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D-Integrity and E-Integrity Numbers in Graphs

Sultan Senan Mahde[1,*], *Alanod M. Sibih* [2] *and Veena Mathad* [3]

[1]Department of Mathematics, Faculty of Education and Science, Al-Baydha University, Al-Baydha, Yemen
[2]Department of Mathematics, Jamoum University College, Umm Al-Qura University, Saudi Arabia
[3]Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysuru, India

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Abstract: Inspired by the definition of integrity and the alternative formulations for integrity, we investigate the $D-$Integrity and $E-$Integrity numbers of a graph in the present study. The D-Integrity number of a graph G is denoted by $DI_k(G)$ defined as: $DI_k(G) = \sum_{k=1}^{p} D_k(G)$, and the E-Integrity number of a graph G, is denoted by $EI_l(G)$ defined as: $EI_l(G) = \sum_{l=0}^{p} E_l(G)$. In this paper, we establish the general formulas for the $D-$Integrity and $E-$Integrity numbers of some classes of graphs. Also, some properties of $D-$Integrity and $E-$Integrity numbers are established.

Keywords: Integrity, D-Integrity number, E-Integrity number

1 Introduction

Throughout this paper, we consider simple and undirected graphs. Let $G = (V,E)$ be such a graph. The number of vertices of G is denoted by p and the number of edges is denoted by q, so $|V(G)| = p$ and $|E(G)| = q$. The degree of a vertex v, denoted by $deg(v)$, is the number of vertices adjacent to v. The contraction of a vertex v in G (denoted by G/v) is the graph obtained by deleting v and putting a clique on the open neighbourhood of v, (note that this operation does not create multiple edges, and if two neighbours of v are already adjacent, they remain simply adjacent) [1]. A spider graph G_s is a tree which is constructed by subdividing each edge once in $K_{1,p-1}, p \geq 3$ [2]. If every pair of vertices of a graph G are adjacent, G is called a complete graph, and it is denoted by K_p with p vertices.

A graph G is called a bipartite graph if the vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 with a vertex of V_2. Furthermore, if every vertex of V_1 joins every vertex of V_2, G is a complete bipartite graph. The complete bipartite graph with two sets of vertices such that $|V_1| = n$, and $|V_2| = m$ is denoted by $K_{n,m}$. The graph $K_{1,p-1}$ is a star, or a star is a tree with at most one non-pendant vertex. If u and v are not adjacent vertices in G, the addition of edge $e = (u,v)$ yields the graph $G+e$. For the vertex set V and edge set $E \cup \{e\}$. For the terminology not defined here, we refer the reader to [3].

A network can be modelled by a graph whose vertices represent the nodes and edges represent the lines of communication. Its efficiency reduces when some vertices or edges are destroyed anyway. Various graph parameters have been used to describe the vulnerability of communication networks (graph), like connectivity, tenacity, and integrity. The concept of integrity of a graph G was introduced in [4] as a useful measure of the vulnerability of G. The authors in [4] compared integrity, connectivity, toughness and binding number for several classes of graphs. Their results suggested that integrity is appropriate for measuring vulnerability, and so it can distinguish between graphs that should have different measures of vulnerability. The integrity of a graph G is defined as

$$I(G) = \min\{|S| + m(G-S) : S \subseteq V(G)\},$$

where $m(G-S)$ denotes the order of the largest component of $G-S$. An I-set of G is any subset S of $V(G)$ for which

$$I(G) = |S| + m(G-S).$$

For more about integrity, see [5,6]. The authors in [7,8,9, 10,11,12] introduced the new concepts of integrity parameter. In (1990), Goddard and Swart [13] introduced two concepts that are useful computationally as follows:

* Corresponding author e-mail: sultan.mahde@gmail.com

Definition 1.1. [13] For any graph G, $D_k(G) = min\{|S| : S \subset V(G) \text{ and } m(G-S) \leq k\}$,

$$k = 1, 2, ..., p-1.$$

$E_l(G) = min\{m(G-S) : S \subset V(G) \text{ and } |S| = l\}$,

$$l = 0, 1, ..., p-1.$$

Motivated by the definition of $D_k(G)$ and $E_l(G)$, we introduce the D-integrity number and E-integrity number of graphs.
In this paper, $D-$Integrity and $E-$Integrity numbers of a graph are taken as a model of network and it is thought that it would become more stable and strong.
The robustness of a distributed system of computers can be represented by the integrity of the graph describing the network. D-Integrity and $E-$Integrity numbers of a graph can be used to describe the stability of communication networks, like telecommunication networks, computer networks, the internet, road and rail networks and other logistic networks.
In a big city, there were a lot of power plants and transformers that distributed electricity to all parts of the city. Those transformers were connected by electric cables. However, the transformers were old, so the municipality replaced them with new ones. Therefore, they had been changed at regular intervals so as not to encounter power cut throughout the whole city. In this way, some transformers were replaced and the electricity remained covering the largest possible number of city neighborhoods. But, such a hard task has been currently solved in that the graphs of the modern transformers consist of several corresponding vertices with some edges that are linked to the electrical cables between the transformers.

We observe that if $k = p$, $D_k(G) = 0$. These definitions prompted us to introduce the concepts of D-Integrity and E-Integrity numbers, as follows:

2 D-Integrity number

Definition 2.1. D-Integrity number of a graph G is denoted by $DI_k(G)$, and defined as:

$$DI_k(G) = \sum_{k=1}^{p} D_k(G).$$

Since $D_p(G) = 0$, we can also define the D-Integrity number as $DI_k(G) = \sum_{k=1}^{p-1} D_k(G)$. In this section, we define the value of D-Integrity number for some standard graphs.

Theorem 2.1. For a complete graph K_p, $DI_k(K_p) = \frac{p(p-1)}{2}$.

Proof. Consider the vertices $v_1, v_2, ..., v_p$ of K_p. For $m(K_p - S) \leq 1$, i.e., $m(K_p - S) = 0$ or 1, so we choose $S = \{v_1, v_2, ..., v_{p-1}, v_p\}$ or $S = \{v_1, v_2, ..., v_{p-1}\}$, correspondingly $|S| = p$, or $|S| = p-1$. So $D_1(K_p) = min\{|S| : m(K_p - S) = 0$ or $m(K_p - S) = 1\} = p-1$. Also, for $m(K_p - S) \leq 2$, $m(K_p - S) = 0, 1$ or 2, we consider $S = \{v_1, v_2, ..., v_{p-1}, v_p\}$, $S = \{v_1, v_2, ..., v_{p-1}\}$ or $S = \{v_1, v_2, ..., v_{p-2}\}$, so $D_2(K_p) = min\{|S| : m(K_p - S) = 0$, or $m(K_p - S) = 1$ or $m(K_p - S) = 2\} = p-2$. Then, $D_2(K_p) = p-2$. That means, if $m(K_p - S) \leq k, 1 \leq k \leq p-1$, then $D_k = p-k$ and finally, for $m(K_p - S) \leq p$, we have $S = \phi$. Thus, $D_p = 0$. Therefore,

$$\begin{aligned}
DI_k(K_p) &= \sum_{k=1}^{p} D_k(K_p) = D_1 + D_2 + D_3 + ... + D_{p-1} + D_p \\
&= p-1+p-2+p-3+...+p-(p-1)+0 \\
&= p-1+p-2+p-3+...+p-(p-1)+p-p \\
&= \sum_{i=1}^{p}(p-i) = \sum_{i=1}^{p} p - \sum_{i=1}^{p} i \\
&= p^2 - \frac{p(p+1)}{2} \\
&= \frac{p(p-1)}{2}.
\end{aligned}$$

Proposition 2.1. For a star graph $K_{1,p-1}$, $DI_k(K_{1,p-1}) = p-1$.

Proof. Let $V(K_{1,p-1}) = \{v, v_1, v_2, ..., v_{p-1}\}$, where v is the central vertex. Since $deg(v) = p-1$, the best chosen for S is $S = \{v\}$ in all cases, and $m(K_{1,p-1} - S) = 1$, so $D_1 = D_2 = D_3 = ... = D_{p-1} = 1$ and $D_p = 0$. Then,

$$\begin{aligned}
DI_k(K_{1,p-1}) &= \sum_{k=1}^{p} D_k(K_{1,p-1}) = D_1 + D_2 + D_3 \\
&+ ... + D_{p-1} + D_p \\
&= \underbrace{1+1+1+...+1}_{p-1 \text{ times}} + 0 \\
&= p-1.
\end{aligned}$$

Theorem 2.2. For a complete bipartite graph $K_{n,m}$,

$$DI_k(K_{n,m}) = \begin{cases} \frac{3n^2-n}{2}, & \text{if } n = m\,; \\ \frac{n^2+2nm-n}{2}, & \text{if } n \neq m, n < m. \end{cases}$$

Proof. Let $V(K_{n,m}) = \{v_1, v_2, ..., v_n, u_1, u_2, ..., u_m\}$, we discuss two cases:
Case 1: $n = m$, for $m(K_{n,n} - S) \leq 1$, i.e., $m(K_{n,n} - S) = 0$ or 1, we can choose $S = V$ or $\{u_1, u_2, ..., u_n\}$. Since $|\{u_1, u_2, ..., u_n\}| < |V|$, we have $min\{|S| : m(K_{n,n} - S) = 0$ or $m(K_{n,n} - S) = 1\} = n$ and so $D_1 = n$. Also, for $m(K_{n,n} - S) \leq k$ and $2 \leq k \leq n$, i.e., $m(K_{n,n} - S) = 2, 3, ..., n-1$ or n, we choose the sets S as follows:
$S = V, \{u_1, u_2, ..., u_n\}, V - \{u_1, v_1\}, V - \{u_1, v_1, u_2\}, V - \{u_1, v_1, u_2, v_2\}, ..., V - \{u_1, u_2, ..., u_{\lceil \frac{n}{2} \rceil}, v_1, v_2, ..., v_{\lfloor \frac{n}{2} \rfloor}\}$. Then, $D_2 = D_3 = ... = D_n = n$. Now, when

$n+1 \le k \le 2n-1$, for $m(K_{n,n}-S) \le k$, we consider $S = \{u_1,u_2,u_3,...,u_{2n-k}\}$ or $S = \{v_1,v_2,v_3,...,v_{2n-k}\}$. Hence, $D_k = \sum_{k=n+1}^{2n}(2n-k)$. Then,

$$\begin{aligned}
DI_k(K_{n,m}) &= D_1+D_2+D_3+D_4+...+D_n+D_{n+1} \\
&+ D_{n+2}+...+D_{2n-1}+D_{2n} \\
&= \underbrace{n+n+n+\ldots+n}_{n\ times}+\sum_{k=n+1}^{2n-1}(2n-k)+0 \\
&= n^2+\sum_{k=1}^{2n-1}(2n-k)-\sum_{k=1}^{n}(2n-k) \\
&= n^2+\sum_{k=1}^{2n-1}2n-\sum_{k=1}^{2n-1}k-\sum_{k=1}^{n}2n+\sum_{k=1}^{n}k \\
&= n^2+2n(2n-1)-\frac{(2n)(2n-1)}{2}-2n^2 \\
&+\frac{n(n+1)}{2} \\
&= \frac{3n^2-n}{2}.
\end{aligned}$$

Case 2: $n \neq m$, if $n<m$, for $m(K_{n,m}-S) \le k$, $1 \le k \le n$, i.e., $m(K_{n,m}-S) = 0,1,2,3,...,n-1$ or n. Consider $S = \{u_1,u_2,...,u_n,v_1,v_2,...,v_m\}$, $S = \{u_1,u_2,...,u_n\}$, $S = \{u_1,u_2,...,u_{n-1},v_1,v_2,...,v_{m-1}\}$, $S = \{u_1,u_2,...,u_{n-2}, v_1,v_2,...,v_{m-1}\}$,..., $S = \{u_1,u_2,v_1,v_2,...,v_{m-1}\}$ or $S = \{u_1,v_1,v_2,...,v_{m-1}\}$, so we have $min\{|S| : m(K_{n,m}-S) \le k, k = 1,2,...,n\} = min\{n+m,n,n+m-2,n+m-3,...,m+1,m\} = n$. Then $D_1 = D_2 = D_3 = ... = D_n = n$. For $m(K_{n,m}-S) \le k$, $n+1 \le k \le m$, consider $S = \{u_1,u_2,...,u_n\}$. Now, for $m(K_{n,m}-S) \le m+k$, $1 \le k \le n-1$, consider $S = \{u_1,u_2,...,u_{n-k}\}$. So $D_{m+k} = n-k$. Thus,

$$\begin{aligned}
DI_k(K_{n,m}) &= D_1+D_2+...+D_n+D_{n+1}+...+D_m \\
&+ D_{m+1}+D_{m+2}+...+D_{m+n-1} \\
&= \underbrace{n+n+n+\ldots+n}_{m\ times}+\underbrace{n+n+n+\ldots+n}_{m-n\ times} \\
&+\sum_{k=m+1}^{n+m-1}(n+m-k) \\
&= nm+n(m-n)+\sum_{k=1}^{n+m-1}(n+m-k) \\
&-\sum_{k=1}^{m}(n+m-k) \\
&= nm+n(m-n)+\sum_{k=1}^{n+m-1}(n+m)-\sum_{k=1}^{n+m-1}k \\
&-\sum_{k=1}^{m}(n+m)+\sum_{k=1}^{m}k \\
&= nm++n(m-n)+(n+m)(n+m-1) \\
&-\frac{(n+m-1)(n+m)}{2}-(n+m)m+\frac{m(m+1)}{2} \\
&= \frac{4nm-n^2-n}{2}.
\end{aligned}$$

Definition 2.2. [14] A double star $S_{n,m}$ is a tree with exactly two vertices that are not pendant vertices, with one adjacent to n pendant vertices and the other to m pendant vertices.

Theorem 2.3. For a double star graph $S_{n,m}$,

$$DI_k(S_{n,m}) = \begin{cases} 3n+1, \text{ if } n=m\,; \\ 2n+m+1, \text{ if } n \neq m, n<m. \end{cases}$$

Proof. Let $V(S_{n,m}) = \{u,u_1,u_2,...,u_n,v,v_1,v_2,...,v_m\}$. Two cases are discussed.

Case 1: $n=m$, for $m(S_{n,n}-S) \le k, 1 \le k \le n$. Consider $S = \{u,u_1,u_2,...,u_n,v,v_1,v_2,...,v_n\}$, $S = \{u,v\}, S = \{u,v_2,v_3,..,v_n\}, S = \{u,v_3,v_4,...,v_n\}$, ..., $S = \{u,v_n\}$, then we have $min\{|S| : m(S_{n,n}-S) \le k, k = 1,2,...,n\} = min\{2n+2,2,n,n-1,...,3,2\} = 2$. Then, $D_1 = D_2 = D_3 = ... = D_n = 2$. Now, if $n+1 \le k < 2n+1$, for $m(S_{n,n}-S) \le k$, consider $S = \{u\}$ or $S = \{v\}$, and when $k = 2n+1$, we consider $S = \{u_i\}$ or $S = \{v_i\}$, $1 \le i \le n$. Thus, $D_{n+1} = D_{n+2} = ... = D_{2n+1} = 1$. Then

$$\begin{aligned}
DI_k(S_{n,n}) &= D_1+D_2+D_3+...+D_n+D_{n+1}+D_{n+2} \\
&+ ...+D_{2n+1} \\
&= \underbrace{2+2+2+\ldots+2}_{n\ times}+\underbrace{1+1+1+\ldots+1}_{n+1\ times} \\
&= 2n+n+1 \\
&= 3n+1.
\end{aligned}$$

Case 2: $n \neq m$ and $n<m$. For $m(S_{n,m}-S) \le k, 1 \le k \le n$, i.e., $m(S_{n,m}-S) = 0,1,2,3,...,n-1$ or n. Consider $S = \{u,u_1,u_2,...,u_n,v,v_1,v_2,...,v_m\}$, $S = \{u,v\}, S = \{u,v_1,v_2,...,v_{n-1}\}, S = \{u,v_1,v_2,...,v_{n-2}\}, S = \{u,v_1,v_2,...,v_{n-3}\}$, ..., $S = \{u,v_1\}$. Hence, $min\{|S| : m(S_{n,m}-S) \le k, k = 1,2,...,n\} = min\{n+m+2,2,n,n-1,...,3,2\} = 2$. Then $D_1 = D_2 = D_3 = ... = D_n = 2$. In case $n+1 \le k \le n+m+1$, for $m(S_{n,m}-S) \le k$, consider $S = \{v\}$, $S = \{u,v_1,v_2,...,v_n\}$, $S = \{u,v_1,v_2,...,v_{n-1}\}$, $S = \{u,v_1,v_2,...,v_{n-2}\}$,..., $S = \{u_1,u_2,...,u_{n-1}\}$, $S = \{u_1,u_2,...,u_{n-2}\}$, it follows that $min\{|S| : m(S_{n,m}-S) \le k, k = n+1,n+2,...,n+m+1\} = min\{1,1,n+1,n,n-1,n-2,...,2,1\} = 1$. So $D_{n+1} = D_{n+2} = ... = D_{n+m+1} = 1$. Then,

$$\begin{aligned}
DI_k(S_{n,m}) &= D_1+D_2+D_3+...+D_n+D_{n+1}+D_{n+2} \\
&+ ...+D_{n+m+1} \\
&= \underbrace{2+2+2+\ldots+2}_{n\ times}+\underbrace{1+1+1+\ldots+1}_{m+1\ times} \\
&= 2n+m+1.
\end{aligned}$$

Theorem 2.4. For the spider graph G_s with $p \ge 3$ vertices,

$$DI_k(G_s) = 4p-4.$$

Proof. Let G_s be a spider graph shown in Figure 1, with $|V(G_s)| = 2p-1$ and $|E(G_s)| = 2p-2$.

Let $V(G_s) = \{u, v_1, v_2, ..., v_{p-1}, u_1, u_2, ..., u_{p-1}\}$. For $m(G_s - S) \leq 1$, then $m(G_s - S) = 0$ or $m(G_s - S) = 1$, consider $S = \{u, u_1, u_2, ..., u_{p-1}, v_1, v_2, ..., v_{p-1}\}$, $|S| = 2p-1$ or $S = \{u_1, u_2, u_3, ..., u_{p-1}\}$, $|S| = p-1$. Thus, $D_1(G) = min\{|S| : m(G_s - S) \leq k, k = 0 \text{ or } 1\} = min\{2p-1, p-1\} = p-1$. Then, $D_1 = p-1$. Now, for $m(G_s - S) \leq k$, $k \geq 2$. It is enough to take $S = \{u\}$ and so $D_2 = D_3 = D_4 = ... = D_{2p-2} = 1$. Therefore,

$$\begin{aligned}
DI_k(G_s) &= D_1 + D_2 + D_3 + D_4 + ... + D_{2p-3} + D_{2p-2} \\
&= p - 1 + \underbrace{1 + 1 + 1 + 1 + \ldots + 1}_{2p-3 \text{ times}} \\
&= p - 1 + 2p - 3 \\
&= 3p - 4.
\end{aligned}$$

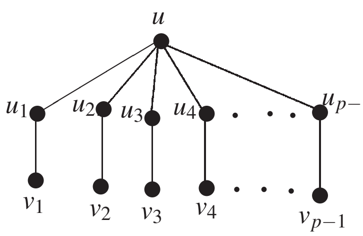

Figure 1: G_s

Observation 2.1. For any graph G,
(1) $D_1 = 1$, if and only if $G = K_{1,p-1}$ or $G = \overline{K_p} \cup K_{1,p-1}$.
(2) $DI_k(G) = p$, if $G = P_4, G = P_5, G = S_{1,2}$ or $G = K_{1,4} + e$.

Remark 2.1. $DI_k(G) = I(G)$ if $G = C_3$, C_5, P_3, $2P_2$, $P_2 \cup P_3$, $C_3 \cup \overline{K_p}$, or $P_3 \cup P_2 \cup \overline{K_p}$.

Proposition 2.2. For any graph G, $p - 1 \leq DI_k(G) \leq \frac{p(p-1)}{2}$. The lower bound is attained if $G = K_{1,p-1}$ and the upper bound holds if $G = K_p$.

Theorem 2.5. Let $G_1, G_2, ..., G_p$ be the components of a graph G. Then, $DI_k(G_1 \cup G_2 \cup ... \cup G_p) = DI_k(G_1) + DI_k(G_2) + ... + DI_k(G_p)$.

Proposition 2.3. For any $x \in V(K_p)$, $DI_k(K_p) = DI_k(K_p/x) + p - 1$.

Proof. Since $K_p/x = K_{p-1}$ and $DI_k(K_{p-1}) = \frac{(p-1)(p-2)}{2}$, then $DI_k(K_{p-1}) + p - 1 = \frac{(p-1)(p-2)}{2} + p - 1 = \frac{p(p-1)}{2} = DI_k(K_p)$.

Remark 2.2. Let S be I-set of a graph G. Then,
1) For any $x \notin S$, $DI_k(G/x) \leq DI_k(G)$
2) For some $x \in S$, there exist graphs G such that $DI_k(G/x) \geq DI_k(G)$. For example, $G = K_{1,p-1}, p \geq 4$, clearly $DI_k(K_{1,p-1}) = p - 1$, but $DI_k(K_{1,p-1}/x) = DI_k(K_{p-1}) = \frac{(p-1)(p-2)}{2}$.

3 E-Integrity number

Definition 3.1. E-Integrity number of a graph G is denoted by $EI_l(G)$ defined as:

$$EI_l(G) = \sum_{l=0}^{p} E_l(G).$$

Since $E_p(G) = 0$, we can define the E-integrity number as $EI_l(G) = \sum_{l=0}^{p-1} E_l(G)$.

Proposition 3.1. For any graph G, $p + 1 \leq EI_l(G) \leq \frac{p^2+p}{2}$. The lower bound is attained if $G = K_2$ and the upper bound holds if $G = K_p$.

Theorem 3.1. For the spider graph G_s with $p \geq 3$ vertices,

$$EI_l(G_s) = 5p - 5.$$

Proof.
Let G_s be a spider graph shown in Figure 1, with $|V(G_s)| = 2p-1$ and $|E(G_s)| = 2p-2$. Consider $V(G_s) = \{u, v_1, v_2, ..., v_{p-1}, u_1, u_2, ..., u_{p-1}\}$. When $|S| = 0$, we have $E_0 = m(G_s - S) = 2p - 1$. Now, for $|S| = 1$ to $|S| = p-2$, i.e., $m(G_s - S) = 2$, at $E_1, E_2, ..., E_{p-2}$. Also, $m(G_s - S) = 1$, when $|S| = p-1$ to $|S| = 2p-2$, so $E_{p-1} = E_p = ... = E_{2p-2} = 1$. Then,

$$\begin{aligned}
EI_l(G_s) &= E_0 + E_1 + E_2 + E_3 + ... + E_{p-2} + E_{p-1} \\
&\quad + E_p + ... + E_{2p-2} \\
&= 2p - 1 + \underbrace{2 + 2 + 2 + \ldots + 2}_{p-2 \text{ times}} + \underbrace{1 + 1 + 1 + \ldots + 1}_{p \text{ times}} \\
&= 2p - 1 + 2(p-2) + p \\
&= 5p - 5.
\end{aligned}$$

Theorem 3.2. For a complete graph K_p, $EI_l(K_p) = \frac{p^2+p}{2}$.
Proof. It is clear that $E_0 = p$. For $1 \leq l \leq p-1$, we have $m(K_p - S) = p - l$, then $E_1 = p-1, E_2 = p-2, E_3 = p-3, ..., E_{p-2} = p-(p-2)$, and $E_{p-1} = p-(p-1)$. Therefore,

$$\begin{aligned}
EI_l(K_p) &= E_0 + E_1 + E_2 + E_3 + E_4 + ... + E_{p-2} + E_{p-1} \\
&= p + \sum_{l=1}^{p-1} (p - l) \\
&= p + \sum_{l=1}^{p-1} p - \sum_{l=1}^{p-1} l \\
&= p + p(p-1) - \frac{p(p-1)}{2} \\
&= \frac{p^2 + p}{2}.
\end{aligned}$$

Corollary 3.1. For a complete graph K_p, $EI_l(K_p) - DI_k(K_p) = p$.

Proposition 3.2. For a star graph $K_{1,p-1}, EI_l(K_{1,p-1}) = 2p - 1$.

Proof. Let $V(K_{1,p-1}) = \{v, v_1, v_2, ..., v_{p-1}\}$ such that v is the central vertex. Put $l = 0$, then $m(K_{1,p-1} - S) = p$. For $|S| = 1$, the best choice for S is $S = \{v\}$, so $E_1 = 1$ and for $2 \le l \le p-1$, put $S = \{v\} \cup \{v_1, v_2, ..., v_{l-1}\}$. Thus $m(K_{1,p-1} - S) = 1$ for all $2 \le l \le p-1$ and hence $E_2 = ... = E_{p-1} = 1$. Then,

$$\begin{aligned} EI_l(K_{1,p-1}) &= E_0 + E_1 + E_2 + E_3 + E_4 + ... + E_{p-2} + E_{p-1} \\ &= p + \underbrace{1 + 1 + 1 + \ldots + 1}_{p-1 \; times} \\ &= p + p - 1 = 2p - 1. \end{aligned}$$

Theorem 3.3. For the double star $S_{n,m}$,

$$EI_l(S_{n,m}) = \begin{cases} 5n+3, \text{ if } n = m\,; \\ 3m+2n+3, \text{ if } n \neq m, n > m. \end{cases}$$

Proof. Let $V(S_{n,m}) = \{v, v_1, v_2, ..., v_n, u, u_1, u_2, ..., u_m\}$, we discuss two cases:
Case 1: $n = m$, $E_0 = 2n+2$ and when $|S| = 1$, the best choice for S is $S = \{u\}$ or $S = \{v\}$. Thus, $m(S_{n,n} - S) = n+1$. Hence, $E_1 = n+1$. In case $|S| = l, l \ge 2$, it is clear $m(S_{n,n} - S) = 1$. Then, $E_2 = E_3 = ... = E_{2n-1} = 1$. Therefore,

$$\begin{aligned} EI_l(S_{n,n}) &= E_0 + E_1 + E_2 + E_3 + E_4 + ... + E_{2n-2} + E_{2n-1} \\ &= 2n + 2 + n + 1 + \underbrace{1 + 1 + 1 + \ldots + 1}_{2n \; times} \\ &= 3n + 3 + 2n \\ &= 5n + 3. \end{aligned}$$

Case 2: $n \neq m, n > m$. $E_0 = n+m+2$. If $|S| = 1$, we have $min\{m(S_{n,m} - S) : |S| = 1\} = min\{n+1, m+1\} = m+1$, hence $E_1 = m+1$. It is clear $E_2 = E_3 = ... = E_{n+m+1} = 1$. Then

$$\begin{aligned} EI_l(S_{n,m}) &= E_0 + E_1 + E_2 + E_3 + E_4 \\ &+ ... + E_{n+m-2} + E_{n+m-1} \\ &= n + m + 2 + m + 1 + \underbrace{1 + 1 + 1 + \ldots + 1}_{n+m \; times} \\ &= 2m + n + 3 + n + m \\ &= 3m + 2n + 3. \end{aligned}$$

Theorem 3.4. For the complete bipartite graph $K_{n,m}$,

$$EI_l(K_{n,m}) = \begin{cases} \frac{3n^2+3n}{2}, \text{ if } n = m\,; \\ \frac{2nm+m^2+m+2n}{2}, \text{ if } n \neq m, \, n > m. \end{cases}$$

Proof. Let $V(K_{n,m}) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_m\}$, we have the following cases:
Case 1: $n = m$, we note that $E_0 = 2n$ and since we can choose any vertex of $V(K_{n,n})$ when $|S| = l, 1 \le l \le n-1$, we get $m(K_{n,n} - S) = 2n - l$, so $E_1 = 2n-1, E_2 = 2n-2, ..., E_{n-1} = 2n-(n-1) = n+1$. When $n \le l \le 2n-1$, we have $m(K_{n,n} - l) = 1$. Then,

$$\begin{aligned} EI_l(K_{n,n}) &= E_0 + E_1 + E_2 + ... + E_{n-1} + E_n \\ &+ ... + E_{2n-2} + E_{2n-1} \\ &= 2n + \sum_{l=1}^{n-1}(2n - l) + \underbrace{1 + 1 + 1 + \ldots + 1}_{n \; times} \\ &= 2n + \sum_{l=1}^{n-1} 2n - \sum_{l=1}^{n-1} l + n \\ &= 2n + 2n(n-1) - \frac{(n-1)n}{2} + n \\ &= \frac{3n^2 + 3n}{2}. \end{aligned}$$

Case 2: $n \neq m, n > m$, $E_0 = n+m$. If $|S| = l, 1 \le l \le m-1$, then $m(K_{n,m} - S) = n+m-l$, so $E_1 = n+m-1, E_2 = n+m-2, ..., E_{m-1} = n+1$. Also, if $|S| \ge m$, then $E_m = E_{m+1} = ... = E_{n+m-1} = 1$. Thus,

$$\begin{aligned} EI_l(K_{n,m}) &= E_0 + E_1 + E_2 + E_3 + E_4 + ... + E_{n+m-2} + E_{n+m-1} \\ &= n + m + \sum_{l=1}^{m-1}(n + m - l) + \underbrace{1 + 1 + 1 + \ldots + 1}_{n \; times} \\ &= n + m + \sum_{l=1}^{m-1}(n+m) - \sum_{l=1}^{m-1} l + n \\ &= m(n+m) - \frac{(m-1)m}{2} + n \\ &= \frac{2nm + m^2 + m + 2n}{2}. \end{aligned}$$

4 Conclusion

In this paper, we introduce the concept of D-integrity and E-integrity numbers in graphs. We also have obtained the D-integrity and E-integrity numbers of some graphs. The DI_k and EI_l of several other families of graphs are an open problem.

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Sultan Senan Mahde was born in Yemen. He got his B. Sc. degree in Mathematics in 2004 from Thamar university, Dhamar, Yemen. He got his M. Sc. degree from King Faisal university, Saudi Arabia and awarded Ph.D (2019) in Mathematics from university of Mysore, Mysuru, India. He has published more than 19 research articles in international and national journals of repute in the field of Graph Theory. His area of interest includes Graph theory, Discrete mathematics, and Analysis.

Alanod Sibih was born in Saudi Arabia. She got her B.Sc. degree in Mathematics in 2008 from Umm Al-Qura University, Makkah, Saudi Arabia. She got her M.Sc. degree from Dalhousie University, Halifax, Canada and awarded Ph.D. (2017) in Mathematics from Dalhousie University, Halifax, Canada. Her research interests are Graph theory, Algebra, Topology, Algebraic Topology.

Veena Mathad was born in India. She completed her M. Sc. (1995), and M.Phil. (1996) degree in Mathematics and awarded Ph.D (2005) in Mathematics from Karnatak university, Dharward, India. Her research interests are graph transformations, Domination in graphs, Distance parameters in graphs, Stability parameters of graphs.