\}$ is not M-Cauchy, then there are $\varepsilon \in (0,1)$, $t_0 > 0$ and $k_0 \in \mathbb{N}$ such that, for each $k \in \mathbb{N}$ with $k \geq k_0$, there exist $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ and $B(x_{m(k)},x_{n(k)},t_0) \leq 1-\varepsilon$.

Let, for each k, $m(k)$ be the least positive integer exceeding $n(k)$ satisfying the above property, that is, $B(x_{m(k)-1},x_{n(k)},t_0) > 1-\varepsilon$ and $B(x_{m(k)},x_{n(k)},t_0) \leq 1-\varepsilon$.

Then, for each positive integer $k \geq k_0$, by condition (NA) we have:

$$\lim_{n\to\infty} B(x_{m(k)+1},x_{n(k)+1},\Omega^{-2}[t_0]) \leq 1-\varepsilon. \tag{16}$$

According to the contractive condition (12)

$$\begin{aligned}\theta\Big(\frac{1}{B(x_{m(k)+1},x_{n(k)+1},\Omega^{-2}[t_0])} - 1\Big) &= \theta\Big(\frac{1}{B(Tx_{m(k)},Tx_{n(k)},\Omega^{-2}[t_0])} - 1\Big)\\ &\leq \theta\Big(\frac{1}{B(x_{m(k)},x_{n(k)},\Omega(t_0))} - 1\Big)^k.\end{aligned} \tag{17}$$

Taking the limit as $k \to \infty$ in the above inequality and using (7) and (9), we get:

$$\Theta\Big(\frac{1}{1-\varepsilon} - 1\Big) \leq \Big(\Theta\Big(\frac{1}{1-\varepsilon} - 1\Big)\Big)^k < \Theta\Big(\frac{1}{1-\varepsilon} - 1\Big)$$

which is a contradiction. This means that $\{x_n\}$ is a Cauchy sequence in X. Since $(X,B,\star,\Omega)$ is a complete extended fuzzy b-metric space, it follows that the sequence $\{x_n\}$ converges to some $x^\star \in X$, that is, $x_n \to x^\star$ as $n \to +\infty$. Now, the continuity of T implies that $Tx_n \to Tx^\star$ as $n \to +\infty$ and so

$\lim_{m,n\to+\infty} B(Tx_n, Tx^\star, t) = 1$ for all $t > 0$. It follows that:

$$\lim_{m,n\to+\infty} B(x_{n+1}, Tx^\star, t) = \lim_{m,n\to+\infty} B(Tx_n, Tx^\star, t) = 1$$

for all $t > 0$, that is, $x_n \to Tx^\star$ as $n \to +\infty$. By the uniqueness of the limit, we get $x^\star = Tx^\star$, i.e. $x^\star$ is a fixed point of T.

*Remark.*In general, in ordered metric spaces, continuity of the mapping T can be replaced by regularity of the space which yields the existence of fixed point.

*Example 3.*Let $X = [0, 1.2)$ and (X, d) be a b-metric space with $d(x,y) = |x-y|^2$ for all $x, y \in X$. Let $\Omega(x) = e^x - 1$ with $\Omega^{-1}(t) = \frac{1}{2}\ln(1+t)$ and $\Omega^{-3}(t) = \frac{1}{2}\ln(1+\frac{1}{2}\ln(1+\frac{1}{2}\ln(1+t)))$. Then $(X, B, *, \Omega)$ is an extended fuzzy b-metric space, where $a \star b = \min\{a,b\}$ for all $a, b \in [0, 1.2]$ and $B(x,y,t) = \frac{2\Omega^{-1}(t)}{2\Omega^{-1}(t)+d(x,y)}$ for all $x, y \in X$ and for all $t > 0$.

Define a mapping $T : X \to X$ by

$$T(x) = \frac{1}{32}\ln(x^2+1)$$

and a function $\beta \in \mathscr{B}_\Omega$ by $\beta(t) = 1 + \frac{3}{t}$. For all $x, y \in X$, we have:

$$\begin{aligned} d(Tx,Ty) &= |\frac{1}{32}\ln(x^2+1) - \frac{1}{32}\ln(y^2+1)|^2 \\ &\leq \frac{1}{32}|x-y|^2 \\ &= \frac{1}{32}d(x,y). \end{aligned}$$

Suppose that

$$\Omega^{-3}(t) = \frac{1}{2}\ln(1+\frac{1}{2}\ln(1+\frac{1}{2}\ln(1+t))) = \alpha(t).$$

So, we have

$$2\alpha(t) + d(Tx,Ty) \leq 2\alpha(t) + \frac{1}{32}d(x,y).$$

Then

$$B(Tx,Ty,\Omega^{-2}(t)) = \frac{2\alpha(t)}{2\alpha(t)+d(Tx,Ty)} \geq \frac{2\alpha(t)}{2\alpha(t)+\frac{1}{32}d(x,y)}.$$

Therefore,

$$\Omega^{-1}(B(Tx,Ty,\Omega^{-2}(t))) \geq \frac{1}{2}\ln(1+\frac{2\alpha(t)}{2\alpha(t)+\frac{1}{32}(d(x,y))}) \geq \frac{1}{4}\cdot\frac{2\alpha(t)}{2\alpha(t)+\frac{1}{32}(d(x,y))}.$$

Then

$$\begin{aligned} \frac{1}{\Omega^{-1}(B(Tx,Ty,\Omega^{-2}(t)))} - 1 &\leq \frac{8\alpha(t)+\frac{1}{8}(d(x,y)}{2\alpha(t)} - 1 \\ &= 3 + \frac{1}{16}\cdot\frac{d(x,y)}{\alpha(t)} \leq 3 + \frac{d(x,y)}{\ln(1+t)}, \end{aligned}$$

as we know that

$$\begin{aligned} \alpha(t) = \ln(1+\frac{1}{2}\ln(1+\frac{1}{2}\ln(1+t))) &\geq \frac{1}{2}\cdot\frac{1}{2}\ln(1+\frac{1}{2}\ln(1+t)) \\ &\geq \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\ln(1+t) = \frac{1}{16}\ln(1+t). \end{aligned}$$

Therefore,

$$\frac{d(x,y)}{\alpha(t)} \leq \frac{16d(x,y)}{\ln(1+t)}$$

which yields that

$$\frac{1}{B(x,y,t)} - 1 = \frac{d(x,y)}{\ln(1+t)}$$

and

$$\beta(t) = 1 + \frac{3}{t}$$

So, from Theorem 1 f has a fixed point.

*Example 4.*Let$X = [0, 1.2)$, (X, d) be a b-metric space with $d(x,y) = |x-y|$ for all $x, y \in X$.Let $\Omega(t) = e^t - 1$ and $\Omega^{-1}(t) = \ln(1+t)$ Then $(X, B, \star, \Omega)$ is an extended fuzzy b-metric space, where $a \star b = \min\{a,b\}$ for all $a, b \in [0, 1.2]$ and $B(x,y,t) = \frac{\Omega^{-1}(t)}{\Omega^{-1}(t)+d(x,y)}$ for all $x, y \in X$ and for all $t > 0$. $(X, B, \star, \Omega)$ is a complete extended fuzzy b-metric space. Here, $\Omega^{-1}(t) = \ln(1+t)$ and $\Omega^{-3}(t) = \ln(1+\ln(1+\ln(1+t)))$.

Define a relation $\preceq$ on X by $x \preceq y$ iff $y \leq x$. Define k and $\theta \in \Theta$ by $k = \frac{1}{\sqrt{2}}$ and $\theta(t) = e^{te^t}$. Let $f : X \to X$ be defined by $fx = \arctan(\frac{x}{64})$. It is easy to see that f is an ordered increasing and continuous self map on X and $0 \leq f0$.

For all comparable elements $x, y \in X$, by mean value theorem, we have,

$$\begin{aligned} d(Tx,Ty) &= |Tx-Ty| \\ &= \left|\arctan\frac{x}{64} - \arctan\frac{y}{64}\right| \\ &\leq \left|\frac{x}{64} - \frac{y}{64}\right| \\ &= \frac{1}{64}d(x,y). \end{aligned}$$

Suppose that

$$\Omega^{-3}(t) = \ln(1+\ln(1+\ln(1+t))) = \alpha(t).$$

So, we have

$$\alpha(t) + d(Tx,Ty) \leq \alpha(t) + \frac{1}{64}d(x,y).$$

Then

$$B(Tx,Ty,\Omega^{-2}(t)) = \frac{\alpha(t)}{\alpha(t)+d(Tx,Ty)} \geq \frac{\alpha(t)}{\alpha(t)+\frac{1}{64}d(x,y)}.$$

Therefore,

$$\frac{1}{B(Tx,Ty,\Omega^{-2}(t))}-1 \le \frac{(\alpha(t)+\frac{1}{64}(d(x,y)}{\alpha(t)}-1$$
$$= \frac{1}{64}\cdot\frac{d(x,y)}{\alpha(t)} \le \frac{1}{16}\cdot\frac{d(x,y)}{\ln(1+t)} = \frac{1}{16}[\frac{1}{B(x,y,t)}-1],$$

because,

$$\alpha(t)=\ln(1+\ln(1+\ln(1+t))) \ge \frac{1}{2}\ln(1+\ln(1+t))$$
$$\ge \frac{1}{2}[\frac{1}{2}\ln(1+t)] = \frac{1}{4}\ln(1+t).$$

Therefore,

$$\theta(\frac{1}{B(Tx,Ty,\Omega^{-2}(t))}-1) \le \theta(\frac{1}{16}(\frac{1}{B(x,y,t)}-1))$$
$$= e^{[\frac{1}{16}[\frac{1}{B(x,y,t)}-1]]e^{[\frac{1}{16}[\frac{1}{B(x,y,t)}-1]]}}$$
$$\le [e^{[\frac{1}{B(x,y,t)}-1]e^{[\frac{1}{B(x,y,t)}-1]}}]^{\frac{1}{\sqrt{2}}}$$
$$= (\theta(\frac{1}{B(x,y,t)}-1))^{\frac{1}{\sqrt{2}}}.$$

Hence, all conditions of Theorem 4 hold and T has a unique fixed point.

3 Conclusion

"Extended fuzzy b-metric space is one of interesting generalizations of the concept of fuzzy metric space. In this paper, we defined this new concept and obtained some fixed point results for three classes of contractive mappings. Our results will be a great motivation for researchers who study in the field of fixed point theory."

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Vahid Parvaneh received the PhD degree in Mathematics at Islamic Azad University of Karj. His research interests are in the areas of applied mathematics, fixed pint theory and its applications and the geometry of Banach spaces. He has published some research articles in reputed international journals of mathematical sciences. He is referee of some mathematical journals.

Roghayyeh Jalahahkoohil S received the PhD degree in Mathematics at Islamic Azad University-Science and Research Branch in Tehran. His research interests are in the areas of pure mathematics, fixed pint theory and its applications and the nonlinear analysis. He work as a faculty member in Islamic Azad University-Aliabad Katoul Branch.

Nawab Hussain received the PhD degree (January 2002) from Bahauddin Zakariya University, Multan, Pakistan. . His research interests are in the areas of Functional Analysis (Fixed Point Theory, Approximation Theory and Applications). He has published more than 200 research articles in international journals of mathematical sciences. He is referee of some mathematical journals.

Zohreh Bagheri received her PhD degree in Mathematics from Islamic Azad University. Her research interest is nonlinear analysis. She has published some research articles in indexed journals of mathematics. She work at Islamic Azad University-Azadshahr Branch.