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A Note on the Characteristic Properties of Geodesic Sub-(α,b,s)-Preinvex Functions

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Abstract: In the present paper, we investigate the properties of geodesic sub-(α,b,s)-preinvex functions on Hadamard manifolds and establish some basic properties in both general and differential cases. Furthermore, we explored sufficient conditions of optimal solutions and proved some new inequalities under geodesic sub-(α,b,s)-preinvexity.

Keywords: preinvex; geodesic; Riemannian manifolds; Hadamard manifolds

1 Introduction

Convexity is an important property in mathematics and economics. Recently, many researchers have developed new generalizations for the classical convexity and also established many properties in new generalized cases. For example, Hanson [1] presented one of the important generalization of convexity (i.e. invexity) in 1981. The preinvexity was presented by Ben-Israel and Mond in [2] which is special case of invex functions. Similarly, some properties of preinvex and α-preinvex functions was considered by Jeyakumar [3] and Noor [4,5], respectively. In 1991, a class of b-vex functions was introduced by Bector and Singh in [6]. Suneja et al. [7] explored the generalizations of preinvex functions, i.e b-preinvex functions. In 2006, a generalization of b-invex function which is known as semi-b-preinvex was presented in [8]. Furthermore, Chao et al. in [9] defined a new class of generalized sub-b-convex functions and discussed sufficient conditions of optimality. s-convex functions of the first type was first introduced by Orlicz in [10] and the second type of s-convexity was introduced by Breckner in [11], then Hudzik and Maligranda in [12] addressed some properties of these types of s-convexity ($s \in (0,1)$). In 2016, sub-b-s-convex functions were defined using modulation s-convexity and sub-b-convexity, see [13].

Thus, various properties of convex functions can be established on Riemannian manifolds. For example, Rapcsák [14] handled smooth nonlinear optimization in R^n and Udeişte [15] investigated some generalizations of convexity as well as optimization problems on Riemannian Manifolds which differ from the others in the use of Riemannian manifold. The convexity along curves and generalizations with applications to duality theory and optimality conditions on Riemannian manifold were considered by Pini [16]. The concept of geodesic invexity in Riemannian manifold was introduced and preinvexity on a geodesic invex set was defined. Moreover, the relationship between geodesic invexity and preinvexity on manifolds was investigated by Barani and Pouryayevali [17], while geodesic α-invexity and α-preinvex functions were defined in [18]. In addition, piece of literature involved more related generalizations of convexity, and new class of generalized convexity such as strongly α-invex and strongly geodesic α-preinvex functions, see [19,20,21,22,23].

Riemannian geometry is considered as generalization of the Euclidean case and smooth Riemannian manifolds accommodate curvature using the tangent planes. Thus, the metric is not trivial and distances need to reconsidered for this curvature, see Petersen et al. [24]. Now, we recall some definitions and results related to Riemannian and

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Hadamard manifolds presented in [14, 15, 18]. In this work, we extend some of these results on Hadamard manifold.Moreover, sufficient conditions of optimal solutions are presented and some new inequalities under new functions are proved.

Consider W m-dimensional Riemannian manifold, and T_pW tangent space to W at point p, if $\mu_p(x_1,x_2)$, the map $\mu: p \longrightarrow \mu_p$ is called a Riemannian metric where μ_p to T_pW. Further, a manifold W is equipped with μ known as a Riemannian manifold, see the details in [14, 15, 18] and [25].

The geodesic property is defined as the shortest possible line between two points on a sphere or another curved surface, or more generally in a Riemannian manifold. Thus, we define the length of curve $\alpha: [a_1,a_2] \longrightarrow W$ as:

$$L(\alpha) = \int_{a_1}^{a_2} \|\acute{\alpha}(x)\| dx.$$

Furthermore, if we let

$$d(p_1,p_2) = \inf\{L(\alpha): \alpha \in C^1 \ curve\ p_1 \to p_2\}$$

for any points $p_1, p_2 \in W$, d is a metric induced by topology on W. Note that for every Riemannian manifold W, there exists only one covariant derivation and it is also known as Levi-Civita connection $\bigtriangledown_X Y$, for $X,Y \in W$. Moreover, a geodesic smooth path α has tangent and satisfies $\bigtriangledown_{\acute{\alpha}(t)}\acute{\alpha}(t) = 0$. Any path α joining p_1 and p_2 in W that $L(\alpha) = d(p_1,p_2)$ is called a minimal geodesic. Similarly, Hadamard manifold is complete, simply connected manifolds and has non-positive sectional curvature on W, i.e having an exponential map $\exp_p : T_pW \longrightarrow W$ such that $\exp_p(v) = \alpha_v(1)$, on the whole tangent space of a point then α_v is also geodesic and applied as velocity of α.

Now, we recall the following definition and the details are found in [18].

Definition 1. *Assume that W is a Hadamard manifold, and $\eta : W \times W \longrightarrow TW$ is a function and while $\alpha : W \times W \longrightarrow \mathbb{R}\setminus\{0\}$ defined such that $\alpha(j_1,j_2)\eta(j_1,j_2) \in T_{j_2}W$, for all $j_1, j_2 \in W$. A non-empty subset $Y \subset W$ is called a geodesic α-invex ($G\alpha$ invex) set with respect to(w.r.t) η if there is a unique geodesic $\alpha_{j_1,j_2} : [0,1] \longrightarrow W$ such that*

$$\alpha_{j_1,j_2}(0) = v, \acute{\alpha}_{j_1,j_2}(0) = \alpha(j_1,j_2)\eta(j_1,j_2)$$

for $\alpha_{j_1,j_2}(t) \in Y$, and $0 \le t \le 1$.

The set Y is called G.α- invex set on a Hadamard manifold if

$$\exp_l(t\alpha(j_1,j_2)\eta(j_1,j_2)) \in Y,$$

for $j_1, j_2 \in Y$ and $0 \le t \le 1$.

Remark. If $\alpha(j_1,j_2) = 1$, the Definition 1 reduces to geodesic invex set [17].

2 Main Results

From now on, let W be a Hadamard manifold; TW be the tangent space of W and Y be a nonempty subset of W. Let $\eta : W \times W \longrightarrow TW$ and $\alpha : W \times W \longrightarrow \mathbb{R}\setminus\{0\}$ be functions such that for every $j_1, j_2 \in W$ then $\alpha(j_1,j_2)\eta(j_1,j_2) \in T_{j_2}W$. Also, let $b(j_1,j_2,t) : Y \times Y \times [0,1] \longrightarrow \mathbb{R}$ be a real value function.

Definition 2. *Assume that Y is G.α- invex set. The function $h : Y \longrightarrow TW$ is called a geodesic sub-(α,b,s)-preinvex, if there exists $b(j_1,j_2,t) : Y \times Y \times [0,1] \longrightarrow \mathbb{R}$ such that*

$$h(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2)) \le t^s h(j_1) + (1-t)^s h(j_2) + b(j_1,j_2,t),$$

for $j_1, j_2 \in Y$, $t \in [0,1]$ and $s \in (0,1]$.

Remark. 1.If $s = 1$ and $b(j_1,j_2,t) \le 0$, Definition 2 reduces to geodesic α-preinvex.
2.If $s = 1$, $\alpha(j_1,j_2) = 1$ and $b(j_1,j_2,t) \le 0$, Definition 2 reduces to geodesic convex.

Example 1. Consider $h : [0,+\infty) \longrightarrow \mathbb{R}$ is defined by

$h(x) = (x^2+4x)^s$, and $b(x,y,t) = tx^2 + 4ty^2$ for $s \in (0,1)$.

Now assume that

$$\alpha(t) = \exp_{j_2}(t\alpha(j_1,j_2)\eta(j_1,j_2))$$

where $\alpha(j_1,j_2) = 1$ and $\eta(j_1,j_2) = exp_{j_2}^{-1} j_1$. Then h is a $sub-(\alpha,b,s)$-preinvex.

Remark. When $\alpha(t) = tj_1 + (1-t)j_2$ in Example 1, h becomes the sub-b-s-convex function [13].

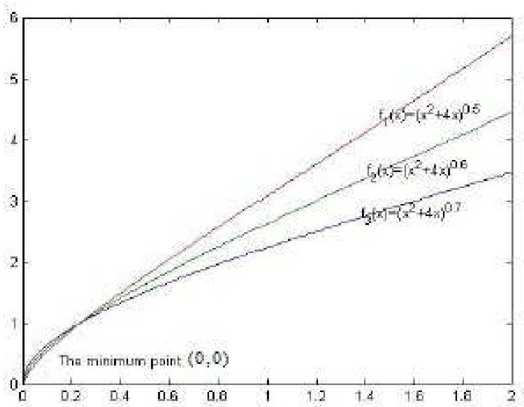

Fig. 1: h is sub-b-s-convex function

Theorem 1.*Assume that $f_1, f_2 : Y \longrightarrow TW$ are geodesic sub-(α,b,s)-preinvex then f_1+f_2 and $\beta f_1, \beta \geq 0$ are also geodesic sub-(α,b,s)-preinvex.*

The above-mentioned theorem defines that the geodesic sub-preinvex property is a linear property. Similarly, we can extend to above theorem and we have the following corollary.

Corollary 1.*If $h_\iota : Y \longrightarrow TW, (\iota = 1,2,\cdots,n)$ are geodesic sub-(α,b,s)-preinvex $b_\iota : Y \times Y \times [0,1] \longrightarrow \mathbb{R}$, $(i = 1,2,\cdots,n)$, respectively, then*

$$h = \sum_{\iota=1}^{n} \lambda_\iota h_\iota, \lambda_\iota \geq 0$$

is also geodesic sub-(α,b,s)-preinvex where $b = \sum_{\iota=1}^{n} \lambda_\iota b_\iota$.

Theorem 2.*Consider $h_1 : Y \longrightarrow TW \subseteq \mathbb{R}$ is a geodesic sub-(α,b,s)-preinvex function and $h_2 : K \longrightarrow \mathbb{R}$ is a non-decreasing convex function where $rang(h_1) \subseteq K$, then $h_1 o h_2$ is a geodesic sub-(α,b,s)-preinvex function b where $b = h_2 o b_1$.*

Proof.

$$\begin{aligned}
&(h_2 o h_1)\left(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2)\right) \\
&= h_2\left(h_1(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2))\right) \\
&\leq h_2\left(t^s h_1(j_1) + (1-t)^s h_1(j_2) + b_1(j_1,j_2,t)\right) \\
&= t^s h_2\left(h_1(j_1)\right) + (1-t)^s h_2\left(h_1(j_2)\right) + h_2\left(b_1(j_1,j_2,t)\right) \\
&= t^s\left(h_2 o h_1\right)(j_1) + (1-t)^s\left(h_2 o h_1\right)(j_2) + b(j_1,j_2,t) \quad (1)
\end{aligned}$$

which means that $h_2 o h_1$ is a geodesic sub-(α,b,s)-preinvex function.

The above-mentioned theorem indicates that under certain conditions the composition is invariant. Next, the definition of a geodesic sub-(α,b,s)-preinvex set.

Definition 3.*A set $Y \subseteq W$ is said to be a geodesic sub-(α,b,s)-preinvex set, if*

$$\left(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2), t^s\beta_1 + (1-t)^s\beta_2 + b(u_1,u_2,t)\right) \in Y,$$

$\forall (j_1,\beta_1),(j_2,\beta_2) \in Y$, $j_1, j_2 \in W$, $t \in [0,1]$, $s \in (0,1]$ and b defined as $b : Y \times Y \times [0,1] \longrightarrow \mathbb{R}$.

The epigraph of a geodesic sub-(α,b,s)-preinvex function $h : Y \longrightarrow TW$ can be explained as

$$\omega(h) = \{(j,r) : j \in Y, \beta \in \mathbb{R}, h(j) \leq \beta\}.$$

Now, in order to prove the sufficient and necessary rule for h to be a geodesic sub-(α,b,s)-preinvexn we need to study properties of geodesic sub-(α,b,s)-preinvex in terms of their epigraph $\omega(h)$.

Proposition 1.*Assume that $h_i : Y \longrightarrow TW$ are geodesic sub-(α,b,s)-preinvex functions with respect to maps $b_i : Y \times Y \times [0,1] \longrightarrow \mathbb{R}$, for $(i = 1,2,\cdots,n)$, then $H = \max h_i$ is also geodesic sub-(α,b,s)-preinvex where $b = \max b_i$.*

Theorem 3.*A function $h : Y \longrightarrow TW$ is geodesic sub-(α,b,s)-preinvex if and only if its epigraph is also a geodesic sub-(α,b,s)-preinvex.*

*Proof.*Assume that h is a geodesic sub-(α,b,s)-preinvex function and $(j_1,\beta_1),(j_2,\beta_2) \in \omega(h)$, then by hypothesis, $h(j_1) \leq \beta_1$ and $h(j_2) \leq \beta_2$. Furthermore,

$$\begin{aligned}
h\left(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2)\right) &\leq t^s h(j_1) \\
&\quad + (1-t)^s h(j_2) + b(j_1,j_2,t) \\
&\leq t^s\beta_1 + (1-t)^s\beta_2 + b(j_1,j_2,t). \quad (2)
\end{aligned}$$

Then,

$$\left(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2), t^s\beta_1 + (1-t)^s\beta_2 + b(j_1,j_2,t)\right)$$

is in $\omega(h)$. Thus, $\omega(h)$ is geodesic sub-(α,b,s)-preinvex set.
Next, let $\omega(h)$ be geodesic sub-(α,b,s)-preinvex set, then

$$(j_1,h(j_1)),(j_2,h(j_2)) \in \omega(h),$$

where $j_1, j_2 \in Y$.
$\left(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2), t^s h(j_1) + (1-t)^s h(j_2) + b(j_1,j_2,\delta)\right)$
is in $\omega(h)$ which shows that

$$h\left(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2)\right) \leq t^s h(j_1) + (1-t)^s h(j_2) + b(j_1,j_2,t).$$

Then h is geodesic sub-(α,b,s)-preinvex function.

Proposition 2.*If Y_i is a family of geodesic sub-(α,b,s)-preinvex sets, the intersection $\cap_{i\in K} Y_i$ is also a geodesic sub-(α,b,s)-preinvex.*

*Proof.*Suppose that $(j_1,\beta_1),(j_2,\beta_2) \in \cap_{i\in K} Y_i$. Then $(j_1,\beta_1),(j_2,\beta_2) \in Y_i, \forall i \in K$

$$\left(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2), t^s\beta_1 + (1-t)^s\beta_2 + b(j_1,j_2,t)\right) \in Y_i$$

$\forall i \in K$. This implies that

$$\left(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2), t^s\beta_1 + (1-t)^s\beta_2 + b(j_1,j_2,t)\right)$$

is in $\cap_{i\in K} Y_i$. Thus, the intersection $\cap_{i\in K} Y_i$ is a geodesic sub-(α,b,s)-preinvex set.

The aforementioned proposition indicates that the arbitrary intersection of geodesic sub-preinvex sets again is geodesic sub-preinvex. As per Theorem 3 and Proposition 2, the following proposition holds:

Proposition 3.*If h_i are geodesic sub-(α,b,s)-preinvex functions then a function $H = \sup_{i\in K} h_i$ is also geodesic sub-(α,b,s)-preinvex function.*

Definition 4. *For a mapping $h: Y \longrightarrow \mathbb{R}$, if the next limit*

$$\lim_{t\longrightarrow 0}\frac{h\left(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2)\right)-h(j_2)}{t\|\alpha(j_1,j_2)\eta(j_1,j_2)\|},$$

exists, h is called a (α,η)-differentiable mapping at $j_2 \in W$.
Also, the (α,η)-differentiable mapping of h at j_2 is given by

$$\begin{aligned} d_{\alpha(j_1,j_2)\eta(j_1,j_2)}h(j_2) &= \lim_{t\longrightarrow 0}\frac{h\left(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2)\right)}{t\|\alpha(j_1,j_2)\eta(j_1,j_2)\|} \\ &- \lim_{t\longrightarrow 0}\frac{h(j_2)}{t\|\alpha(j_1,j_2)\eta(j_1,j_2)\|}. \end{aligned}$$

Theorem 4. *Assume that Y is a G.α-invex set. If $h: Y \longrightarrow \mathbb{R}$ is (α,η)-differentiable geodesic sub-(α,b,s)-preinvex, the following holds*

$$\begin{aligned} d_{\alpha(j_1,j_2)\eta(j_1,j_2)}h(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\| &\le t^{s-1}h(j_1) \\ &+\frac{h(j_2)}{2t}+\lim_{t\longrightarrow 0_+}\frac{b(j_1,j_2,t)}{t}. \end{aligned}$$

Proof. Since h is a geodesic sub-(α,b,s)-preinvex, then it follows that

$$\begin{aligned} h\left(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2)\right) &\le t^s h(j_1)+(1-t)^s h(j_2) \\ &+b(j_1,j_2,t), \end{aligned}$$

$\forall j_1,j_2\in Y, t\in[0,1]$ and for some $s\in(0,1]$. Also, since h is (α,η)-differentiable, then

$$\begin{aligned} d_{\alpha(j_1,j_2)\eta(j_1,j_2)}h(j_2) &= \lim_{t\longrightarrow 0}\frac{h\left(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2)\right)}{t\|\alpha(j_1,j_2)\eta(j_1,j_2)\|} \\ &- \lim_{t\longrightarrow 0}\frac{h(j_2)}{t\|\alpha(j_1,j_2)\eta(j_1,j_2)\|}. \end{aligned}$$

Hence

$$\begin{aligned} h(j_2)+td_{\alpha(j_1,j_2)\eta(j_1,j_2)}h(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\| &+ O^2(t) \\ &= h\left(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2)\right) \\ &\le t^s h(j_1)+(1-t)^s h(j_2)+b(j_1,j_2,t) \\ &\le t^s h(j_1)+(1+t^s)h(j_2)+b(j_1,j_2,t). \end{aligned}$$

Then

$$\begin{aligned} td_{\alpha(j_1,j_2)\eta(j_1,j_2)}h(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\| &+ O^2(t) \\ &\le t^s\left[h(j_1)+h(j_2)\right]+b(j_1,j_2,t). \end{aligned}$$

Since $\lim_{t\longrightarrow 0_+}\frac{b(j_1,j_2,t)}{t}$ is the maximum of $\frac{b(j_1,j_2,t)}{t}-\frac{O^2(t)}{t}$, then we obtain that

$$\begin{aligned} &d_{\alpha(j_1,j_2)\eta(j_1,j_2)}h(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\| \\ &\le t^{s-1}\left[h(j_1)+h(j_2)\right]+\lim_{t\longrightarrow 0_+}\frac{b(j_1,j_2,t)}{t}. \end{aligned} \tag{3}$$

On the other hand, because of

$$\begin{aligned} h(j_2)+td_{\alpha(j_1,j_2)\eta(j_1,j_2)}h(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\|+O^2(t) \\ \le t^s h(j_1)+(1-t)^s h(j_2)+b(j_1,j_2,t) \\ = t^s h(j_1)+(1-t)^s h(j_2)-t^s h(j_2)+t^s h(j_2)+b(j_1,j_2,t) \\ = t^s\left(h(j_1)-h(j_2)\right)+b(j_1,j_2,t)+\left((1-t)^s+t^s\right)h(j_2). \end{aligned}$$

Hence, $((1-t)^s+t^s)\le 2$ for $t\in[0,1]$ and for $s\in(0,1]$. Here h is a non-negative function, then we have

$$\begin{aligned} h(j_2)+td_{\alpha(j_1,j_2)\eta(j_1,j_2)}h(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\|+O^2(t) \\ \le t^s\left(h(j_1)-h(j_2)\right)+b(j_1,j_2,t)+2h(j_2), \end{aligned}$$

which implies that

$$\begin{aligned} td_{\alpha(j_1,j_2)\eta(j_1,j_2)}h(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\|+O^2(t) \\ \le t^s\left(h(j_1)-h(j_2)\right)+h(j_2)+b(j_1,j_2,t), \end{aligned}$$

it follows that

$$\begin{aligned} &d_{\alpha(j_1,j_2)\eta(j_1,j_2)}h(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\| \\ &\le t^{s-1}\left(h(j_1)-h(j_2)\right)+\frac{h(j_2)}{t}+\lim_{t\longrightarrow 0_+}\frac{b(j_1,j_2,t)}{t}. \end{aligned} \tag{4}$$

Hence, by adding equations (3) and (4), the result is obtained.

Theorem 5. *Assume that $g: Y \longrightarrow \mathbb{R}$ is (α,η)-differentiable geodesic sub-(α,b,s)-preinvex then*

$$\begin{aligned} &d_{\alpha(j_1,j_2)\eta(j_1,j_2)}g(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\| \\ &\le t^{s-1}\left(g(j_1)-g(j_2)\right)+\lim_{t\longrightarrow 0_+}\frac{b(j_1,j_2,t)}{t}. \end{aligned}$$

Proof. If g is a geodesic sub-(α,b,s)-preinvex and also (α,η)-differentiable, then

$$\begin{aligned} d_{\alpha(j_1,j_2)\eta(j_1,j_2)}g(j_2) &= \lim_{t\longrightarrow 0}\frac{g\left(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2)\right)}{t\|\alpha(j_1,j_2)\eta(j_1,j_2)\|} \\ &- \lim_{t\longrightarrow 0}\frac{g(j_2)}{t\|\alpha(j_1,j_2)\eta(j_1,j_2)\|}, \end{aligned}$$

so

$$\begin{aligned} g(j_2)+td_{\alpha(j_1,j_2)\eta(j_1,j_2)}g(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\|+O^2(t) \\ \le t^s g(j_1)+(1-t)^s g(j_2)+b(j_1,j_2,t). \end{aligned}$$

Since $t\in[0,1]$ and $s\in(0,1]$, then $(t^s+(1-t)^s\ge 1)$, which implies that

$$\begin{aligned} g(j_2)+td_{\alpha(j_1,j_2)\eta(j_1,j_2)}g(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\|+O^2(t) \\ \le t^s g(j_1)+(1-t^s)g(j_2)+b(j_1,j_2,t), \end{aligned}$$

$$\begin{aligned} td_{\alpha(j_1,j_2)\eta(j_1,j_2)}g(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\|+O^2(t) \\ \le t^s\left(g(j_1)-g(j_2)\right)+b(j_1,j_2,t), \end{aligned}$$

$$\begin{aligned} &d_{\alpha(j_1,j_2)\eta(j_1,j_2)}g(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\| \\ &\le t^{s-1}\left(g(j_1)-g(j_2)\right)+\lim_{t\longrightarrow 0_+}\frac{b(j_1,j_2,t)}{t}. \end{aligned}$$

Next, we apply the aforementioned associated results to the non-linear programming. First, the following unconstrain problem (P) is considered

$$(P): \min f(x), x \in Y.$$

Theorem 6. *Assume that $g: Y \longrightarrow \mathbb{R}$ is a non-negative (α,η)-differentiable and sub-(α,b,s). If $\tilde{j} \in Y$ and the inequality*

$$d_{\alpha(j,\tilde{j})\eta(j,\tilde{j})} f(\tilde{j}) \|\alpha(j,\tilde{j})\eta(j,\tilde{j})\| \geq \frac{g(\tilde{j})}{t} + \lim_{t\longrightarrow 0_+} \frac{b(j,\tilde{j},t)}{t} \tag{5}$$

holds for $j \in Y$, $t \in (0,1]$ and $s \in (0,1]$, then $\tilde{j}$ is the optimal solution for problem (P) w.r.t. g on Y.

Proof. Using (4), we have

$$d_{\alpha(j,\tilde{j})\eta(j,\tilde{j})} g(\tilde{j}) \|\alpha(j,\tilde{j})\eta(j,\tilde{j})\| \leq t^{s-1}[g(j)-g(\tilde{j})] + \frac{g(\tilde{j})}{t} + \lim_{t\longrightarrow 0_+} \frac{b(j,\tilde{j},t)}{t},$$

$$d_{\alpha(j,\tilde{j})\eta(j,\tilde{j})} g(\tilde{j}) \|\alpha(j,\tilde{j})\eta(j,\tilde{j})\| - \frac{g(\tilde{j})}{t} - \lim_{t\longrightarrow 0_+} \frac{b(j,\tilde{j},t)}{t} \leq t^{s-1}[g(j)-g(\tilde{j})],$$

holds for $t \in (0,1]$ and $s \in (0,1]$ On the other hand,

$$d_{\alpha(j,\tilde{j})\eta(j,\tilde{j})} g(\tilde{j}) \|\alpha(j,\tilde{j})\eta(j,\tilde{j})\| \geq \frac{g(\tilde{j})}{t} + \lim_{t\longrightarrow 0_+} \frac{b(j,\tilde{j},t)}{t},$$

then get $g(j) - g(\tilde{j}) \geq 0$. Hence, $\tilde{j}$ is the optimal solution of g on Y.

Corollary 2. *Considering that $g: Y \longrightarrow \mathbb{R}$ is a strictly non-negative sub-(α,b,s)-preinvex. If $\tilde{j} \in Y$ satisfies (5), $\tilde{j}$ is a unique optimal solution.*

Proof. From (4) if g is a strictly non-negative and sub-(α,b,s)-preinvex,

$$d_{\alpha(j_1,j_2)\eta(j_1,j_2)} g(j_2) \|\alpha(j_1,j_2)\eta(j_1,j_2)\| \leq t^{s-1}[g(j_1)-g(j_2)] + \frac{g(j_2)}{t} + \lim_{t\longrightarrow 0_+} \frac{b(j_1,j_2,t)}{t}.$$

Assume that $u_1, v_1 \in Y$ are two different optimal solutions for (P). Then $g(u_1) = g(v_1)$, so

$$d_{\alpha(u_1,v_1)\eta(u_1,v_1)} g(v_1) \|\alpha(u_1,v_1)\eta(u_1,v_1)\| - \frac{g(v_1)}{t} - \lim_{t\longrightarrow 0_+} \frac{b(u_1,v_1,t)}{t} \leq t^{s-1}[g(u_1)-g(v_1)].$$

Applying 5, we get

$$t^{s-1}[g(u_1)-g(v_1)] > 0,$$

and since $g(u_1) = g(v_1)$, then it follows that $u_1 = v_1 = \tilde{j}$. Therefore, $\tilde{j}$ is the unique optimal solution of g on Y. Thus the corollary is proved.

Next, the following non-linear programming problem will be given

$$(P_Y): \min\{f(u) : u \in W, g_i(u) \leq 0, i \in I\}, I = \{1,2,\cdots,m\}.$$

Now assume feasible set of (P_Y) is given by $M = \{u \in W : g_i(u) \leq 0, i \in I\}$, and f and g_i are all differentiable and W_1 is a non-empty set in W. Then we have the next theorem.

Theorem 7(Karush-Kuhn-Tucher condition). *Assume that $f: W \longrightarrow \mathbb{R}$ is a non-negative (α,η)-differentiable sub-(α,b,s)-preinvex, and $g_i: W \longrightarrow \mathbb{R}$ $(i \in I)$ are (α,η)-differentiable sub-(α,b,s)-preinvex and*

$$d_{\alpha(j_1,j^*)\eta(j_1,j^*)} f(j^*) + \sum_{i\in I} z_i d_{\alpha(j_1,j^*)\eta(j_1,j^*)} g_i(j^*) = 0, \quad z_i g_i(j^*) = 0, \tag{6}$$

where $j^* \in M$ and $z_i \geq 0 (i \in I)$.
If

$$\frac{f(j^*)}{t} + \lim_{t\longrightarrow 0_+} \frac{b(j_1,j^*,t)}{t} \leq -\sum_{i\in I} \lim_{t\longrightarrow 0_+} \frac{b(j_1,j^*,t)}{t}, \tag{7}$$

then j^* is an optimal solution of (P_Y).

Proof. Assume that $j_1 \in P_Y$, then

$$g_i(j_1) \leq 0 = g_i(j^*), i \in I(j^*) = \{i \in I : g_i(j^*) = 0\}.$$

Since g_i are $sub-(\alpha,b,s)$-preinvex and by Theorem 5, we have

$$d_{\alpha(j_1,j^*)\eta(j_1,j^*)} g_i(j^*) \|\alpha(j_1,j^*)\eta(j_1,j^*)\| \leq t^{s-1}[g_i(j_1)-g_i(j^*)] + \lim_{t\longrightarrow 0_+} \frac{b(j_1,j^*,t)}{t},$$

which means that

$$d_{\alpha(j_1,j^*)\eta(u,v^*)} g_i(j^*) \|\alpha(j_1,j^*)\eta(j_1,j^*)\| - \lim_{t\longrightarrow 0_+} \frac{b(j_1,j^*,t)}{t} \leq t^{s-1}[g_i(j_1)-g_i(j^*)] \leq 0.$$

From 6, we get

$$\begin{aligned} &d_{\alpha(j_1,j^*)\eta(j_1,j^*)} f(j^*) \|\alpha(j_1,j^*)\eta(j_1,j^*)\| \\ &= -\sum_{i\in I} z_i d_{\alpha(j_1,j^*)\eta(j_1,j^*)} g_i(j^*) \|\alpha(j_1,j^*)\eta(j_1,j^*)\| \\ &= -\sum_{i\in I(j^*)} z_i d_{\alpha(j_1,j^*)\eta(j_1,j^*)} g_i(j^*) \|\alpha(j_1,j^*)\eta(j_1,j^*)\| \end{aligned} \tag{8}$$

Using equation 7, then

$$\begin{aligned} &d_{\alpha(j_1,j^*)\eta(j_1,j^*)} f(j^*) \|\alpha(j_1,j^*)\eta(j_1,j^*)\| - \frac{f(j^*)}{t} - \lim_{t\longrightarrow 0_+} \frac{b(j_1,j^*,t)}{t} \\ &\geq -\sum_{i\in I(j^*)} z_i \left[d_{\alpha(j_1,j^*)\eta(j_1,j^*)} g_i(j^*) \|\alpha(j_1,j^*)\eta(j_1,j^*)\| - \lim_{t\longrightarrow 0_+} \frac{b(j_1,j^*,t)}{t} \right]. \end{aligned} \tag{9}$$

From equations (8) and (9), we have

$$d_{\alpha(j_1,j^*)\eta(j_1,j^*)}f(j^*)\|\alpha(j_1,j^*)\eta(j_1,j^*)\| - \frac{f(j^*)}{t} - \lim_{t\longrightarrow 0_+}\frac{b(j_1,v^*,t)}{t} \geq 0.$$

From Theorem 6, we get $f(j_1) - f(j^*) \geq 0, \forall j_1 \in M$. Hence, j^* is an optimal solution of the problem (P_Y).

3 Conclusion

In this work, the properties of geodesic sub-(α,b,s)-preinvex functions on Hadamard manifolds are presented and some basic properties were studied in both general and differential cases. Furthere, sufficient conditions of optimal solutions was also studied and some new inequalities under geodesic sub-(α,b,s)-preinvexity such as the invariant of compositions were proved.

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