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New Prospective of the Equiconvergence Theorem

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Abstract: We prove a general form of the equiconvergence theorem using the method of V. A. Il'n. Horváth, Joó and Komornik which provides a general theorem for the one dimensional Schrödinger operator. We prove that the theorem for certain situation is more general than the previous theorem. In particular, we write down the difference of the trigonometric kernel of the general expansion and estimate the resulting infinite sums. For the terms of these sums we used different and sharper estimates than in the previous investigations.

Keywords: Differential operators, Equiconvergence, Eigenfunction Expansions, Schrödinger operator, Riesz bases, Fejér means.

1 Introduction

Many central problems of spectral theory of linear operators concentrate on the problem of eigenfunction expansions. On one hand, it accumulates questions of eigenvalues and eigenfunctions asymptotic and on the other it connects mathematics with many physical problems of string and membrane vibrations of quantum mechanics [1]. The difference of eigenfunction expansions converges to zero in any interior point of the main interval. This phenomenon was called equiconvergence and it makes possible to reduce numerous questions of point and uniform convergence to those of some model, usually, trigonometric system.

The equiconvergence theorems are very useful in the spectral investigation of differential operators, because many results known for the most special operators may be transferred by their applications to more general ones. One of the first results of this type was proved by A. Haar [2,3] in 1910-1911, and then by N. Wiener and J. L. Walsh in 1921 see [4].

In order to investigate eigenfunctions expansions, the three following estimates are essential:

1. Upper estimate of one eigenfunction.
2. Upper estimate of sum of squares of eigenfunctions.
3. Titchmarsh type mean value formula.

In 1977, a fruitful method was developed by V. A. Il'n (cf. [5,6]). His method works for the first and second estimates in the case of ordinary differential operator of second order (Schrödinger operator) with $q \in L^2$. For n^{th}-order differential operator (with smooth coefficients and using the fundamental solution) only the first estimate was proved, and he assumed that the second was fulfilled. I. Joó gave for both estimates a new procedure (cf. [7,8,9]) without using the fundamental solution in case $q \in L^1_{loc}$. These results led I. Joó and V. Komornik [10] to very general equiconvergence theorem for the Schrödinger operator. This theorem concerns expansions by Riesz bases formed by eigenfunctions of higher order of the Schrödinger operator. The existence of Riesz basis consisting of eigenfunctions of higher order was proved by V. P. Mikhailov [11] and G. m. Keselman [12]. As another illustration, we state Riesz bases $\{c_n e^{i\lambda_n t}\}_{n\in\mathbb{N}}$ with $\sup\lim \lambda = +\infty$; the construction is described in M. Horváth dissertation (cf. [13]). Joó's method made it possible to extend (1) and (2) for the differential operators of n^{th}-order with any (smooth or not) coefficients. This was conducted by V. Komornik as well as several authors (cf. [14,15,16]) using and developing some ideas and results of Joó–Komornik [10] and Joó's papers [7,8,9]. The results of Komornik are also new when the coefficients of the differential operator are smooth.

A generalization of the mentioned paper Joó and Komornik [10] is given in Komornik [17] (for higher order differential operators) and it is based on the results of (cf. [14,15,16]). Komornik had to extend the Titchmarsh formula [16] and needed the explicit formulas for their coefficients given by Joó [18]. A general equiconvergence theorem was published in [19] by Horváth, Joó and Komornik for the one dimensional Schrödinger operator without any restriction of the

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distribution of the eigenvalues on the complex plane, generalizing some known classical results of the field. The proof uses some estimates of [20] given by Joó.

2 The Methodology

The proof demonstrates that we write down the difference of the trigonometric kernel of the general expansion considered, and we have to estimate the resulting infinite sums. For the terms of these sums we used different and sharper estimates than in the previous pieces of literature (the most exact estimates were given by V. A. Il'n, I. Joó and V. Komornik). Now we explain the reason why our proof is harder and longer than Komornik's:
$f \in L^2$, (u_n) is not (known) Riesz bases:

$$\begin{aligned} &\int_G f(y)\left[w^R(|x-y|,\mu) - \sum_{\sqrt{\lambda_n}<\mu} u_n(x)v_n(y)\right].dy \\ &= \int \left[\sum_{n=1} c(\mu,\lambda_n)u_n(x)v_n(y)f(y)\right].dy \\ &= \sum_{n=1}^{\infty} c(\mu,\lambda_n) \underbrace{u_n(x)}_{\le \|u_n\|_{L^2(G)}} \int_G \underbrace{v_n(y)f(y)}_{\le \|v_n\|_{L^2(G)}\|f\|_{L^2(G)}} .dy \\ &\le \|f\|_{L^2(G)} \sum_{n=1}^{\infty} c(\mu,\lambda_n)\|u_n\|_{L^2(G)}\|v_n\|_{L^2(G)} \\ &\le c\|f\|_{L^2(G)} \sum_{k=1}^{\infty} |c(\mu,k^2)| \sum_{k\le\sqrt{\lambda_n}\le k+1} 1 \\ &\le c(x), \end{aligned} \tag{1}$$

$c(x)$ is dependent on μ, (we proved this). But if $f \in L^2$, and (u_n) is Riesz bases, $\sum |f|^2 < \infty$, and we can estimate as follows:

$$\begin{aligned} &\int_G \left[\sum_{n=1}^{\infty} c(\mu,\lambda_n)u_n(x)v(y)\right] f(y).dy \\ &= \sum_{n=1}^{\infty} c(\mu,\lambda_n)u_n(x) \int v(y)f(y).dy \\ &\le \left[\sum_{n=1}^{\infty} \left|c(\mu,\lambda_n)\underbrace{u_n(x)}_{\le c}\right|^2\right]^{\frac{1}{2}} \left[\sum_{n=1}^{\infty} |f_n|^2\right]^{\frac{1}{2}} \\ &\le c\sum_{n=1}^{\infty} |c(\mu,\lambda_n)|^2 \\ &= c\sum_{k=1} |c(\mu,k^2)|^2 . \sum_{k\le\sqrt{\lambda_n}\le k+1} 1 \\ &\le c(x), \end{aligned} \tag{2}$$

$c(x)$ is dependent on μ, has been proved by Komornik [16].

3 The Results

In [22], we generalized [19] using the estimates developed by Komornik in these papers [16,17], and by Joó in his paper [20]. Let G be an open interval (finite or infinite) on the real line, n is a natural number, $q_s \in L^1_{loc}(G)$ are complex functions, $s = 2,3,\ldots,n$, and consider the differential operator:

$$Lu := u^{(n)} + q_2(x)u^{(n-2)} + \ldots + q_n(x)u, \quad n \ge 2, \tag{3}$$

defined on $H^n_{loc}(G)$, (Recall that, by definition, $H^n_{loc}(G)$ is the set of all complex functions $v \in L^2_{loc}(G)$ having distributional derivatives in $L^2_{loc}(G)$ of order up to k). Given a complex number λ, the function, $u : G \to \mathbb{C}$, $u \equiv 0$, is called an eigenfunction of order -1 of the operator L with eigenvalue λ. Furthermore, a function $u : G \to \mathbb{C}$ is called an eigenfunction of order k, $k = 0,1,\ldots$, of the operator L with the eigenvalues λ if the function $u^* = Lu - \lambda u$ is an eigenfunction of order $(k-1)$ with the same eigenvalues λ. Now, let us give a complete and minimal system $(u_\alpha) \subset L^2(G)$ of eigenfunctions of the operator L, denoted by $\lambda_\alpha(resp.\ o_\alpha)$ the eigenvalue $(resp.\ order)$ of u_α and assume:
1. $\sup_\alpha o_\alpha < \infty$.
2. In case $o_\alpha > 0$, $\lambda_\alpha u_\alpha - Lu_\alpha = u_{\alpha-1}$.

We introduce some notations: Index the n^{th}-roots of λ_α such that $Re\ \mu_{1,\alpha} \ge \ldots \ge Re\ \mu_{n,\alpha}$, and put $\mu_\alpha := \mu_{m,\alpha}$, $Im\ \mu_{j,\alpha} > Im\ \mu_{j+1,\alpha}$ in case $Re\ \mu_{j,\alpha} = Re\ \mu_{j+1,\alpha}$, and put $\mu_\alpha := \mu_{m,\alpha}$, $\rho_\alpha := |Re\ \mu_\alpha|$, $\nu_\alpha := |Im\ \mu_\alpha|$, where $m = \left[\frac{n+1}{2}\right]$,

$$\delta(\nu,\nu_\alpha) = \begin{cases} 1, & \text{if } \nu > \nu_\alpha \\ \frac{1}{2}, & \text{if } \nu = \nu_\alpha \\ 0, & \text{if } \nu < \nu_\alpha, \end{cases} \tag{4}$$

$$W_R(t) = \begin{cases} \frac{\sin\nu(x-t)}{\pi(x-t)}, & \text{if } |x-t| \le R \\ 0, & \text{if } |x-t| > R, \end{cases} \tag{5}$$

where $x \in K$, K is an arbitrary fixed compact interval $K \subset G$, and $R \in (0, dist\ (K,\partial G))$;

$$D_{R_0}f := \frac{2}{R_0}\int_{\frac{R_0}{2}}^{R_0} f(R).dR,\ 0 < R_0 < dist\ (K,\partial G) \tag{6}$$

$$W(t) := D_{R_0}(W_R), \tag{7}$$

$$\sigma_\nu(f,x) := \sum_{\nu_\alpha<\nu} (f,\nu_\alpha)u_\alpha(x) + \sum_{\nu_\alpha=\nu} c_\alpha(f,\nu_\alpha)u_\alpha(x), \tag{8}$$

where c_α are arbitrary constants, $|c_\alpha| \le C$, and $\sum^*$ denotes the sum for any subset of $\{\alpha : \nu_\alpha = \nu\}$, $f \in L^2(G)$, $\nu > 0$, $x \in G$, (ν_α) is the dual system of (u_α), (i.e. $(\nu_\alpha) \subset L^2(G)$ and $\langle \nu_\alpha, u_j\rangle = \delta_{k,j}$);

$$S_\nu(f,x) := \int_{x-R}^{x+R} \frac{\sin\nu(y-x)}{\pi(y-x)} f(y).dy, \tag{9}$$

where $f \in L^2(G)$, $v > 0$, $x \pm R \in G$; $K_b := \{x \in G : dist(x,K) \le b\}$, and where $K \subset G$ is a compact interval and $0 < b < dist(K,\partial G)$. We prove the following:

Theorem 3.1. Assume that the above-mentioned assumptions (1) and (2) are satisfied, $q \equiv 0$, $u^*_\alpha \equiv 0$ and $\sup_{t>0} \sum_{t \le v_\alpha \le t+1} 1 < \infty$ are fulfilled. Then the following three statements are equivalent:

(a). For any compact interval $K \subset G$,

$$\sup_\alpha \|v_\alpha\|_{L^2(G)} \|u_\alpha\|_{L^2(G)} < \infty. \tag{10}$$

(b). For any compact interval $K \subset G$ and any subsume $\sum^*$,

$$\lim_{v\to\infty} \sup_{x\in K} |S_v(f,x) - \sigma_\alpha(f,x)| = 0, \tag{11}$$

for every $f \in L^2(G)$ and every $0 < R < dist(K,\partial G))$.

(c). For any compact interval $K \subset G$ and any subsume $\sum^*$,

$$\lim_{v\to\infty} \|f - \sigma_v(f)\|_{L^2(G)} = 0, \tag{12}$$

for every $f \in L^2(G)$.

In [22], we have proved the above-mentioned result. Now, we are working to eliminate the conditions $q_2 = 0$ and $u^* = 0$. The case $n = 4$, was proved by the author in [23] earlier, and in [24], we prove a general equiconvergence theorem for Fejér means. We will give some notations and theorems of our results in [24]:

Let G be an arbitrary (finite or infinite) open interval on the real line, $q, \hat{q} \in L^1_{loc}(G)$ be arbitrary complex functions. Let $(u_k)(resp.\ (\hat{u}_k))$ be a Riesz-basis in $L^2(G)$ consisting of eigenfunctions of the operator

$$Lu = -u'' + qu (\text{resp. } \hat{L}u = -u'' + \hat{q}u), \tag{13}$$

and having the following properties:

1. $\sup o_k < \infty$, $\sup \hat{o}_k < \infty$.
2. In case $o_k > 0$, $(resp.\ \hat{o}_k > 0)$, $\lambda_k u_k - Lu_k = u_{k-1}$, $(resp.\ \hat{\lambda}_k \hat{u}_k - \hat{L}\hat{u}_k = \hat{u}_{k-1})$, where λ_k and o_k $(resp.\ \hat{\lambda}_k$ and $\hat{o}_k)$, are the eigenvalues and the order of $(u_k)(resp.\ (\hat{u}_k))$.

Now, let us introduce some notations:

3. $R_\mu(f,x) := \sum_{\left|Re \sqrt{\lambda_k}\right| < 2\mu} \langle f, v_k \rangle u_k(x) \big(1 - \frac{\mu_k}{2\mu}\big)$, $\left(\mu_k = \sqrt{\lambda_k}\right)$,

$\hat{R}_\mu(f,x) := \sum_{\left|Re \sqrt{\hat{\lambda}_k}\right| < 2\mu} \langle f, \hat{v}_k \rangle \hat{u}_k(x) \big(1 - \frac{\hat{\mu}_k}{2\mu}\big)$, $\left(\hat{\mu}_k = \sqrt{\hat{\lambda}_k}\right)$,

where $f \in L^2(G)$, $x \in G$, $\mu > 0$, and $(v_k)(resp.\ (\hat{v}_k))$ is the dual system of $(u_k)(resp.\ (\hat{u}_k))$, i.e. $(v_k), (\hat{v}_k) \subset L^2(G)$ and $\langle v_k, u_j \rangle = \langle \hat{v}_k, \hat{u}_j \rangle = \delta_{k,j}$.

The following result holds:

Theorem 3.2. Given any compact interval $K \subset G$ for all $f \in L^2(G)$, (K is finite or infinite), then:

$$\lim_{\mu\to\infty} \sup_{x\in K} \left| R_\mu(f,x) - \hat{R}_\mu(f,x) \right| = 0. \tag{14}$$

Furthermore, we have proved the same theorem for $f \in L^1(G)$, (G is finite or infinite).

Remark: If we modify the definition of R_μ,

$$R^*_\mu(f,x) := \sum_{\left|Re \sqrt{\lambda_k}\right| < 2\mu} \langle f, v_k \rangle u_k(x) \big(1 - \frac{\rho_k}{2\mu}\big), \left(\rho_k = \sqrt{\lambda_k}\right). \tag{15}$$

Then, Theorem 3.2 remains true.

After that we investigate a special case. Denote $G = (0,+\infty)$, and

$$u_k(x) := \sqrt{2} x^{\alpha+1/2} e^{-x^2/2} l_k^{(\alpha)}(x^2), \tag{16}$$

$$q(x) := x^2 - 2\alpha - 2 + \frac{\alpha^2 - 1/2}{x^2}, \tag{17}$$

$$\lambda_k := 4k, \tag{18}$$

where $\alpha \ge -\frac{1}{2}$ and $l_k^{(\alpha)}(x)$ is named the Laguerre polynomial. We have proved the following theorem:

Theorem 3.3. If $f \in L^1(G)$, $f'(t)(1+t^2) \in L^1(G)$, and $\lim_{+\infty} f = 0$, for any compact interval $K \subset G$ and for any sufficient small $R > 0$, we have:

$$\sup_{x\in K} \left| F_\mu(f,x) - R_\mu(f,x) \right| = O\big(\frac{1}{\mu}\big), \tag{19}$$

where for $f \in L^2(G)$, $\mu > 0$ and $x \pm R \in G$, define

$$F_\mu(f,x) = F_\mu(f,x,R) := \frac{1}{\mu\pi} \int_{x-R}^{x+R} \Big(\frac{\sin\mu(y-x)}{y-x}\Big) f(y) dy \tag{20}$$

For the proof, we need the following Lemma.

Lemma: If $\alpha > -1$, then:

$$\sum_{a \le k < b} \left(\int_{x_1}^{x_2} u_k(x).dx \right)^2 \le c \frac{\sqrt{b-a}}{a} (x_2^4 + 1), \tag{21}$$

such that $0 \le x_1 \le x_2 < \infty$.

In addition, we frequently use the formulas of Szegö's book [29]. For example, we use the formula:

$$\sum_{a \le k < b} \left(\int_d^t x^\beta u_k(x).dx \right)^2 = \frac{1}{4\pi i^{\alpha-1}} \int_0^{2\pi} \int_d^t \int_d^t \frac{\exp\{-i\frac{x^2+y^2}{2} \text{ctg} \frac{\varphi}{2}\}}{\sin\frac{\varphi}{2}} x^{\beta+1/2} y^{\beta+1/2} .J_\alpha\left(-\frac{xy}{\sin\frac{\varphi}{2}}\right) .e^{-i\frac{\alpha}{2}\varphi - i\frac{b+a}{2}\varphi} \frac{\sin\frac{b-a}{2}\varphi}{\sin\frac{\varphi}{2}} .dxdyd\varphi. \tag{22}$$

For Applications, see [30,31].

4 Conclusion

Motivated by the results of the previous pieces of literature, [17,19,20,21], we investigated a general form of the equiconvergence theorem, using the method of V. A. Il'n. It is shown that the theorem for certain situation is more general than the previous theorem. We also proved a general equiconvergence theorem for Fejér means. We consider the Schrödinger operator with any complex potential function $q : G \to \mathbb{C}$ on any (finite or infinite) interval G, with arbitrary (complex) eigenvalues λ_n, see [21,24]. Finally, it is necessary to be stressed that the coefficients of the differential operators don't need to be assumed sufficiently smooth. Furthermore, there is no assumption on the distribution of the eigenvalues in the complex plane.

References

[1] A.M. Minkin, Equiconvergence theorems for differential operators, *Journal of Mathematical Sciences volume*, **96**, 3631–3715 (1999).

[2] A. Haar, Zur theorie der orthogonalen funktionensysteme I, *Math. Ann.*, **69**, 331-371 (1910).

[3] A. Haar, Zur theorie der orthogonalen funktionensysteme II, *Math. Ann.*, **71**, 38-53 (1911).

[4] J.L. Walsh and N. Wiener, The equivalence of expansions in terms of orthogonal functions, *Journal of Mathematics and Physics*, **2**, 103-122 (1922).

[5] S.A. Alimov, V.A. Il'in and E.M. Nikishin, Problems of convergence of multiple trigonometric series and spectral decompositions II, *Russian Mathematical Surveys*, **32**, 115-139 (1977).

[6] V.A. Il'n, On convergence of eigenfunction expansions in discontinuity points of the coefficients of the differential operator, *Mat. Zam.*, **22**, 679–698 (1977).

[7] I. Joó, On some question of spectral theory of one dimensional non-self adjoint Schrödinger operator with potential from L^1_{loc}, *DAN SSSR*, **250**, 29–31 (1980).

[8] I. Joó, An equiconvergence theorem, *DAN USSSR*, **4**, 6–8 (1983).

[9] I. Joó, Upper estimates for the eigenfunctions of the Schrödinger operator, *Acta Sci. Math. (Szeged)*, **44**, 87–93 (1982).

[10] I. Joó and V. Komornik, On the equiconvergence of expansions by Riesz bases formed by eigenfunctions of the Schrödinger operator, *Acta Sci. Math.*, **46**, 357–375 (1983).

[11] E.S. Panakhov and H. Koyunbakan, Inverse problem for singular Sturm–Liouville operator, *Proceeding of IMM of NAS of Azerbaijan*, 113-126, (2003).

[12] G.M. Keselman, On unconditional convergence of expansions with respect to the eigenfunctions of some differential operator, *Izv. SSSR Math.*, **2**, 82–93 (1964).

[13] M. Horváth. *On the Spectral Expansions of Laplace and Schrödinger Operator*. Ph.D. dissertation, The Budapest University of Technology and Economics, Hungary, (1991).

[14] V. Komornik, The asymptotic behaviour of the eigenfunctions of higher order of the Schrödinger operator, *Acta Mathematica Hungarica*, **40**, 287-293 (1982).

[15] V. Komornik, On the distribution of the eigenvalues of an orthonormal system consisting of eigenfunctions of higher order of a linear differential operator, *Acta Mathematica Hungarica*, **42**, 171-175 (1983).

[16] V. Komornik, Upper estimates for the eigenfunctions of higher order of a linear differential operator, *Acta Sci. Math. Seged*, **45**, 260–271 (1983).

[17] V. Komornik, On the equiconvergence of eigenfunction expansions associated with ordinary linear differential operators, *Acta Mathematica Hungarica*, **47**, 261-280 (1986).

[18] I. Joó, Remarks to a paper of V. Komornik, *Acta Sci. Math.*, **47**, 201-204 (1984).

[19] M. Horváth, I. Joó and V. Komornik, An equiconvergence theorem, *Ann. Univ. Sci. Budapest Sect. Math.*, **31**, 19–26 (1988).

[20] I. Joó, On the divergence of eigenfunction expansions, *Ann. Univ. Sci. Budapest Sect. Math.*, **32**, 3-36 (1989).

[21] I. Joó, Saturation theorems for Hermite-Fourier series, *Acta Mathematica Hungarica*, **57**, 169-179 (1991).

[22] I. Joó, M.B. Tahir and S. Szabó, A basis theorem for ordinary differential operators of n^{th}-order, *Ann. Univ. Sci. Budapest Sect. Math.*, **36**, 103–127 (1993).

[23] M.B. Tahir, An equiconvergence theorem, *Studia Sci. Math. Hung.*, **29**, 233–239 (1994).

[24] M.B. Tahir and S. Szabó, On the Fejér summability of eigenfunction expansins, *Ann. Univ. Sci. Budapest Sect. Math.*, **35**, 157-188 (1992).

[25] V.A. Il'n, *Spectral Theory of Differential Operators*, Nauka, Moscow, (1991).

[26] V. Komornik, Upper estimate for eigenfunctions, *Ann. Univ. Sci. Budapest Sect. Math.*, **27**, 125-135 (1984).

[27] A.M. Minkin. *Equiconvergence Theorems for Differential Operators*. Ph.D. Thesis, Saratov State University, Russia, (1982).

[28] U. Goginava and K. Nagy, Maximal operators of Fejér means of Walsh-Kaczmarz-Fourier series, *Journal of Function Spaces*, **8**, 181-200 (2010).

[29] G. Szegö, *Orthogonal Polynomials*, American Math. Soc., Providence USA, (1975).

[30] V.A. Marchenko, *Sturm-Liouville Operators and Applications*, American Mathematical Soc., Providence USA, (2011).

[31] L. Colzani, A. Crespi, G. Travaglini and M. Vignati, Equiconvergence theorems for Fourier-Bessel expansions with applications to the harmonic analysis of radial functions in Euclidean and non-Euclidean spaces, *Transactions of the American Mathematical Society*, **338**, 43-55 (1993).

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