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Perturbation Differential A-Infinity Algebra

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Abstract: In the present paper, we investigate and introduce the perturbation of dA_∞-algebra and the homotopy property (SDR-case). We also verify the homotopy theory of dA_∞-algebras and A_∞- differential module. In addition, We construct a property of homotopy invariant property of A_∞-differential algebras.

Keywords: A-infinity-Differential module -Homotopy -Homology theory.

1 Introduction

According to perturbation hypothesis, speculation is a peneficial procession to get relatively small differential complexes expressing an assumed chain homotopy type. The use of perturbation method in differential homological algebra has a long history, much of it was indicated in [1]. Stasheff in [2] started the possibility of an A-infinity space, since it is continuous associative multiplication and homotopy invariant parallel to topological space. One of the primary homes of the structure in an A- infinity space is its homotopy invariance, as the stability of this structure which is estimation to the arbitrary homotopy equivalence to topological spaces. In [3], [4] and [5], the graded A-infinity algebras applications to the sort of homologies of twisted tensor products and homologies of differential algebras are addicted. In [1] and [3] applications of differential A-infinity algebras to mathematical physics, topology, and geometry are stated. In [1], they induce the universal of the D-infinity differential A-infinity algebra, that is a homotopically invariant quantum analogue of the universal of a differential A-infinity algebra. Lodder, Lambe, and Stasheff ([4],[6] and [2]) began the perturbation of differential module intention and established the homotopy invariance property of differential module perturbation. They ordained the dependence between the homotopy of a structure of the differential A-infinity (to short A_∞-algebra) and differential perturbations. In [1], Lapin presented the concept of D-infinity differential module (shortly D_∞-module), and detected the relation between D-infinity differential module and perturbations differential module. In the coincident work, we define and ponder the perturbation of D-infinity A-infinity algebra and its homotopy invariant characteristic (SDR-case).
We recollect some fundamental facts existent in the sequel.

Definition 1.1 [7] We can define a differential algebra (A,d,π) as (A,d) which is differential module over an algebra with the multiplication map $\pi : A\otimes A \longrightarrow A$ such satisfy the associate law, $(1\otimes\pi)\pi = (\pi\otimes 1)\pi$ holds.

Definition 1.2 For any arbitrary algebra A, the form (A,d,π_i) is referred to as A - infinity algebra, since the graded module over algebra (A,d) such that:

$$\sum_{i=0}^{n}(-1)^{\varepsilon}\pi_i(1\otimes\ldots\otimes\pi_{n-1}\otimes\ldots\otimes 1)=0,\quad \varepsilon = nk+ik+n+k$$

Definition 1.3[8] A D_∞-module A together with a set of the operations $\pi_n : A^{\otimes n+2} \longrightarrow A, n\geq 0$ is called differential A_∞-algebra (dA_∞-algebra), with the following identity;

$$d(\pi_{n-1}) = \sum_{i=0}^{n}(-1)^{\varepsilon}\pi_i(1\otimes\ldots\otimes\pi_{n-1}\otimes\ldots\otimes 1)=0, \\ \varepsilon = nk+ik+n+k \tag{1}$$

Definition 1.4 The homomorphism $f : (X_1,d,\pi_n) \longrightarrow (X_2,d,\pi_n)$ of the dA_∞-algebras is the set; $f = f_n : X_1^{\otimes(n+1)} \longrightarrow X_2 | f_n(X_1^{\otimes(n+1)})_* \subseteq X_{2_{*+n}}, n\in Z, n\geq 0$

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that fulfill the accompanying connection: for integer, $n \geq -1$,

$$df_{n+1} + (-1)^n f_{n+1} = \sum_{m=0}^{n} (-1)^{n_2+n_4+\cdots} f_{n-m}(1 \otimes \ldots \otimes 1 \otimes \pi_m \otimes 1 \otimes \ldots \otimes 1) + \sum_{m=0}^{n} (-1)^{t(m+1)+n} \pi_m (f_{n_1} \otimes \ldots \otimes f_{n_{m+2}}) \quad (2)$$

where $n_1 + \ldots + n_{n-m}, \pi_m$ can be located and the sum is appropriated over all locations t.

Definition 1.5 A family of morphisms $f = f_n : X \longrightarrow Y$ and $g = g_n : Y \longrightarrow Z$ of dA_∞-algebras, its composition $gf = (gf)_n : (X, d, \pi_n) \longrightarrow (Z, d, \pi_n)$ is defined by:

$$((gf)_n = \sum_{m=0}^{n} g_m (f_{n_1} \otimes \ldots \otimes f_{n_{m+2}})), \quad n \geq 0$$

and $n_1 + \ldots + n_{m+1} = n - m.$

Definition 1.6 We can define the family of maps as follows:

$$h = h_n : X_1^{\otimes(n+1)} \longrightarrow X_2 | h_2(X_1^{\otimes(n+1)})_* \subseteq X_{2_{*+n+1}} \quad n \in Z, n \geq 0$$

as the homotopy morphism, $h : X_1 \longrightarrow X_2$ between the morphisms of the dA_∞-algebras, $f = f_n, g = g_n : (X_1, d, \pi_n) \longrightarrow (X_2, d, \pi_n)$ such that satisfy the relation: $\forall n \geq -1$,

$$dh_{n+1} + (-1)^{n+1} h_{n+1} = f_{n+1} + g_{n+1} + \sum_{m=0}^{n} (-1)^{t(m+1)+n+1} h_{n-m}(1 \otimes \ldots \otimes 1 \otimes \pi_m \otimes 1) + \sum_{m=0}^{n} (-1)^{(m+1)+\varepsilon(t)} \pi_m (g_{n_1} \otimes \ldots \otimes g_{n_{t-1}} \otimes h_{n_t} \otimes f_{n_{t+1}} \otimes \ldots \otimes f_{n_{m+2}})$$

and the sum over t. And π_m and h_m can be situated, and;

$$\varepsilon(t) = n_2 + n_4 + \ldots + n_{2[t/2]} + n_{2[(t+1)/2]+1} + n_{2[(t+1)/2]+3} + \ldots + n_{2[(m+1)/2]+1}.$$

Definition 1.7[9] Consider two arbitrary differentials A-infinity algebras X_1 and X_2. The triple system $\eta : X_1 \leftrightharpoons X_2$ defines the strong differential retract of a dA_∞-algebras, since the maps; $\eta : X_1 \longrightarrow X_2$, $\xi : X_2 \longrightarrow X_1$ are differential module morphisms and satisfy; $\eta\xi = 1_{X_2}$ and h is defined to be the homotopy between η, ξ and 1_X. If $h = 0, h\xi = 0, hh = 0$ hold, we can call the triple: $\eta : X_1 \leftrightharpoons X_2 : h, \xi$ SDR-case of a differential modules.

We give an excellent precedent of a differential SDR-case as a homology of the differential module X_1 over the field K, defined by $H(X_1) = Ker\ d/Im\ d$, to be the homology module of the differential module (X_1, d) over K. If $H(X_1)$ defined as the differential module such the differential is zero, then SDR-case $\eta : X_1 \leftrightharpoons X_2 : h, \xi$ for the differential modules, referred to as homology SDR-case of differential modules, via the use of the decomposition of fixed direct sum $(Ker\ d = H(X_1) \oplus Im\ d)$.

Definition 1.8 The differential perturbation of the dA_∞-algebra (X_1, d, π_i) is the differential perturbation of the differential (X_1, d) modules satisfying;

$$t^i \pi = \pi(1 \otimes t^i + t^i \otimes 1)$$

$t : X_1 \longrightarrow X_1$ with differential module $(X_1, d + t)$, i.e., the mapping $d + t : X_1 \longrightarrow X_1$ satisfies the rule: $(d + t)^2 = 0$. Clearly, any $t : X_1 \longrightarrow X_1$ of the differential module (X_1, d) satisfies, $dt + td = -t^2$. For any (X_1, d) module there is a new map $D : X_1 \longrightarrow X_1 \quad s.h. \quad t = D - d : X_1 \longrightarrow X_1$.

Definition 1.9 A graded D_∞-module (X, d^i) with the set of the maps;

$$(\pi_n^i : X^{\otimes(n+2)} \longrightarrow X_* \mid \pi_n^i (X^{\otimes(n+1)})_* \subseteq X_{*+n}, \ n \geq 0, \ i \geq 0)$$

of modules is defined a differential A_∞- algebra if for the integer numbers;$n \geq 0$ and $k \geq 0$, the following relations hold:

$$\sum_{i+j=k} d^i \pi_0^j = \sum_{i+j=k} \pi_0^i d^j$$

where

$$\sum_{i+j=k} d^i \pi_{n+1}^j + (-1)^n \pi_{n+1}^i d^j = \sum_{i+j=k} \sum_{m=0}^{n} (-1)^{t(m+1)+n} \pi_{n-m}^i (1 \otimes \ldots \otimes 1 \otimes \pi_m^j \otimes 1 \otimes \ldots \otimes 1)$$

With the sum is over all t,and π_m^j can be situated.

Example 1.10 For an A_∞-algebras (X, d^i, π). We hold the A_∞ of dA_∞-algebra (X, d^i, π^i) if we set: $\pi_0^0 = \pi, \ \ \pi_n^i = \pi$ for $(n, i) \neq (0, 0)$.

Definition 1.11 The homomorphism; $f : X_1 \longrightarrow X_2$ of the dA_∞-algebra X_1 and X_2 is the set;

$$f = (f_n^i : X_1^{\otimes(n+1)} \longrightarrow X_{2_\bullet} | f_n^i (X_1^{\otimes(n+1)})_\bullet \subseteq X_{2_{\bullet+n}}, \ n \geq, i \geq 0)$$

of mappings of modules such that for any integer $n \geq 0$ and $k \geq 0$, then: $\sum_{i+j=k} d^i f_0^j = \sum_{i+j=k} f_0^i d^j$ where,

$$\sum_{i+j=k} d^i f_{n+1}^j + (-1)^n f(n+1)^i d^j = \sum_{i+j=k} \sum_{m=0}^{n} (-1)^{t(m+1)+n} f_{n-m}^i (1 \otimes \ldots \otimes 1 \otimes \pi_m^j \otimes 1 \otimes \ldots \otimes 1)$$

$$+ \sum_{i+j=k} \sum_{m=0}^{n} (-1)^{n_2+n_4+\cdots} \pi_m^i (f_{n_1}^{j_1} \otimes \ldots \otimes f_{n_{m+2}}^{j_{m+2}})$$

Where; $n_1 + \ldots + n_{m+2} = n - m, \quad j_1 + \ldots + j_{m+2} = j$, since the sum over t, and π_m^j can be situated.

Definition 1.12 The composition $gf = (gf)_n^i : (X, d^i, \pi_n^i) \longrightarrow (Z, d^i, \pi_n^i)$ of morphisms

$f = f_n^i : X \longrightarrow Y$ and $g = g_n^i : Y \longrightarrow Z$ of the dA_∞-algebras is defined as:

$$((gf)_n^k = \sum_{i+j=k} \sum_{m=0}^{n} g_m^i (f_{n_1}^{j_1}) \otimes ... \otimes f_{n_{m+1}}^{j_{m+1}}), \ n \geq 0, \ k \geq 0,$$

where, $n_1 + ... + n_{m+1} = n - m$ and $j_1 + ... + j_{m+1} = j$.

Definition 1.13 The homotopy between amorphism $f, g : X_1 \longrightarrow X_2$ of the dA_∞-algebras is the set of a function;

$$h = h_n^i : X_1^{\otimes(n+1)} \longrightarrow X_{2_\bullet} | (h_n^i X_1^{\otimes(n+1)})_\bullet \subseteq X_{2_{\bullet+n}} \quad n \geq 0, \ i \geq 0$$

that fulfills the accompanying connection: for any integer numbers, $k \geq 0, \ n \geq 0$;

$$\sum_{i+j=k} dh_0^j + h_0^i d^j = f_0^k - g_0^k,$$

$$\begin{aligned} &\textstyle\sum_{i+j=k} d^i h_{n+1}^j + (-1)^{t(m+1)+n+1} h_{n+1}^i d^j \\ &= f_{n+1}^k - g_{n+1}^k \textstyle\sum_{i+j=k} \sum_{m=0}^{n} (-1)^{t(m+1)+n+1} h_{n-m}^i (1 \otimes \\ &\quad ... \otimes 1 \otimes \pi_m^j \otimes 1 \otimes ... \otimes 1) \\ &+ \textstyle\sum_{i+j=k} \sum_{m=0}^{n} (-1)^{(m+1)+\varepsilon(t)} \pi_m^i (g_{n_1}^{j_1} \otimes g_{n_{t-1}}^{j_{t-1}} \otimes h_{n_t}^{j_t} \\ &\quad \otimes f_{n_{t+1}}^{j_{t+1}} \otimes ... \otimes f_{n_{m+2}}^{j_{m+2}}) \end{aligned}$$

the sum over all t, since π_m^j and $h_{n_t}^{j_t}$ can be situated, $n_1 + ... + n_{m+2} = n - m$ and $j_1 + ... + j_{m+2} = j$ and

$$\varepsilon(t) = n_2 + n_4 + ... + n_{2[t/2]} + n_{2[(t+1)/2]+1} + n_{2[(t+1)/2]+3} + ... + n_{2[m+1/2]+1}.$$

2 Results

We describe and discuss the homotopy of dA_∞-algebras properties. Let any two dA_∞-algebras X, Y and $(\eta : X \leftrightharpoons Y : \xi, h^i)$ be an SDR-case of A_∞-modules, where ξ, η are the morphisms of A_∞-algebras, the map h is a homotopy map between ξ, η and 1_X of A_∞ of dA_∞-algebras. Then SDR-situation of A_∞-modules defines dA_∞-algebras SDR-situation.

Definition 2.1 The differential perturbation of A_∞-algebras (X_1, d^i, π^i) is characterized to be the set of the maps formula; $t^i : X_1 \longrightarrow X_1, i \geq 1, \ i \in Z, \ t^0 = 0$ such that,

$$\sum_{i+j=k} d^i t^j + \sum_{i+j=k} t^i d^j = - \sum_{i+j=k} t^i t^j, \forall k \geq 1 \qquad (3)$$

From this relation, any differential perturbation in the form, $t^i : X_1 \longrightarrow X_1, t^0 = 0, i \in Z, i \geq 1$ of an arbitrary A_∞-algebras (X_1, d^i, π^i), there exists a new dA_∞ $s.h.$ $D^i = d^i + t^i, i \geq 0, i \in Z$.

Notethat :

i- If we put $k = 1$, equation (3) satisfies the relation: $d^0 t^1 + t^1 d^0 = 0$, that is anti-commutative $(i.e. \ d^0 t^1 = -t^1 d^0)$.

ii- If $k = 2$, then $d^0 t^2 + t^2 d^0 = -(d^1 t^1 + t^1 d^1 + (t^1)^2$: Then the map $t^1 : X_1 \longrightarrow X_1$ is homotopic to the map $d^1 : X_1 \longrightarrow X_1$. Along these lines we introduce a new definition of differential D_∞-module (X_1, d^i) the perturbation t^i $s.h.$ $t^1 = t, d^0 = 0$, and $t^i = 0, i \geq 2$.

Example 2.2 The perturbation A_∞-algebras (X_1, d^i, π^i) can be built up by taking the filtration differential module over a self-assertive field, the filtration differential algebra, X_1^n, $d(X_1^n) \subseteq X_1^n$, $n \geq 0$ and there is (X, d) which is the differential module for differential perturbation such that satisfies the condition $(X_1^n) \subseteq X_1^{n-1}, n \geq 1$. Suppose the sub-module X_2^n on X_1^n, such that $X_1^n = X_2^n \oplus X_1^{n-1}$, then: $t_n^i : X_2^n \longrightarrow X_2^{n-1}$, such that

$$t : X_2^n \longrightarrow X_1^{n-1} = X_2^{n-1} \oplus ... \oplus X_2^{n-i} \oplus ... \oplus X_2^0.$$

Clearly, the set $t^i : X_1 \longrightarrow X_1$, $i \geq 1$, where $t^i = 0$, $t^i = \oplus_{n \geq 0} \oplus t_n^i$, $i \geq 1$ is a perturbation of dA_∞-algebras (X_1, d^i, π^i) , since; $td + dt = -t^2$.

To examine the perturbation homotopy invariant of A_∞-algebra (X_1, d^i, π^i) let the deformation be as follows:

$$\eta^i : ((X, d^i) \longrightarrow (Y, d^i) : \xi^i, h^i)$$

of differential D_∞- module, and the perturbation $t^i : X_1 \longrightarrow X_1$ for differential D_∞- module (X_1, d^i). Our plan to set up the perturbation $t_*^i : X_2 \longrightarrow X_2$ of dA_∞-module (X_2, d^i). Clearly, $t_*^0 = 0$. Let, $t_*^1 = \eta^0 \ t^1 \ \xi^0$, and using the relation $d^0 t^1 + t^1 d^0 = 0$ we get

$$\begin{aligned} d^0 t_*^1 + t_*^1 d^0 &= d^0(\eta^0 t^1 \xi^0) + (\eta^0 t^1 \xi^0) d^0 = \eta^0 d^0 t^1 \xi^0 \\ &+ \eta^0 t^1 d^0 \xi^0 = \eta^0 (d^0 t^1 + t^1 d^0) \xi^0 = 0 \end{aligned} \qquad (i)$$

Let us define the map t_*^2 by the

$$t_*^2 = \eta^0 t^2 \xi^0 + \eta^1 t^1 \xi^0 + \eta^0 t^2 \xi^1 + \eta^0 t^1 h^0 t^1 \xi^0.$$

From the relation (i) we have

$$d^0 t^2 + d^1 t^1 + t^2 d^0 = -t^1 t^1 \qquad (ii)$$

for given maps t_*^1 and t_*^2 we get the accompanying connection,

$$d^0 t_*^2 + d^1 t_*^1 + t_*^1 d^1 + t_*^2 d^0 = -t^1 t_*^1,$$

since

$$\begin{aligned} d^1 t_*^1 &= d^1(\eta^0 t^1 \xi^0) = \eta^1 d^0 t^1 \xi^0, \\ t_*^1 d^1 &= (\eta^0 t^1 \xi^0) d^1 = \eta^0 t^1 d^0 t^1 \xi^1, \\ d^0 t_*^2 &= \eta^0 d^0 t^2 \xi^0 + \eta^0 d^1 t^1 \xi^0 + \eta^0 t^1 d^1 \xi^0 + d^0 \eta^0 t^1 h^0 t^1 \xi^0, \\ t_*^2 d^0 &= \eta^0 t^2 d^0 \xi^0 + \eta^0 t^1 d^0 t^1 \xi^0 + \eta^0 t^1 d^1 \xi^0 + \eta^0 t^1 h^0 t^1 \xi^0 d^0, \\ t_*^1 t_*^1 &= (\eta^0 t^1 \xi^0)(\eta^0 t^1 \xi^0) = \eta^0 t^1 (d^0 h^0 + h^0 d^0 - 1) t^1 \xi^0 \\ &= \eta^0 t^1 d^0 h^0 t^1 \xi^0 + \eta^0 t^1 h^0 d^0 t^1 \xi^0 - \eta^0 t^1 t^1 \xi^0 \\ &= -d^0 \eta^0 t^1 h^0 t^1 \xi^0 - \eta^0 t^1 h^0 t^1 \xi^0 d^0 - \eta^0 t^1 t^1 \xi^0 \end{aligned}$$

Subsequently by thinking about the relations $(i),(ii)$ and:

$$d^0t^3+d^1t^2+d^2t^1+t^2d^1+d^1t^2+t^3d^0=-(t^1t^2+t^2t^1)$$

we get t_*^3 as;

$$t_*^3=\eta^0t^3\xi^0+\eta^1t^2\xi^0+\eta^2t^1\xi^0+\eta^0t^1\xi^2 +\eta^1t^1\xi^1+\eta^0t^2h^0t^1\xi^0+\eta^0t^1h^0t^2\xi^0$$

$$+\eta^0t^1h^1t^1\xi^0+\eta^0t^1h^0t^1\xi^0+\eta^0t^1h^0t^1\xi^0+\eta^0t^1h^0t^1h^0t^1\xi^0,$$

such that

$$d^0t_*^3+d^1t_*^2+t_*^2d^1+t_*^1d^2+t_*^3d^1=-(t^1t_*^2+t_*^2t^1).$$

The accompanying statement gives a perturbation; $t_*^i: i\geq 0$, of differential D_∞ by aiding the homotopy idea.

Theorem 2.3 Let a strong deformation retraction;

$$\eta^i:((X_1,d^i)\leftrightharpoons(X_2,d^i):\xi^i,h^i)$$

of A_∞-algebra (X_1,d^i,π^i) and let the differential perturbation $t^i: X_1\longrightarrow X_1$, then we have the following statements:
On the A_∞-algebra (X_2,d^i,π^i) we can establish the perturbation $\tilde{\eta}^i: X_2\longrightarrow X_2$ as follows

$$t_*^i=\Sigma_{1\leq k\leq i,\ i_1+\ldots+i_k+j_1,\ +j_2+\ldots+j_{k+1}=i}(h^{j_1}t^{i_1})(h^{j_2}t^{i_2})\ldots(h^{j_k}t^{i_k})\xi^{j_{k+1}},\ \ t_*^0=0 \tag{4}$$

The strong deformation retraction

$$(\tilde{\eta}^i:(X_1,d^i+t^i)\leftrightharpoons(X_2,d^i+t_*^i):\tilde{\xi}^i,\tilde{h}^i)$$

such that

$$\begin{cases}\tilde{\xi}^0=\xi^0\\ \tilde{\xi}^i=\Sigma_{1\leq k\leq i,\ i_1+\ldots+i_k+j_1+j_2+\ldots+j_{k+1}=i}(h^{j_1}t^{i_1})\\ (h^{j_2}t^{i_2})\ldots(h^{j_k}t^{i_k})\xi^{j_{k+1}}, i\geq 1\end{cases} \tag{5}$$

$$\begin{cases}\tilde{\eta}^0=\eta^0\\ \tilde{\eta}^i=\Sigma_{1\leq k\leq i,\ i_1+\ldots i_k+j_1+j_2+\ldots+j_{k+1}=i}(h^{j_i}t^{i_1})\\ (h^{j_2}t^{i_2})\ldots(h^{j_k}t^{i_k})\xi^{j_{k+1}},\ i\geq 1\end{cases} \tag{6}$$

is a strong deformation retraction $\eta^i:(X_1,d^i)\leftrightharpoons(X_2,d^i):\xi^i,h^i)$ is SDR-case of dA_∞- module, then

$$\tilde{\eta}^i:(X,d^i+t^i)\leftrightharpoons(Y,d^i+t_*^i):\tilde{\xi}^i,\tilde{h}^i) \tag{7}$$

is also SDR-case of A_∞-algebra (X_1,d^i,π^i). **Proof.** A deformation of strong retraction, $(\eta_*^i:(X_1,d^i):(X_2,d_*^i):\xi_*^i,h_*^i)$ of A_∞-module which defined relations$(7)-(4)$ and is the deformation of strong retraction $\eta^0:(X_1,d^0)\leftrightharpoons(X_2,d^0):\xi^0,h^0)$.
Most importantly, the deformation of strong retraction

$$\bar{\eta}^i:(X_1,D^i=d^i+t^i)\leftrightharpoons((X_2,\bar{D}^i=d^i+\bar{t}^i)):(\bar{\xi}^i,\bar{h}^i) \tag{8}$$

Where, $\bar{t}^i=\bar{D}^i-d_*^i$ is a great deformation of strong retraction $(i=0)$:

$$(\bar{\eta}^0:(X_1,D^0=d^0+t^0)\leftrightharpoons(X_2,\bar{D}^0=d^0+\bar{t}^0):\bar{\xi}^0,\bar{h}^0).$$

□

By thinking about the isomorphism

$$\eta*\xi=(\eta*\xi)^i:(X,d^0)\leftrightharpoons(Y,d_*^0):g^i=g,$$

from equation (8) we get:

$$\bar{\eta}^i:(X_1,D^i=d^i+t^i)\leftrightharpoons((X_2,\bar{D}^i=d^i+\bar{t}^i)):(\bar{\xi}^i,\bar{h}^i) \tag{9}$$

as follows:

$$\begin{cases}\tilde{\eta}^0=\eta^0\\ \tilde{\eta}^i=\Sigma_{1\leq k\leq i,\ i_1+\ldots i_k+j_1+j_2+\ldots+j_{k+1}=i}(h^{j_i}t^{i_1})\\ (h^{j_2}t^{i_2})\ldots(h^{j_k}t^{i_k})\xi^{j_{k+1}},\ i\geq 1\end{cases} \tag{10}$$

The immediate estimation shows that the deformation of strong retraction of A_∞-algebra [9] is obscure (Equation (10) identical to recipe $(4)-(7)$).
Example 2.4 The homology $H_*(A)$ has a graded A_∞-algebras structure if A is a differential graded algebra A over field.

Example 2.5 The graded space $A=M[\varepsilon]/(\varepsilon^2)$ with the trivial A_∞-structure given by the map m_2 induced by the multiplication of M and the maps $m_n=0$ for all $n\neq 2$,where M is an ordinary algebra for $N\geq 1$ and ε be an indeterminate of degree $2-N$. We define the linear map $f: M^{\otimes N}\longrightarrow M$ and also the deformed multiplication

$$m_n^{'}=\begin{cases}m_n & n\neq N\\ m_N+\varepsilon f & n=N\end{cases}$$

A endowed with the $m_n^{'}$ is an A_∞-algebra iff f is Hochschild cocycle for M.

3 Conclusion

In our work, we studied the derived E-infinity algebra and the homology of differential graded algebra. We define the minimal derived E-infinity algebra and studied some properties of differential graded algebra.

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