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Stability and Existence Analysis to a Coupled System of Caputo Type Fractional Differential Equations with Erdelyi-Kober Integral Boundary Conditions

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Abstract: This article focuses on the Hyers-Ulam type stability, existence and uniqueness of solutions for new types of coupled boundary value problems involving fractional differential equations of Caputo type and augmented with Erdelyi-Kober fractional integral boundary conditions. The nonlinearity relies on the unknown functions. The consequence of the existence is obtained through the Leray-Schauder alternative, whereas the uniqueness of the solution relies on the Banach contraction mapping principle. We analyze the stability of the solutions concerned in the Hyers-Ulam form. As an application, some examples are presented to illustrate the main results. Finally, some variants of the problem are addressed.

Keywords: Coupled system, Caputo derivatives, Erdelyi-Kober fractional integral, Existence, Stability, Fixed point.

1 Introduction

Fractional differential equations have been used to model some biological, chemical, technological, physical, economic, and other applications problems. They have become a useful tool for explaining science and engineering models of nonlinear phenomena. In addition, researchers have found that fractional calculus is very useful for explaining long memory and hereditary properties of different materials and processes, see [1–6] and the references cited therein.
The fractional calculus with the concept of expanding the calculus was still an elegant yet exotic theory. Nevertheless, various examples were gathered at the end of the last century for useful applications of this theory to solve practical problems in different areas of natural sciences, engineering, control theory, economics, biomedicine, etc. See [7–15] and the references for examples and information. Mathematicians, physicists, chemists, engineers, etc. have begun to talk to one another in the language of differentiation and integration with a clearer and more accurate explanation of the real world and life phenomena of arbitrary order and models based on fractional calculus. Nevertheless, it was found that most work encompasses either the fractional derivatives of Riemann-Liouville or Caputo form. As introduced by Arthur Erdelyi and Hermann Kober in 1940, the Erdelyi-Kober fractional integral operator [16] is useful in the solution of single, dual and triple integral equations with special functions of mathematical physics in their kernels. For example, see [17, 18] and the references cited therein. The present paper aim to investigate the existence and uniqueness of coupled system of fractional differential equations for the Caputo type:

$$\begin{aligned} &\mathfrak{D}^{\delta}u(z) = f(z,u(z),v(z)), \quad z\in[0,T], \quad 1<\delta\leq 2,\\ &\mathfrak{D}^{\gamma}v(z) = g(z,u(z),v(z)), \quad z\in[0,T], \quad 1<\gamma\leq 2, \end{aligned} \tag{1}$$

supplemented by Erdelyi-Kober fractional integral boundary conditions:

$$\begin{aligned} &u(T)=\xi\mathfrak{I}_{\omega}^{\theta,\vartheta}v(\alpha), \quad v(T)=\zeta\mathfrak{I}_{\rho}^{\sigma,\varsigma}u(\beta),\\ &u(0) = 0, \quad v(0) = 0, \end{aligned} \tag{2}$$

where $\mathfrak{D}^{\delta}$, $\mathfrak{D}^{\gamma}$ denote the Caputo fractional derivatives and f,g: $[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous

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functions. $\mathfrak{I}_{\omega}^{\theta,\vartheta}$, $\mathfrak{I}_{\rho}^{\sigma,\varsigma}$ are the Erdelyi-Kober fractional integral of order $\vartheta,\varsigma > 0$, $\omega,\rho > 0$, $\theta,\sigma \in \mathbb{R}$. We define spaces $\mathfrak{U} = \{u(z) : u(z) \in C([0,T],\mathbb{R})\}$ endowed with the norm $\|u\| = \sup\{|u(z)|, z \in [0,T]\}$. Obviously $(\mathfrak{U}, \|\cdot\|)$ is a Banach space. Also $\mathfrak{V} = \{v(z) : v(z) \in C([0,T],\mathbb{R})\}$ endowed with the norm $\|v\| = \sup\{|v(z)|, z \in [0,T]\}$ is a Banach space. Then the product space $(\mathfrak{U}\times\mathfrak{V}, \|(u,v)\|)$ is also a Banach space equipped with norm $\|(u,v)\| = \|u\| + \|v\|$. The rest of the paper is organized as follows: Section Two handles certain basic concepts of fractional calculus with the fundamental lemmas associated with this problem. The existence and unique results can be accomplished using the fixed point theorems Leray-Schauder alternative and Banach in section Three. Section Four addresses stability of solutions in Hyers-Ulam and provides appropriate stability conditions. Examples are given in section Five to verify the results. Two new problems are considered similar to (1)-(2), and Section Six defines the strategy for resolving them.

2 Preliminaries

We begin with some basic definitions, properties, and lemmas derived from [19–21].

Definition 1. *The fractional integral of order δ with the lower limit zero for a function f is defined as*

$$\mathfrak{I}^{\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\tau)}{(z-\tau)^{1-\delta}} d\tau, \ \ z > 0, \ \delta > 0,$$

provided the right hand-side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(\delta) = \int_0^{\infty} z^{\delta-1} e^{-z} dz$.

Definition 2. *The Caputo derivative of order δ for a function $f : [0,\infty) \to \mathbb{R}$ can be written as*

$${}^{C}\mathfrak{D}^{\delta} f(z) = \frac{1}{\Gamma(n-\delta)} \int_0^z \frac{f^n(\tau)}{(z-\tau)^{\delta+1-n}} d\tau, \ n-1 < \delta < n.$$

where δ denotes the integer part of the order δ.

Definition 3. *The Erdelyi-Kober fractional integral of order $\delta > 0$ and $\rho > 0$, of a function $f(z)$, for all $0 < z < \infty$, is defined as*

$$\mathfrak{I}_{\omega}^{\theta,\vartheta} f(z) = \frac{\omega z^{-\omega(\vartheta+\theta)}}{\Gamma(\vartheta)} \int_0^z \frac{\tau^{\omega\theta+\omega-1}}{(z^{\omega}-\tau^{\omega})^{1-\vartheta}} f(\tau) d\tau,$$

provided the right hand-side is point-wise defined on $(0,\infty)$.

Remark. For $\omega = 1$ the above-mentioned operator is reduced to the Kober operator

$$\mathfrak{I}_{1}^{\theta,\vartheta} f(z) = \frac{z^{-(\vartheta+\theta)}}{\Gamma(\vartheta)} \int_0^z \frac{\tau^{\theta}}{(z-\tau)^{1-\vartheta}} f(\tau) d\tau,$$

that was introduced for the first time by Kober in [22]. For $\theta = 0$, the Kober operator is reduced to the Riemann-Liouville fractional integral with a power weight:

$$\mathfrak{I}_{1}^{0,\vartheta} f(z) = \frac{z^{-(\vartheta)}}{\Gamma(\vartheta)} \int_0^z \frac{1}{(z-\tau)^{1-\vartheta}} f(\tau) d\tau, \ \ \vartheta > 0.$$

Lemma 1. *Let $\vartheta,\omega > 0$, and $\theta,\delta \in \mathbb{R}$. Then we have*

$$\mathfrak{I}_{\omega}^{\theta,\vartheta} z^{\delta} = \frac{z^{\delta}\Gamma\left(\theta + (\frac{\delta}{\omega}) + 1\right)}{\Gamma\left(\theta + (\frac{\delta}{\omega}) + \vartheta + 1\right)}. \quad (3)$$

Lemma 2. *For $\delta > 0$, the general solution of the fractional differential equation ${}^{C}\mathfrak{D}^{\delta} u(z) = 0$ is given by*

$$u(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1},$$

where $a_i \in \mathbb{R}$, $i = 1,2,...,n-1$, $(n = [\delta]+1)$.
In view of Lemma 2, it follows that

$$\mathfrak{I}^{\delta C}\mathfrak{D}^{\delta} u(z) = u(z) + a_0 + a_1 z + \cdots + a_{n-1} z^{n-1},$$

for some $a_i \in \mathbb{R}$, $i = 1,2,...,n-1$ $(n = [\delta]+1)$.

Theorem 1. *[Leray-Schauder alternative* (see [23])] Let $\mathscr{T} : \mathscr{G} \to \mathscr{G}$ be a completely continuous operator(i.e., a map restricted to any bounded set in $\mathscr{G}$ is compact). Let $\Theta(\mathscr{T}) = \{y \in \mathscr{G} : y = \mu\mathscr{T}(y)$ for some $0 < \mu < 1\}$. Then either the set $\Theta(\mathscr{T})$ is unbounded, or $\mathscr{T}$ has at least one fixed point.

Theorem 2. *[Arzela-Ascoli Theorem* (see [24])] A subset F in $\mathscr{C}([a,b],\mathbb{R})$ is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a,b]$.

Theorem 3. *[Banach Fixed Point Theorem* (see [24])] Let $(\mathscr{G},d)$ be a complete metric space, and $\mathscr{P} : \mathscr{H} \to \mathscr{H}$ a contraction mapping: $d(\mathscr{P}y,\mathscr{P}z) \le \kappa d(y,z)$, where $0 < \kappa < 1$, for each $y,z \in \mathscr{H}$. Then, $\exists$ a unique fixed point y of $\mathscr{P}$ in $\mathscr{H}$.

Lemma 3. *For $\hat{f},\hat{g} \in C([0,T],\mathbb{R})$, the solution of the linear system of fractional differential equations*

$$\begin{aligned} \mathfrak{D}^{\delta} u(z) &= \hat{f}(z), \ \ 1 < \delta \le 2, \\ \mathfrak{D}^{\gamma} v(z) &= \hat{g}(z), \ \ 1 < \gamma \le 2, \end{aligned} \quad (4)$$

supplemented with the coupled integral boundary conditions (2) is equivalent to the system of integral equations

$$u(z) = \kappa_1(z)\Big[\varpi_1\zeta\mathfrak{I}_{\rho}^{\sigma,\varsigma}\mathfrak{I}^{\delta}\hat{f}(\beta) - T\mathfrak{I}^{\delta}\hat{f}(T) + T\xi\mathfrak{I}_{\omega}^{\theta,\vartheta}\mathfrak{I}^{\gamma}\hat{g}(\alpha) - \varpi_1\mathfrak{I}^{\gamma}\hat{g}(T)\Big] + \mathfrak{I}^{\delta}\hat{f}(z), \quad (5)$$

and

$$v(z) = \kappa_1(z)\Big[\varpi_2\xi\mathfrak{I}_{\omega}^{\theta,\vartheta}\mathfrak{I}^{\gamma}\hat{g}(\alpha) - T\mathfrak{I}^{\gamma}\hat{g}(T)$$

$$+T\zeta\mathfrak{J}_{\rho}^{\sigma,\varsigma}\mathfrak{J}^{\delta}\hat{f}(\beta)-\varpi_2\mathfrak{J}^{\delta}\hat{f}(T)\Bigg]+\mathfrak{J}^{\gamma}\hat{g}(z), \quad (6)$$

where

$$\kappa_1(z)=\frac{z}{T^2-(\varpi_1\varpi_2)}, \text{ where } T^2-(\varpi_1\varpi_2)\neq 0, \quad (7)$$

$$\varpi_1=\frac{\alpha\xi\Gamma(\theta+\frac{1}{\omega}+1)}{\Gamma(\theta+\frac{1}{\omega}+\vartheta+1)},\ \varpi_2=\frac{\beta\zeta\Gamma(\sigma+\frac{1}{\rho}+1)}{\Gamma(\sigma+\frac{1}{\rho}+\varsigma+1)} \quad (8)$$

Proof. By standard solution of the fractional differential equations (4), we get

$$u(z)=\mathfrak{J}^{\delta}\hat{f}(z)+a_1+a_2z, \quad (9)$$
$$v(z)=\mathfrak{J}^{\gamma}\hat{g}(z)+b_1+b_2z, \quad (10)$$

where $a_i,b_i\in\mathbb{R},\ i=1,2$, are arbitrary constants. Using the conditions $u(0)=v(0)=0$, it is found that $a_1=b_1=0$. Using the boundary conditions (2) in (9) and (10) respectively, as well as Lemma 2, we get

$$a_2T-b_2\varpi_1=\xi\mathfrak{J}_{\omega}^{\theta,\vartheta}\mathfrak{J}^{\gamma}\hat{g}(\alpha)-\mathfrak{J}^{\delta}\hat{f}(T), \quad (11)$$
$$b_2T-a_2\varpi_2=\zeta\mathfrak{J}_{\rho}^{\sigma,\varsigma}\mathfrak{J}^{\delta}\hat{f}(\beta)-\mathfrak{J}^{\gamma}\hat{g}(T). \quad (12)$$

Solving the system (11) and (12), we get

$$a_2=\frac{1}{T^2-(\varpi_1\varpi_2)}\Bigg[T\times\Big\{\xi\mathfrak{J}_{\omega}^{\theta,\vartheta}\mathfrak{J}^{\gamma}\hat{g}(\alpha)-\mathfrak{J}^{\delta}\hat{f}(T)\Big\}+\varpi_1\times\Big\{\zeta\mathfrak{J}_{\rho}^{\sigma,\varsigma}\mathfrak{J}^{\delta}\hat{f}(\beta)-\mathfrak{J}^{\gamma}\hat{g}(T)\Big\}\Bigg],$$

$$b_2=\frac{1}{T^2-(\varpi_1\varpi_2)}\Bigg[T\times\Big\{\zeta\mathfrak{J}_{\rho}^{\sigma,\varsigma}\mathfrak{J}^{\delta}\hat{f}(\beta)-\mathfrak{J}^{\gamma}\hat{g}(T)\Big\}+\varpi_2\times\Big\{\xi\mathfrak{J}_{\omega}^{\theta,\vartheta}\mathfrak{J}^{\gamma}\hat{g}(\alpha)-\mathfrak{J}^{\delta}\hat{f}(T)\Big\}\Bigg].$$

Substituting the values of a_1, a_2, b_1 and b_2 in (9) and (10) respectively, we get the solution (5) and (6).

3 Existence and Uniqueness Results

In view of Lemma 3, we define an operator $\mathfrak{G}:\mathfrak{U}\times\mathfrak{V}\to\mathfrak{U}\times\mathfrak{V}$ as

$$\mathfrak{G}(u,v)(z)=(\mathfrak{G}_1(u,v)(z),\mathfrak{G}_2(u,v)(z)), \quad (13)$$

where

$$\mathfrak{G}_1(u,v)(z)=\kappa_1(z)\Bigg[\varpi_1\zeta\mathfrak{J}_{\rho}^{\sigma,\varsigma}\mathfrak{J}^{\delta}f(\tau,u(\tau),v(\tau))(\beta)-T\mathfrak{J}^{\delta}f(\tau,u(\tau),v(\tau))(T)+T\xi\mathfrak{J}_{\omega}^{\theta,\vartheta}\mathfrak{J}^{\gamma}g(\tau,u(\tau),v(\tau))(\alpha)-\varpi_1\mathfrak{J}^{\gamma}g(\tau,u(\tau),v(\tau))(T)\Bigg]+\mathfrak{J}^{\delta}f(\tau,u(\tau),v(\tau))(z), \quad (14)$$

and

$$\mathfrak{G}_2(u,v)(z)=\kappa_1(z)\Bigg[\varpi_2\xi\mathfrak{J}_{\omega}^{\theta,\vartheta}\mathfrak{J}^{\gamma}g(\tau,u(\tau),v(\tau))(\alpha)-T\mathfrak{J}^{\gamma}g(\tau,u(\tau),v(\tau))(T)+T\zeta\mathfrak{J}_{\rho}^{\sigma,\varsigma}\mathfrak{J}^{\delta}f(\tau,u(\tau),v(\tau))(\beta)-\varpi_2\mathfrak{J}^{\delta}f(\tau,u(\tau),v(\tau))(T)\Bigg]+\mathfrak{J}^{\gamma}g(\tau,u(\tau),v(\tau))(z). \quad (15)$$

The following terms are used in the sequel:

$$\mathfrak{J}^{\delta}f(\tau,u(\tau),v(\tau))(z)=\frac{1}{\Gamma(\delta)}\int_0^z\frac{f(\tau,u(\tau),v(\tau))d\tau}{(z-\tau)^{1-\delta}},$$

$$\mathfrak{J}^{\gamma}g(\tau,u(\tau),v(\tau))(z)=\frac{1}{\Gamma(\gamma)}\int_0^z\frac{g(\tau,u(\tau),v(\tau))d\tau}{(z-\tau)^{1-\gamma}},$$

$$\mathfrak{J}_{\rho}^{\sigma,\varsigma}\mathfrak{J}^{\delta}f(\tau,u(\tau),v(\tau))(\beta)=\frac{\rho\beta^{-\rho(\varsigma+\sigma)}}{\Gamma(\delta)\Gamma(\varsigma)}\times\int_0^{\beta}\int_0^s\frac{s^{\rho\sigma+\rho-1}(s-\tau)^{\delta-1}}{(\beta^{\rho}-s^{\rho})^{1-\varsigma}}\times f(\tau,u(\tau),v(\tau))d\tau ds,$$

$$\mathfrak{J}_{\omega}^{\theta,\vartheta}\mathfrak{J}^{\gamma,}g(\tau,u(\tau),v(\tau))(\alpha)=\frac{\omega\alpha^{-\omega(\vartheta+\theta)}}{\Gamma(\gamma)\Gamma(\vartheta)}\times\int_0^{\alpha}\int_0^s\frac{s^{\omega\theta+\omega-1}(s-\tau)^{\gamma-1}}{(\alpha^{\omega}-s^{\omega})^{1-\vartheta}}\times g(\tau,u(\tau),v(\tau))d\tau ds.$$

In order to perform the interference with the proof, we introduce the notations:

$$\mathfrak{P}_1=\frac{T^{\delta}}{\Gamma(\delta+1)}+\hat{\kappa}_1\Bigg[\frac{T^{\delta+1}}{\Gamma(\delta+1)}+\frac{\varpi_1|\zeta|\beta^{\delta}\Gamma(\sigma+\frac{\delta}{\rho}+1)}{\Gamma(\delta+1)\Gamma(\sigma+\frac{\delta}{\rho}+\varsigma+1)}\Bigg], \quad (16)$$

$$\mathfrak{P}_2=\hat{\kappa}_1\Bigg[\frac{|\varpi_2|T^{\delta}}{\Gamma(\delta+1)}+\frac{T|\zeta|\beta^{\delta}\Gamma(\sigma+\frac{\delta}{\rho}+1)}{\Gamma(\delta+1)\Gamma(\sigma+\frac{\delta}{\rho}+\varsigma+1)}\Bigg], \quad (17)$$

$$\mathfrak{Q}_1=\hat{\kappa}_1\Bigg[\frac{|\varpi_1|T^{\gamma}}{\Gamma(\gamma+1)}+\frac{T|\xi|\alpha^{\gamma}\Gamma(\theta+\frac{\gamma}{\omega}+1)}{\Gamma(\gamma+1)\Gamma(\theta+\frac{\gamma}{\omega}+\vartheta+1)}\Bigg], \quad (18)$$

$$\mathfrak{Q}_2=\frac{T^{\gamma}}{\Gamma(\gamma+1)}+\hat{\kappa}_1\Bigg[\frac{T^{\gamma+1}}{\Gamma(\gamma+1)}+\frac{\varpi_2|\xi|\alpha^{\gamma}\Gamma(\theta+\frac{\gamma}{\omega}+1)}{\Gamma(\gamma+1)\Gamma(\theta+\frac{\gamma}{\omega}+\vartheta+1)}\Bigg], \quad (19)$$

$$\Lambda=min\{1-((\mathfrak{P}_1+\mathfrak{P}_2)\eta_1+(\mathfrak{Q}_1+\mathfrak{Q}_2)\hat{\eta}_1),\ 1-((\mathfrak{P}_1+\mathfrak{P}_2)\eta_2+(\mathfrak{Q}_1+\mathfrak{Q}_2)\hat{\eta}_2)\}. \quad (20)$$

Theorem 4. *Assume that* $f,g:[0,T]\times\mathbb{R}^2$ *to* $\mathbb{R}$ *are continuous and there exist positive constants* η_i *and*

$\hat{\eta}_i \geq 0$, $i = 1,2$, $\eta_0 > 0, \hat{\eta}_0 > 0$ such that $\forall\ u_i \in \mathbb{R}$, $i = 1,2$. $|f(z,u_1,u_2)| \leq \eta_0 + \eta_1|u_1| + \eta_2|u_2|$, $|g(z,u_1,u_2)| \leq \hat{\eta}_0 + \hat{\eta}_1|u_1| + \hat{\eta}_2|u_2|$. Furthermore, understand that $(\mathfrak{P}_1+\mathfrak{P}_2)\eta_1 + (\mathfrak{Q}_1+\mathfrak{Q}_2)\hat{\eta}_1 < 1$ and $(\mathfrak{P}_1+\mathfrak{P}_2)\eta_2 + (\mathfrak{Q}_1+\mathfrak{Q}_2)\hat{\eta}_2 < 1$. Then there exists at least one solution for problem (1) and (2) on [0,T], where $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are given by (16)-(19) respectively.

Proof. First of all, we show that operator $\mathfrak{G} : \mathfrak{U}\times\mathfrak{V} \to \mathfrak{U}\times\mathfrak{V}$ is completely continuous. By continuity of the f,g functions, operator $\mathfrak{G}$ is continuous.

$\Theta \subset \mathfrak{U}\times\mathfrak{V}$ are bounded. Then there exist positive constants $\mathfrak{M}_f$ and $\mathfrak{M}_g$ such that $|f(z,u(z),v(z))| \leq \mathfrak{M}_f$, $|g(z,p(z),q(z))| \leq \mathfrak{M}_g$, $\forall\ (u,v) \in \Theta$. Then for any $(u,v) \in \Theta$, we can find that

$$
\begin{aligned}
&|\mathfrak{G}_1(u,v)(z)|\\
&\leq \kappa_1(z)\Big[\varpi_1\zeta\mathfrak{I}_\rho^{\sigma,\varsigma}\mathfrak{I}^\delta|f(\tau,u(\tau),v(\tau))|(\beta)\\
&\qquad +T\mathfrak{I}^\delta|f(\tau,u(\tau),v(\tau))|(T)\\
&\qquad +T\xi\mathfrak{I}_\omega^{\theta,\vartheta}\mathfrak{I}^\gamma|g(\tau,u(\tau),v(\tau))|(\alpha)\\
&\qquad +\varpi_1\mathfrak{I}^\gamma|g(\tau,u(\tau),v(\tau))|(T)\Big]\\
&\quad +\mathfrak{I}^\delta|f(\tau,u(\tau),v(\tau))|(z)\\
&\leq \kappa_1(z)\Big[\varpi_1\zeta\mathfrak{I}_\rho^{\sigma,\varsigma}\mathfrak{I}^\delta\mathfrak{M}_f(1)(\beta)+T\mathfrak{I}^\delta\mathfrak{M}_f(1)(T)\\
&\qquad +T\xi\mathfrak{I}_\omega^{\theta,\vartheta}\mathfrak{I}^\gamma\mathfrak{M}_g(1)(\alpha)+\varpi_1\mathfrak{I}^\gamma\mathfrak{M}_g(1)(T)\Big]\\
&\quad +\mathfrak{I}^\delta\mathfrak{M}_f(1)(z)\\
&\leq \frac{\mathfrak{M}_f\hat{\kappa}_1}{\Gamma(\delta+1)}\left[\frac{\varpi_1|\zeta|\beta^\delta\Gamma(\sigma+\frac{\delta}{\rho}+1)}{\Gamma(\sigma+\frac{\delta}{\rho}+\varsigma+1)}+T^{\delta+1}\right]\\
&\quad +\frac{\mathfrak{M}_fT^\delta}{\Gamma(\delta+1)}+\frac{\mathfrak{M}_g\hat{\kappa}_1}{\Gamma(\gamma+1)}\Bigg[\frac{T|\xi|\alpha^\gamma\Gamma(\theta+\frac{\gamma}{\omega}+1)}{\Gamma(\theta+\frac{\gamma}{\omega}+\vartheta+1)}\\
&\qquad +|\varpi_1|T^\gamma\Bigg]\\
&= \mathfrak{M}_f\mathfrak{P}_1+\mathfrak{M}_g\mathfrak{Q}_1. \qquad (21)
\end{aligned}
$$

Equivalently, we obtain

$$
\begin{aligned}
&|\mathfrak{G}_2(u,v)(z)|\\
&\leq \kappa_1(z)\Big[\varpi_2\xi\mathfrak{I}_\omega^{\theta,\vartheta}\mathfrak{I}^\gamma|g(\tau,u(\tau),v(\tau))|(\alpha)\\
&\qquad +T\mathfrak{I}^\gamma|g(\tau,u(\tau),v(\tau))|(T)\\
&\qquad +T\zeta\mathfrak{I}_\rho^{\sigma,\varsigma}\mathfrak{I}^\delta|f(\tau,u(\tau),v(\tau))|(\beta)\\
&\qquad +\varpi_2\mathfrak{I}^\delta|f(\tau,u(\tau),v(\tau))|(T)\Big]\\
&\quad +\mathfrak{I}^\gamma|g(\tau,u(\tau),v(\tau))|(z)\\
&\leq \kappa_1(z)\Big[\varpi_2\xi\mathfrak{I}_\omega^{\theta,\vartheta}\mathfrak{I}^\gamma\mathfrak{M}_g(1)(\alpha)+T\mathfrak{I}^\gamma\mathfrak{M}_g(1)(T)\\
&\qquad +T\zeta\mathfrak{I}_\rho^{\sigma,\varsigma}\mathfrak{I}^\delta\mathfrak{M}_f(1)(\beta)+\varpi_2\mathfrak{I}^\delta\mathfrak{M}_f(1)(T)\Big]\\
&\quad +\mathfrak{I}^\gamma\mathfrak{M}_g(1)(z)\\
&\leq \frac{\mathfrak{M}_g\hat{\kappa}_1}{\Gamma(\gamma+1)}\left[\frac{\varpi_2|\xi|\alpha^\gamma\Gamma(\theta+\frac{\gamma}{\omega}+1)}{\Gamma(\theta+\frac{\gamma}{\omega}+\vartheta+1)}+T^{\gamma+1}\right]\\
&\quad +\frac{\mathfrak{M}_gT^\gamma}{\Gamma(\gamma+1)}+\frac{\mathfrak{M}_f\hat{\kappa}_1}{\Gamma(\delta+1)}\Bigg[\frac{T|\zeta|\beta^\delta\Gamma(\sigma+\frac{\delta}{\rho}+1)}{\Gamma(\sigma+\frac{\delta}{\rho}+\varsigma+1)}\\
&\qquad +|\varpi_2|T^\delta\Bigg]\\
&= \mathfrak{M}_g\mathfrak{Q}_2+\mathfrak{M}_f\mathfrak{P}_2. \qquad (22)
\end{aligned}
$$

It follows from (21) and (22) that $\mathfrak{G}$ is uniformly bound. We will prove the equicontinuity of operator $\mathfrak{G}$. For $z_1,z_2 \in [0,T]$ with $z_1 < z_2$, we have

$$
\begin{aligned}
&|\mathfrak{G}_1(u(z_2),v(z_2))-\mathfrak{G}_1(u(z_1),v(z_1))|\\
&\leq |\kappa_1(z_2)-\kappa_1(z_1)|\Big[\varpi_1\zeta\mathfrak{I}_\rho^{\sigma,\varsigma}\mathfrak{I}^\delta|f(\tau,u(\tau),v(\tau))|(\beta)\\
&\qquad +T\mathfrak{I}^\delta|f(\tau,u(\tau),v(\tau))|(T)\\
&\qquad +T\xi\mathfrak{I}_\omega^{\theta,\vartheta}\mathfrak{I}^\gamma|g(\tau,u(\tau),v(\tau))|(\alpha)\\
&\qquad +\varpi_1\mathfrak{I}^\gamma|g(\tau,u(\tau),v(\tau))|(T)\Big]\\
&\quad +\left|\int_0^{z_1}\frac{[(z_2-\tau)^{\delta-1}-(z_1-\tau)^{\delta-1}]}{\Gamma(\delta)}f(\tau,u(\tau),v(\tau))d\tau\right|\\
&\quad +\left|\int_{z_1}^{z_2}\frac{(z_2-\tau)^{\delta-1}}{\Gamma(\delta)}f(\tau,u(\tau),v(\tau))d\tau\right|\\
&\leq \frac{|z_2-z_1|}{T^2-(\varpi_1\varpi_2)}\Bigg\{\frac{\mathfrak{M}_f}{\Gamma(\delta+1)}\Bigg[\frac{\varpi_1|\zeta|\beta^\delta\Gamma(\sigma+\frac{\delta}{\rho}+1)}{\Gamma(\sigma+\frac{\delta}{\rho}+\varsigma+1)}\\
&\qquad +T^{\delta+1}\Bigg]+\frac{\mathfrak{M}_g}{\Gamma(\gamma+1)}\Bigg[\frac{T|\xi|\alpha^\gamma\Gamma(\theta+\frac{\gamma}{\omega}+1)}{\Gamma(\theta+\frac{\gamma}{\omega}+\vartheta+1)}\\
&\qquad +|\varpi_1|T^\gamma\Bigg]\Bigg\}+\frac{\mathfrak{M}_f}{\Gamma(\delta+1)}[(z_2-z_1)^\delta+(z_2^\delta-z_1^\delta)].
\end{aligned}
$$

Hence, we got that $\|\mathfrak{G}_1(u,v)-\mathfrak{G}_1(u,v)\|_{\mathfrak{U}} \to 0$ independent of u and v as $z_2 \to z_1$. Accordingly, we get

$$
\begin{aligned}
&|\mathfrak{G}_2(u(z_2),v(z_2))-\mathfrak{G}_2(u(z_1),v(z_1))|\\
&\leq |\kappa_1(z_2)-\kappa_1(z_1)|\Big[\varpi_2\xi\mathfrak{I}_\omega^{\theta,\vartheta}\mathfrak{I}^\gamma|g(\tau,u(\tau),v(\tau))|(\alpha)\\
&\qquad +T\mathfrak{I}^\gamma|g(\tau,u(\tau),v(\tau))|(T)\\
&\qquad +T\zeta\mathfrak{I}_\rho^{\sigma,\varsigma}\mathfrak{I}^\delta|f(\tau,u(\tau),v(\tau))|(\beta)
\end{aligned}
$$

$$+\varpi_2\mathfrak{J}^{\delta}|f(\tau,u(\tau),v(\tau))|(T)\Big]$$

$$+\left|\int_0^{z_1}\frac{[(z_2-\tau)^{\gamma-1}-(z_1-\tau)^{\gamma-1}]}{\Gamma(\gamma)}g(\tau,u(\tau),v(\tau))d\tau\right|$$

$$+\left|\int_{z_1}^{z_2}\frac{(z_2-\tau)^{\gamma-1}}{\Gamma(\gamma)}g(\tau,u(\tau),v(\tau))d\tau\right|$$

$$\leq\frac{|z_2-z_1|}{T^2-(\varpi_1\varpi_2)}\left\{\frac{\mathfrak{M}_g}{\Gamma(\gamma+1)}\left[\frac{\varpi_2|\xi|\alpha^{\gamma}\Gamma(\theta+\frac{\gamma}{\omega}+1)}{\Gamma(\theta+\frac{\gamma}{\omega}+\vartheta+1)}+T^{\gamma+1}\right]+\frac{\mathfrak{M}_f}{\Gamma(\delta+1)}\left[\frac{T|\zeta|\beta^{\delta}\Gamma(\sigma+\frac{\delta}{\rho}+1)}{\Gamma(\sigma+\frac{\delta}{\rho}+\varsigma+1)}+|\varpi_2|T^{\delta}\right]\right\}+\frac{\mathfrak{M}_g}{\Gamma(\gamma+1)}[(z_2-z_1)^{\gamma}+(z_2^{\gamma}-z_1^{\gamma})],$$

which imply that $\|\mathfrak{G}_2(u,v)-\mathfrak{G}_2(u,v)\|_{\mathfrak{V}}\to 0$ independent of u and v as $z_2\to z_1$. The $\mathfrak{G}(u,v)$ operator is therefore equicontinuous, which makes the operator completely continuous according to Thoerem 2. Next, we show that the set $\mathfrak{S}=\{(u,v)\in\mathfrak{U}\times\mathfrak{V}|(u,v)=\varepsilon\mathfrak{G}(u,v),0<\varepsilon<1\}$ is bounded. Let $(u,v)\in\mathfrak{S}$, then $(u,v)=\varepsilon\mathfrak{G}(u,v)$ and for any $z\in[0,T]$, we have
$u(z)=\varepsilon\mathfrak{G}_1(u,v)(z)$, $v(z)=\varepsilon\mathfrak{G}_2(u,v)(z)$.
Thus,

$$|u(z)|\leq(\eta_0+\eta_1\|u\|+\eta_2\|v\|)\times\left\{\frac{\hat{\kappa}_1}{\Gamma(\delta+1)}\left[\frac{\varpi_1|\zeta|\beta^{\delta}\Gamma(\sigma+\frac{\delta}{\rho}+1)}{\Gamma(\sigma+\frac{\delta}{\rho}+\varsigma+1)}+T^{\delta+1}\right]+\frac{T^{\delta}}{\Gamma(\delta+1)}\right\}+(\hat{\eta}_0+\hat{\eta}_1\|u\|+\hat{\eta}_2\|v\|)\times\left\{\frac{\hat{\kappa}_1}{\Gamma(\gamma+1)}\left[\frac{T|\xi|\alpha^{\gamma}\Gamma(\theta+\frac{\gamma}{\omega}+1)}{\Gamma(\theta+\frac{\gamma}{\omega}+\vartheta+1)}+|\varpi_1|T^{\gamma}\right]\right\}.$$

Identical to the above-mentioned we can have

$$|v(z)|\leq(\hat{\eta}_0+\hat{\eta}_1\|u\|+\hat{\eta}_2\|v\|)\times\left\{\frac{\hat{\kappa}_1}{\Gamma(\gamma+1)}\left[\frac{\varpi_2|\xi|\alpha^{\gamma}\Gamma(\theta+\frac{\gamma}{\omega}+1)}{\Gamma(\theta+\frac{\gamma}{\omega}+\vartheta+1)}+T^{\gamma+1}\right]+\frac{T^{\gamma}}{\Gamma(\gamma+1)}\right\}+(\hat{\eta}_0+\hat{\eta}_1\|u\|+\hat{\eta}_2\|v\|)\times\left\{\frac{\hat{\kappa}_1}{\Gamma(\delta+1)}\left[\frac{T|\zeta|\beta^{\delta}\Gamma(\sigma+\frac{\delta}{\rho}+1)}{\Gamma(\sigma+\frac{\delta}{\rho}+\varsigma+1)}+|\varpi_2|T^{\delta}\right]\right\}.$$

Hence we have

$$\|u\|\leq(\eta_0+\eta_1\|u\|+\eta_2\|v\|)\mathfrak{P}_1+(\hat{\eta}_0+\hat{\eta}_1\|u\|+\hat{\eta}_2\|v\|)\mathfrak{Q}_1. \quad (23)$$

Identical to the above-mentioned we can have

$$\|v\|\leq(\hat{\eta}_0+\hat{\eta}_1\|u\|+\hat{\eta}_2\|v\|)\mathfrak{Q}_2+(\eta_0+\eta_1\|u\|+\eta_2\|v\|)\mathfrak{P}_2. \quad (24)$$

In conjunction with the notations (23)-(24), we deduce the following results by supporting the aforementioned inequalities.

$$\|u\|+\|v\|\leq(\mathfrak{P}_1+\mathfrak{P}_2)\eta_0+(\mathfrak{Q}_1+\mathfrak{Q}_2)\hat{\eta}_0 nonumber +((\mathfrak{P}_1+\mathfrak{P}_2)\eta_1+(\mathfrak{Q}_1+\mathfrak{Q}_2)\hat{\eta}_1+((\mathfrak{P}_1+\mathfrak{P}_2)\eta_2+(\mathfrak{Q}_1+\mathfrak{Q}_2)\hat{\eta}_2,$$

which leads to $\|(u,v)\|\leq\frac{(\mathfrak{P}_1+\mathfrak{P}_2)\eta_0+(\mathfrak{Q}_1+\mathfrak{Q}_2)\hat{\eta}_0}{\Lambda}$.
This results in the set $\mathfrak{S}$ which is bounded. Thus, operator $\mathfrak{G}$ has at least one fixed point by Theorem 1 implying that problem (1)-(2) has at least one solution on [0,T].

Next, we'll roll the ball into solutions that make it unique with the fixed point theorem of Banach for the problem (1)-(2).
In order to avoid computational complexity, we presume

$$\Delta_1=\mathfrak{N}_1\mathfrak{P}_1+\mathfrak{W}_1\mathfrak{Q}_1+\mathfrak{N}_2\mathfrak{P}_1+\mathfrak{W}_2\mathfrak{Q}_1, \quad (25)$$

$$\Delta_2=\mathfrak{W}_1\mathfrak{Q}_2+\mathfrak{N}_1\mathfrak{P}_2+\mathfrak{W}_2\mathfrak{Q}_2+\mathfrak{N}_2\mathfrak{P}_2, \quad (26)$$

$$\chi_1=\mathfrak{K}_1(\mathfrak{P}_1+\mathfrak{P}_2),\quad \chi_2=\mathfrak{K}_2(\mathfrak{Q}_1+\mathfrak{Q}_2), \quad (27)$$

$$\mathfrak{K}_1=\sup_{z\in[0,T]}f(z,0,0)<\infty,\ \mathfrak{K}_2=\sup_{z\in[0,T]}g(z,0,0)<\infty \quad (28)$$

Theorem 5. *Assume that $f,g:[0,T]\times\mathbb{R}^2$ to $\mathbb{R}$ are continuous and there exist positive constants $\mathfrak{N}_i$ $(i=1,2)$ and $\mathfrak{W}_i$ $(i=1,2)$ such that for all $z\in[0,T]$ and $u_i,v_i\in\mathbb{R}(i=1,2)$, we have*
$|f(z,x_1,x_2)-f(z,y_1,y_2)|\leq\mathfrak{N}_1|x_1-y_1|+\mathfrak{N}_2|x_2-y_2|$,
$|g(z,x_1,x_2)-g(z,y_1,y_2)|\leq\mathfrak{W}_1|x_1-y_1|+\mathfrak{W}_2|x_2-y_2|$.
Moreover, let us understand that

$$\Delta_1+\Delta_2<1, \quad (29)$$

where Δ_1,Δ_2 are specified by (25)-(26) respectively. Then there exists a unique solution for problem (1)-(2) on [0,T].

Proof. Let us define $\hat{r}>\frac{\chi_1+\chi_2}{1-(\Delta_1+\Delta_2)}$. where Δ_1,Δ_2 and χ_1,χ_2 are respectively given by (27), and show that $\mathfrak{G}\mathfrak{B}_{\hat{r}}\subset\mathfrak{B}_{\hat{r}}$, where the operator $\mathfrak{G}$ is given by (13) and $\mathfrak{B}_{\hat{r}}=\{(u,v)\in\mathfrak{U}\times\mathfrak{V}:\|(u,v)\|\leq\hat{r}\}$. For $(u,v)\in\mathfrak{B}_{\hat{r}}$, $z\in[0,T]$, we have

$$|f(z,u(z),v(z))|\leq\mathfrak{N}_1|u(z)|+\mathfrak{N}_2|v(z)|+\mathfrak{K}_1\leq\mathfrak{N}_1\|u\|+\mathfrak{N}_2\|v\|+\mathfrak{K}_1,$$

$$|g(z,u(z),v(z))|\leq\mathfrak{W}_1\|u\|+\mathfrak{W}_2\|v\|+\mathfrak{K}_2.$$

This guides to

$$|\mathfrak{G}_1(u,v)(z)|\leq\mathfrak{J}^{\delta}|f(z,u(z),v(z))-f(z,0,0,0)|+|f(z,0,0,0)|+\kappa_1(z)\Big[\varpi_1\zeta\mathfrak{J}_{\rho}^{\sigma,\varsigma}\mathfrak{J}^{\delta}|f(\beta,u(\beta),v(\beta))-f(\beta,0,0,0)|$$

$$+|f(\beta,0,0,0)|$$
$$+T\mathfrak{J}^{\delta}|f(T,u(T),v(T))-f(T,0,0,0)|+|f(T,0,0,0)|$$
$$+T\xi\mathfrak{I}_{\omega}^{\theta,\vartheta}\mathfrak{J}^{\gamma}|g(\alpha,u(\alpha),v(\alpha))-g(\alpha,0,0,0)|$$
$$+|g(\alpha,0,0,0)|$$
$$\left.+\varpi_1\mathfrak{J}^{\gamma}|g(T,u(T),v(T))-g(T,0,0,0)|+|g(T,0,0,0)|\right]$$
$$\leq(\mathfrak{N}_1\|u\|+\mathfrak{N}_2\|v\|+\mathfrak{K}_1)\left[\hat{\kappa}_1\left(\frac{T^{\delta+1}}{\Gamma(\delta+1)}\right.\right.$$
$$\left.\left.+\frac{\varpi_1|\zeta|\beta^{\delta}\Gamma(\sigma+\frac{\delta}{\rho}+1)}{\Gamma(\delta+1)\Gamma(\sigma+\frac{\delta}{\rho}+\varsigma+1)}\right)+\frac{T^{\delta}}{\Gamma(\delta+1)}\right]$$
$$+(\mathfrak{W}_1\|u\|+\mathfrak{W}_2\|v\|+\mathfrak{K}_2)$$
$$\times\hat{\kappa}_1\left[\frac{T|\xi|\alpha^{\gamma}\Gamma(\theta+\frac{\gamma}{\omega}+1)}{\Gamma(\gamma+1)\Gamma(\theta+\frac{\gamma}{\omega}+\vartheta+1)}+\frac{|\varpi_1|T^{\gamma}}{\Gamma(\gamma+1)}\right]$$
$$\leq(\mathfrak{N}_1\|u\|+\mathfrak{N}_2\|v\|+\mathfrak{K}_1)\mathfrak{P}_1$$
$$+(\mathfrak{W}_1\|u\|+\mathfrak{W}_2\|v\|+\mathfrak{K}_2)\mathfrak{Q}_1. \tag{30}$$

Equivalently, we obtain

$$|\mathfrak{G}_2(p,q)(z)|$$
$$\leq\mathfrak{J}^{\gamma}|g(z,u(z),v(z))-g(z,0,0,0)|+|g(z,0,0,0)|$$
$$+\kappa_1(z)\left[\varpi_2\xi\mathfrak{I}_{\omega}^{\theta,\vartheta}\mathfrak{J}^{\gamma}|g(\alpha,u(\alpha),v(\alpha))-g(\alpha,0,0,0)|\right.$$
$$+|g(\alpha,0,0,0)|$$
$$+T\mathfrak{J}^{\gamma}|g(T,u(T),v(T))-g(T,0,0,0)|+|g(T,0,0,0)|$$
$$+T\zeta\mathfrak{I}_{\rho}^{\sigma,\varsigma}\mathfrak{J}^{\delta}|f(\beta,u(\beta),v(\beta))-f(\beta,0,0,0)|$$
$$+|f(\beta,0,0,0)|$$
$$\left.+\varpi_2\mathfrak{J}^{\delta}|f(T,u(T),v(T))-f(T,0,0,0)|+|f(T,0,0,0)|\right]$$
$$\leq(\mathfrak{W}_1\|u\|+\mathfrak{W}_2\|v\|+\mathfrak{K}_2)\mathfrak{Q}_2$$
$$+(\mathfrak{N}_1\|u\|+\mathfrak{N}_2\|v\|+\mathfrak{K}_1)\mathfrak{P}_2. \tag{31}$$

Thus, it follows from (30) and (31) that $\|\mathfrak{G}(u,v)\|\leq\hat{r}$, and consequently, $\mathfrak{G}\mathfrak{B}_{\hat{r}}\subset\mathfrak{B}_{\hat{r}}$.
Now, for (u_1,v_1), $(u_2,v_2)\in\mathfrak{U}\times\mathfrak{V}$ and any $z\in[0,T]$, we get

$$|\mathfrak{G}_1(u_1,v_1)(z)-\mathfrak{G}_1(u_2,v_2)(z)|$$
$$\leq\kappa_1(z)\left[\varpi_1\zeta\mathfrak{I}_{\rho}^{\sigma,\varsigma}\mathfrak{J}^{\delta}|f(\beta,u_1(\beta),v_1(\beta))\right.$$
$$-f(\beta,u_2(\beta),v_2(\beta))|$$
$$+T\mathfrak{J}^{\delta}|f(T,u_1(T),v_1(T))-f(T,u_2(T),v_2(T))|$$
$$+T\xi\mathfrak{I}_{\omega}^{\theta,\vartheta}\mathfrak{J}^{\gamma}|g(\alpha,u_1(\alpha),v_1(\alpha))-g(\alpha,u_2(\alpha),v_2(\alpha))|$$
$$\left.+\varpi_1\mathfrak{J}^{\gamma}|g(T,u_1(T),v_1(T))-g(T,u_2(T),v_2(T))|\right]$$
$$+\mathfrak{J}^{\delta}|f(z,u_1(z),v_1(z))-f(z,u_2(z),v_2(z))|$$
$$\leq(\mathfrak{N}_1\|u_1-u_2\|+\mathfrak{N}_2\|v_1-v_2\|)$$
$$\left\{\frac{\hat{\kappa}_1}{\Gamma(\delta+1)}\left[\frac{\varpi_1|\zeta|\beta^{\delta}\Gamma(\sigma+\frac{\delta}{\rho}+1)}{\Gamma(\sigma+\frac{\delta}{\rho}+\varsigma+1)}+T^{\delta+1}\right]\right.$$
$$\left.+\frac{T^{\delta}}{\Gamma(\delta+1)}\right\}+(\mathfrak{W}_1\|u_1-u_2\|+\mathfrak{W}_2\|v_1-v_2\|)$$
$$\left\{\frac{\hat{\kappa}_1}{\Gamma(\gamma+1)}\left[\frac{T|\xi|\alpha^{\gamma}\Gamma(\theta+\frac{\gamma}{\omega}+1)}{\Gamma(\theta+\frac{\gamma}{\omega}+\vartheta+1)}+|\varpi_1|T^{\gamma}\right]\right\}$$
$$\leq(\mathfrak{N}_1\|u_1-u_2\|+\mathfrak{N}_2\|v_1-v_2\|)\mathfrak{P}_1$$
$$+(\mathfrak{W}_1\|u_1-u_2\|+\mathfrak{W}_2\|v_1-v_2\|)\mathfrak{Q}_1$$
$$\leq\mathfrak{N}_1\mathfrak{P}_1+\mathfrak{W}_1\mathfrak{Q}_1+\mathfrak{N}_2\mathfrak{P}_1+\mathfrak{W}_2\mathfrak{Q}_1.$$

Equivalently, we obtain

$$|\mathfrak{G}_2(u_1,v_1)(z)-\mathfrak{G}_2(u_2,v_2)(z)|$$
$$\leq(\mathfrak{W}_1\|u_1-u_2\|+\mathfrak{W}_2\|v_1-v_2\|)$$
$$\left\{\frac{\hat{\kappa}_1}{\Gamma(\gamma+1)}\left[\frac{\varpi_2|\xi|\alpha^{\gamma}\Gamma(\theta+\frac{\gamma}{\omega}+1)}{\Gamma(\theta+\frac{\gamma}{\omega}+\vartheta+1)}+T^{\gamma+1}\right]\right.$$
$$\left.+\frac{T^{\gamma}}{\Gamma(\gamma+1)}\right\}+(\mathfrak{N}_1\|u_1-u_2\|+\mathfrak{N}_2\|v_1-v_2\|)$$
$$\left\{\frac{\hat{\kappa}_1}{\Gamma(\delta+1)}\left[\frac{T|\zeta|\beta^{\delta}\Gamma(\sigma+\frac{\delta}{\rho}+1)}{\Gamma(\sigma+\frac{\delta}{\rho}+\varsigma+1)}+|\varpi_2|T^{\delta}\right]\right\}$$
$$\leq(\mathfrak{W}_1\|u_1-u_2\|+\mathfrak{W}_2\|v_1-v_2\|)\mathfrak{Q}_2$$
$$+(\mathfrak{N}_1\|u_1-u_2\|+\mathfrak{N}_2\|v_1-v_2\|)\mathfrak{P}_2$$
$$\leq\mathfrak{W}_1\mathfrak{Q}_2+\mathfrak{N}_1\mathfrak{P}_2+\mathfrak{W}_2\mathfrak{Q}_2+\mathfrak{N}_2\mathfrak{P}_2,$$

so we obtain

$$\|\mathfrak{G}_1(u_1,v_1)-\mathfrak{G}_1(u_2,v_2)\|\leq\Delta_1[\|u_1-u_2\|+\|v_1-v_2\|]. \tag{32}$$

Similarly,

$$\|\mathfrak{G}_2(u_1,v_1)-\mathfrak{G}_2(u_2,v_2)\|\leq\Delta_2[\|u_1-u_2\|+\|v_1-v_2\|]. \tag{33}$$

Thus, we obtain

$$\|\mathfrak{G}(u_1,v_1)-\mathfrak{G}(u_2,v_2)\|\leq(\Delta_1+\Delta_2)(\|u_1-u_2\|+\|v_1-v_2\|).$$

Thus, in view of the condition (29), it follows that the operator $\mathfrak{G}$ is a contraction. Hence it follows by Theorem 3 that the system (1)-(2) has a unique solution on $[0,T]$.

4 Hyers-Ulam Stability

This section discusses the stability of boundary value problem solutions for Hyers-Ulam (1)-(2) by integral representing its solution provided by

$$u(z)=\mathfrak{G}_1(u,v)(z),\quad v(z)=\mathfrak{G}_2(u,v)(z), \tag{34}$$

where $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are defined by (14) and (15). Define the following nonlinear operators $\mathfrak{M}_1,\mathfrak{M}_2\in C([0,T],\mathbb{R})\times C([0,T],\mathbb{R})\to C([0,T],\mathbb{R})$;

$$\mathfrak{D}^{\delta}u(z)-f(z,u(z),v(z))=\mathfrak{M}_1(u,v)(z),\quad z\in[0,T],$$
$$\mathfrak{D}^{\gamma}v(z)-g(z,u(z),v(z))=\mathfrak{M}_2(u,v)(z),\quad z\in[0,T],$$

For some $\lambda_1, \lambda_2 > 0$, the following inequalities are considered:

$$\|\mathfrak{M}_1(u,v)\| \leq \lambda_1, \ \ \|\mathfrak{M}_2(u,v)\| \leq \lambda_2. \tag{35}$$

Definition 4. *The coupled system (1)-(2) is said to be Hyers-Ulam stable, if there exist $\mathfrak{S}_1, \mathfrak{S}_2 > 0$ such that for every solution $(u^*, v^*) \in C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$ of the inequality (35), there exists a unique solution $(u,v) \in C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$ of problems (1)-(2) with*

$$\|(u,v) - (u^*, v^*)\| \leq \mathfrak{S}_1\lambda_1 + \mathfrak{S}_2\lambda_2.$$

Theorem 6. *Assume that Theorem 5 assumptions hold. Then the boundary value problem (1)-(2) is Hyers-Ulam-stable.*

Proof. Let $(u,v) \in C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$ be the solution of (1)-(2) the problems that satisfying (14) and (15). Let (u^*, v^*) be any solution satisfying (35):

$$\mathfrak{D}^{\delta}u(z) = f(z,u(z),v(z)) + \mathfrak{M}_1(u,v)(z), \ \ z \in [0,T],$$
$$\mathfrak{D}^{\gamma}v(z) = g(z,u(z),v(z)) + \mathfrak{M}_2(u,v)(z), \ \ z \in [0,T],$$

So,

$$\begin{aligned} u^*(z) &= \mathfrak{G}_1(u^*,v^*)(z) + \kappa_1(z)\Bigg[\varpi_1\zeta\frac{\rho\beta^{-\rho(\varsigma+\sigma)}}{\Gamma(\delta)\Gamma(\varsigma)} \\ &\times \int_0^{\beta}\int_0^{s}\frac{s^{\rho\sigma+\rho-1}(s-\tau)^{\delta-1}}{(\beta^{\rho}-s^{\rho})^{1-\varsigma}}\mathfrak{M}_1(u^*,v^*)(\tau)d\tau ds \\ &-T\frac{1}{\Gamma(\delta)}\int_0^T\frac{1}{(T-\tau)^{1-\delta}}\mathfrak{M}_1(u^*,v^*)(\tau)d\tau \\ &+T\xi\frac{\omega\alpha^{-\omega(\vartheta+\theta)}}{\Gamma(\gamma)\Gamma(\vartheta)} \\ &\times \int_0^{\alpha}\int_0^{s}\frac{s^{\omega\theta+\omega-1}(s-\tau)^{\gamma-1}}{(\alpha^{\omega}-s^{\omega})^{1-\vartheta}}\mathfrak{M}_2(u^*,v^*)(\tau)d\tau ds \\ &-\varpi_1\frac{1}{\Gamma(\gamma)}\int_0^T\frac{1}{(T-\tau)^{1-\gamma}}\mathfrak{M}_2(u^*,v^*)(\tau)d\tau\Bigg] \\ &+\frac{1}{\Gamma(\delta)}\int_0^z\frac{1}{(z-\tau)^{1-\delta}}\mathfrak{M}_1(u^*,v^*)(\tau)d\tau \end{aligned}$$

It follows that

$$\begin{aligned} &|\mathfrak{G}_1(u^*,v^*)(z) - u^*(z)| \\ &\leq |\kappa_1(z)|\Bigg[|\varpi_1\zeta|\frac{\rho\beta^{-\rho(\varsigma+\sigma)}}{\Gamma(\delta)\Gamma(\varsigma)} \\ &\times \int_0^{\beta}\int_0^{s}\frac{s^{\rho\sigma+\rho-1}(s-\tau)^{\delta-1}}{(\beta^{\rho}-s^{\rho})^{1-\varsigma}}\lambda_1 d\tau ds \\ &+|T|\frac{1}{\Gamma(\delta)}\int_0^T\frac{1}{(T-\tau)^{1-\delta}}\lambda_1 d\tau \\ &+|T\xi|\frac{\omega\alpha^{-\omega(\vartheta+\theta)}}{\Gamma(\gamma)\Gamma(\vartheta)} \\ &\times \int_0^{\alpha}\int_0^{s}\frac{s^{\omega\theta+\omega-1}(s-\tau)^{\gamma-1}}{(\alpha^{\omega}-s^{\omega})^{1-\vartheta}}\lambda_2 d\tau ds \\ &+|\varpi_1|\frac{1}{\Gamma(\gamma)}\int_0^T\frac{1}{(T-\tau)^{1-\gamma}}\lambda_2 d\tau\Bigg] \\ &+\frac{1}{\Gamma(\delta)}\int_0^z\frac{1}{(z-\tau)^{1-\delta}}\lambda_1 d\tau \\ &\leq \Bigg[\hat{\kappa}_1\left(\frac{T^{\delta+1}}{\Gamma(\delta+1)} + \frac{\varpi_1|\zeta|\beta^{\delta}\Gamma(\sigma+\frac{\delta}{\rho}+1)}{\Gamma(\delta+1)\Gamma(\sigma+\frac{\delta}{\rho}+\varsigma+1)}\right) \\ &+\frac{T^{\delta}}{\Gamma(\delta+1)}\Bigg]\lambda_1 + \Bigg[\frac{T|\xi|\alpha^{\gamma}\Gamma(\theta+\frac{\gamma}{\omega}+1)}{\Gamma(\gamma+1)\Gamma(\theta+\frac{\gamma}{\omega}+\vartheta+1)} \\ &+\frac{|\varpi_1|T^{\gamma}}{\Gamma(\gamma+1)}\Bigg]\lambda_2 \\ &\leq \mathfrak{P}_1\lambda_1 + \mathfrak{Q}_1\lambda_2. \end{aligned}$$

Similarly,

$$\begin{aligned} &|\mathfrak{G}_2(u^*,v^*)(z) - v^*(z)| \\ &\leq \Bigg[\hat{\kappa}_1\left(\frac{T^{\gamma+1}}{\Gamma(\gamma+1)} + \frac{\varpi_2|\xi|\alpha^{\gamma}\Gamma(\theta+\frac{\gamma}{\omega}+1)}{\Gamma(\gamma+1)\Gamma(\theta+\frac{\gamma}{\omega}+\vartheta+1)}\right) \\ &+\frac{T^{\gamma}}{\Gamma(\gamma+1)}\Bigg]\lambda_2 + \Bigg[\hat{\kappa}_1\Bigg(\frac{|\varpi_2|T^{\delta}}{\Gamma(\delta+1)} \\ &+\frac{T|\zeta|\beta^{\delta}\Gamma(\sigma+\frac{\delta}{\rho}+1)}{\Gamma(\delta+1)\Gamma(\sigma+\frac{\delta}{\rho}+\varsigma+1)}\Bigg)\Bigg]\lambda_1 \\ &\leq \mathfrak{Q}_2\lambda_2 + \mathfrak{P}_2\lambda_1, \end{aligned}$$

where $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are defined in (16)-(19). Thus, the operator $\mathfrak{G}$, which is given by (14) and (15), can be extracted from the fixed-point property, as follows:

$$\begin{aligned} &|u(z) - u^*(z)| \\ &= |u(z) - \mathfrak{G}_1(u^*,v^*)(z) + \mathfrak{G}_1(u^*,v^*)(z) - u^*(z)| \\ &\leq |\mathfrak{G}_1(u,v)(z) - \mathfrak{G}_1(u^*,v^*)(z)| + |\mathfrak{G}_1(u^*,v^*)(z) - u^*(z)| \\ &\leq \mathfrak{N}_1\mathfrak{P}_1 + \mathfrak{W}_1\mathfrak{Q}_1 + \mathfrak{N}_2\mathfrak{P}_1 + \mathfrak{W}_2\mathfrak{Q}_1 + \mathfrak{P}_1\lambda_1 + \mathfrak{Q}_1\lambda_2. \end{aligned} \tag{36}$$

Similarly,

$$\begin{aligned} &|v(z) - v^*(z)| \\ &= |v(z) - \mathfrak{G}_2(u^*,v^*)(z) + \mathfrak{G}_2(u^*,v^*)(z) - v^*(z)| \\ &\leq |\mathfrak{G}_2(u,v)(z) - \mathfrak{G}_2(u^*,v^*)(z)| + |\mathfrak{G}_2(u^*,v^*)(z) - v^*(z)| \\ &\leq \mathfrak{W}_1\mathfrak{Q}_2 + \mathfrak{N}_1\mathfrak{P}_2 + \mathfrak{W}_2\mathfrak{Q}_2 + \mathfrak{N}_2\mathfrak{P}_2 + \mathfrak{Q}_2\lambda_2 + \mathfrak{P}_2\lambda_1. \end{aligned} \tag{37}$$

According to (36) and (37)

$$\begin{aligned} \|(u,v) - (u^*,v^*)\| &\leq (\Delta_1 + \Delta_2)\|(u,v) - (u^*,v^*)\| \\ &+ (\mathfrak{P}_1 + \mathfrak{P}_2)\lambda_1 + (\mathfrak{Q}_1 + \mathfrak{Q}_2)\lambda_2, \end{aligned}$$

$$\begin{aligned} \|(u,v) - (u^*,v^*)\| &\leq \frac{(\mathfrak{P}_1 + \mathfrak{P}_2)\lambda_1 + (\mathfrak{Q}_1 + \mathfrak{Q}_2)\lambda_2}{1 - (\Delta_1 + \Delta_2)}, \\ &\leq \mathfrak{S}_1\lambda_1 + \mathfrak{S}_2\lambda_2, \end{aligned}$$

with

$$\mathfrak{S}_1 = \frac{(\mathfrak{P}_1+\mathfrak{P}_2)}{1-(\Delta_1+\Delta_2)}, \quad \mathfrak{S}_2 = \frac{(\mathfrak{Q}_1+\mathfrak{Q}_2)}{1-(\Delta_1+\Delta_2)}.$$

Thus, the boundary value problem 1)-(2) is Hyers-Ulam stable.

5 Examples

Example 1. Consider the fractional differential equations given by

$$\begin{aligned} \mathfrak{D}^{\frac{29}{20}}u(z) &= \frac{z^2+1}{4} + \frac{19}{200}\frac{|u(z)|}{(1+|u(z)|)} + \frac{6}{125}\sin|v(z)|, \\ \mathfrak{D}^{\frac{37}{20}}v(z) &= \frac{1+\sqrt{z}}{30} + \frac{29}{400}\tan^{-1}(|u(z)|) + \frac{1}{25}\frac{9|v(z)|}{(1+|v(z)|)}, \end{aligned} \tag{38}$$

subject to the boundary conditions,

$$\begin{aligned} u(1) &= \frac{1}{4}\mathfrak{I}^{\frac{1}{9},\frac{\sqrt{5}}{4}}_{\frac{3}{5}} v\left(\frac{6}{5}\right), \quad v(1) = \frac{1}{6}\mathfrak{I}^{\frac{\sqrt{4}}{5},\frac{1}{11}}_{\frac{4}{7}} u\left(\frac{7}{6}\right), \\ u(0) &= 0, \quad v(0) = 0. \end{aligned} \tag{39}$$

Clearly,

$$\begin{aligned} |f(z,u_1,v_1) - f(z,u_2,v_2)| &= \frac{19}{200}|u_1-u_2| + \frac{6}{125}|v_1-v_2|, \\ |g(z,u_1,v_1) - g(z,u_2,v_2)| &= \frac{29}{400}|u_1-u_2| + \frac{9}{25}|v_1-v_2|. \end{aligned}$$

Here, $\delta = \frac{29}{20}$, $\gamma = \frac{37}{20}$, $T = 1$, $\xi = \frac{1}{4}$, $\zeta = \frac{1}{6}$, $\theta = \frac{1}{9}$, $\vartheta = \frac{\sqrt{5}}{4}$, $\omega = \frac{3}{5}$, $\sigma = \frac{\sqrt{4}}{5}$, $\varsigma = \frac{1}{11}$, $\rho = \frac{4}{7}$, $\alpha = \frac{6}{5}$, $\beta = \frac{7}{6}$. Using the given data, it is found that

$\mathfrak{N}_1 = \frac{19}{200}$, $\mathfrak{N}_2 = \frac{6}{125}$, $\mathfrak{W}_1 = \frac{29}{400}$, $\mathfrak{W}_2 = \frac{9}{25}$,
$\varpi_1 = 0.17706194242075243$,
$\varpi_2 = 0.17759374715250995$, $\hat{\kappa}_1 = 1.032465989932396$,
$\mathfrak{P}_1 = 1.6091378876654243$,
$\mathfrak{P}_2 = 0.29232136778641965$,
$\mathfrak{Q}_1 = 0.2000003974982166$, $\mathfrak{Q}_2 = 0.7716312686506392$,
$\Delta_1 = 0.3166068898541344$,
$\Delta_2 = 0.37553247928485944$.

With $\Delta_1 + \Delta_2 \cong 0.6921393691389939 < 1$, so all requirements of Theorem 6, then problem (38)-(39) has a unique solution for $[0,1]$, which is stable for Hyers-Ulam.

Example 2. Consider the fractional differential equation given by

$$\begin{aligned} \mathfrak{D}^{\frac{34}{25}}u(z) &= \frac{1}{z+1} + \frac{1}{88}\sin(|u(z)|) + \frac{|v(z)|}{25(1+|v(z)|)}, \\ \mathfrak{D}^{\frac{83}{50}}v(z) &= \sqrt{z+1} + \frac{|u(z)|}{20(1+|u(z)|)} + \frac{1}{60}\tan^{-1}(|v(z)|), \end{aligned} \tag{40}$$

subject to the boundary conditions,

$$\begin{aligned} u(1) &= \frac{1}{5}\mathfrak{I}^{\frac{\sqrt{3}}{2},\frac{1}{7}}_{\frac{2}{5}} v\left(\frac{3}{2}\right), \quad v(1) = \frac{1}{7}\mathfrak{I}^{\frac{1}{8},\frac{\sqrt{2}}{3}}_{\frac{3}{7}} u\left(\frac{5}{4}\right), \\ u(0) &= 0, \quad v(0) = 0. \end{aligned} \tag{41}$$

Clearly,

$$\begin{aligned} |f(z,u(z),v(z))| &= \left|\frac{1}{z+1} + \frac{1}{88}\sin(|u|) + \frac{1}{25}\frac{|v|}{(1+|v|)}\right| \\ &\le \frac{5}{14} + \frac{1}{88}|u| + \frac{1}{25}|v|, \\ |g(z,u(z),v(z))| &= \left|\sqrt{(}z+1) + \frac{|u|}{20(1+|u|)} + \frac{1}{60}\tan^{-1}(|v|)\right| \\ &\le \sqrt{\frac{14}{5}} + \frac{1}{20}|u| + \frac{1}{60}|v|. \end{aligned}$$

Here, $\delta = \frac{34}{25}$, $\gamma = \frac{83}{50}$, $T = \frac{9}{5}$, $\xi = \frac{1}{5}$, $\zeta = \frac{1}{7}$, $\theta = \frac{\sqrt{3}}{2}$, $\vartheta = \frac{1}{7}$, $\omega = \frac{2}{5}$, $\sigma = \frac{1}{8}$, $\varsigma = \frac{\sqrt{2}}{3}$, $\rho = \frac{3}{7}$, $\alpha = \frac{3}{2}$, $\beta = \frac{5}{4}$. With the given data, we find that

$\eta_0 = \frac{5}{14}$, $\eta_1 = \frac{1}{88}$, $\eta_2 = \frac{1}{25}$, $\hat{\eta}_0 = \sqrt{\frac{14}{5}}$, $\hat{\eta}_1 = \frac{1}{20}$, $\hat{\eta}_2 = \frac{1}{60}$,
$\varpi_1 = 0.2465635652094569$,
$\varpi_2 = 0.10313907312479913$,
$\hat{\kappa}_1 = 0.5599505342951776$, $\mathfrak{P}_1 = 3.700317103237051$,
$\mathfrak{P}_2 = 0.18949925057891354$,
$\mathfrak{Q}_1 = 0.45115862438514553$, $\mathfrak{Q}_2 = 2.22370429860805$,
we find that

$\Lambda = min\{1-((\mathfrak{P}_1+\mathfrak{P}_2)\eta_1+(\mathfrak{Q}_1+\mathfrak{Q}_2)\hat{\eta}_1), 1-((\mathfrak{P}_1+\mathfrak{P}_2)\eta_2+(\mathfrak{Q}_1+\mathfrak{Q}_2)\hat{\eta}_2)\} \cong 0.7998262971308081 < 1$. Thus, Theorem 4 holds assumption and the problem (40)-(41) has at least one solution at $[0, \frac{9}{5}]$.

Example 3. Consider the fractional differential equations given by

$$\begin{aligned} \mathfrak{D}^{\frac{32}{25}}u(z) &= \frac{z+1}{8} + \frac{1}{12}\frac{|u(z)|}{(1+|u(z)|)} + \frac{1}{45}\sin|v(z)|, \\ \mathfrak{D}^{\frac{37}{25}}v(z) &= \frac{z}{25} + \frac{1}{16}\tan^{-1}|u(z)| + \frac{1}{15}\frac{4|v(z)|}{(1+|v(z)|)}, \end{aligned} \tag{42}$$

subject to the boundary conditions,

$$\begin{aligned} u(1) &= \frac{1}{4}\mathfrak{I}^{\frac{1}{9},\frac{\sqrt{5}}{4}}_{\frac{3}{5}} v\left(\frac{6}{5}\right), \quad v(1) = \frac{1}{6}\mathfrak{I}^{\frac{\sqrt{4}}{5},\frac{1}{11}}_{\frac{4}{7}} u\left(\frac{7}{6}\right), \\ u(0) &= 0, \quad v(0) = 0. \end{aligned} \tag{43}$$

Clearly,

$$\begin{aligned} |f(z,u_1,v_1) - f(z,u_2,v_2)| &= \frac{1}{12}|u_1-u_2| + \frac{1}{45}|v_1-v_2|, \\ |g(z,u_1,v_1) - g(z,u_2,v_2)| &= \frac{1}{16}|u_1-u_2| + \frac{4}{15}|v_1-v_2|. \end{aligned}$$

Here, $\delta = \frac{32}{25}$, $\gamma = \frac{37}{25}$, $T = \frac{4}{3}$, $\xi = \frac{1}{4}$, $\zeta = \frac{1}{6}$, $\theta = \frac{1}{9}$, $\vartheta = \frac{\sqrt{5}}{4}$, $\omega = \frac{3}{5}$, $\sigma = \frac{\sqrt{4}}{5}$, $\varsigma = \frac{1}{11}$, $\rho = \frac{4}{7}$, $\alpha = \frac{6}{5}$, $\beta = \frac{7}{6}$. Using the given data, it is found that

$\mathfrak{N}_1 = \frac{1}{12}$, $\mathfrak{N}_2 = \frac{1}{45}$, $\mathfrak{W}_1 = \frac{1}{16}$, $\mathfrak{W}_2 = \frac{4}{15}$,
$\varpi_1 = 0.17706194242075243$,
$\varpi_2 = 0.17759374715250995$,
$\hat{\kappa}_1 = 0.7635047695042699$, $\mathfrak{P}_1 = 2.5509784407625897$,
$\mathfrak{P}_2 = 0.33124989196843574$,
$\mathfrak{Q}_1 = 0.2868824581156235$, $\mathfrak{Q}_2 = 1.4546003701614751$,
$\Delta_1 = 0.3637020889879994$, $\Delta_2 = 0.5137712215525982$.

With $\Delta_1 + \Delta_2 \cong 0.8774733105405976 < 1$. Hence, the Theorem 5 is satisfied and here the problem (42)-(43) has a unique solution on $[0, \frac{4}{3}]$.

6 Discussion

We discussed existence, uniqueness and stability of solutions for a coupled system of Caputo type fractional differential equations supplemented by Erdelyi-Kober fractional integral conditions through Leray-Schauder alternative, Banach fixed-point theorem, Hyer-Ulam stable respectively. When we have fixed the parameters involved in the problem (ξ, ζ), our results have corresponded to certain specific problems. Suppose that taking $\xi = \zeta = 0$ in the results provided, we present the problems (1)-(2) with the form:

$$u(0) = 0, \;\; v(0) = 0, \; u(T) = 0, \;\; v(T) = 0,$$

Next, modifying the condition, we considered two new problems:

$$\begin{aligned} &u(T) = \xi \mathfrak{I}^{\theta,\vartheta}_{\omega} v(\alpha), \;\; v(T) = \zeta \mathfrak{I}^{\sigma,\varsigma}_{\rho} u(\beta), \\ &u(0) = 0, \;\; v(0) = 0. \end{aligned} \tag{44}$$

in problem (1)-(2) with

$$\begin{aligned} &u(T) = \xi \mathfrak{I}^{\vartheta} v(\alpha), \;\; v(T) = \zeta \mathfrak{I}^{\varsigma} u(\beta), \\ &u(0) = 0, \quad v(0) = 0. \end{aligned} \tag{45}$$

$$\begin{aligned} &u(T) = \xi \int_0^{\alpha} v(\tau) d\tau, \;\; v(T) = \zeta \int_0^{\beta} u(\tau) d\tau, \\ &u(0) = 0, \;\; v(0) = 0. \end{aligned} \tag{46}$$

In relation to the problem (1)-(2) with (45) instead of (44), we obtain the operator $\mathfrak{T} : \mathfrak{U} \times \mathfrak{V} \to \mathfrak{U} \times \mathfrak{V}$ defined by

$$\begin{aligned} \mathfrak{T}_1(u,v)(z) = \kappa_1(z) \Big[&\varpi_1 \zeta \mathfrak{I}^{\sigma+\delta} f(\tau, u(\tau), v(\tau))(\beta) \\ &- T \mathfrak{I}^{\delta} f(\tau, u(\tau), v(\tau))(T) \\ &+ T \xi \mathfrak{I}^{\theta+\gamma} g(\tau, u(\tau), v(\tau))(\alpha) \\ &- \varpi_1 \mathfrak{I}^{\gamma} g(\tau, u(\tau), v(\tau))(T) \Big] \\ &+ \mathfrak{I}^{\delta} f(\tau, u(\tau), v(\tau))(z), \end{aligned}$$

and

$$\begin{aligned} \mathfrak{T}_2(u,v)(z) = \kappa_1(z) \Big[&\varpi_2 \xi \mathfrak{I}^{\theta+\gamma} g(\tau, u(\tau), v(\tau))(\alpha) \\ &- T \mathfrak{I}^{\gamma} g(\tau, u(\tau), v(\tau))(T) \\ &+ T \zeta \mathfrak{I}^{\sigma+\delta} f(\tau, u(\tau), v(\tau))(\beta) \\ &- \varpi_2 \mathfrak{I}^{\delta} f(\tau, u(\tau), v(\tau))(T) \Big] \\ &+ \mathfrak{I}^{\gamma} g(\tau, u(\tau), v(\tau))(z). \end{aligned}$$

where

$$\kappa_1(z) = \frac{z}{T^2 - (\varpi_1 \varpi_2)}, \;\; \text{where} \;\; T^2 - (\varpi_1 \varpi_2) \neq 0,$$
$$\varpi_1 = \frac{\xi \alpha^{\vartheta+1}}{\vartheta + 2}, \; \varpi_2 = \frac{\zeta \beta^{\varsigma+1}}{\varsigma + 2}.$$

On the other hand, the operator $\widehat{\mathfrak{T}} : \mathfrak{U} \times \mathfrak{V} \to \mathfrak{U} \times \mathfrak{V}$ associated with the problem (1)-(2) involving (46) instead of (44) is

$$\begin{aligned} \widehat{\mathfrak{T}}_1(u,v)(z) = \kappa_1(z) \Big[&\varpi_1 \zeta \int_0^{\beta} \mathfrak{I}^{\delta} f(\tau, u(\tau), v(\tau)) d\tau \\ &- T \mathfrak{I}^{\delta} f(\tau, u(\tau), v(\tau))(T) \\ &+ T \xi \int_0^{\alpha} \mathfrak{I}^{\gamma} g(\tau, u(\tau), v(\tau)) d\tau \\ &- \varpi_1 \mathfrak{I}^{\gamma} g(\tau, u(\tau), v(\tau))(T) \Big] \\ &+ \mathfrak{I}^{\delta} f(\tau, u(\tau), v(\tau))(z), \end{aligned}$$

and

$$\begin{aligned} \widehat{\mathfrak{T}}_2(u,v)(z) = \kappa_1(z) \Big[&\varpi_2 \xi \int_0^{\alpha} \mathfrak{I}^{\gamma} g(\tau, u(\tau), v(\tau)) d\tau \\ &- T \mathfrak{I}^{\gamma} g(\tau, u(\tau), v(\tau))(T) \\ &+ T \zeta \int_0^{\beta} \mathfrak{I}^{\delta} f(\tau, u(\tau), v(\tau)) d\tau \\ &- \varpi_2 \mathfrak{I}^{\delta} f(\tau, u(\tau), v(\tau))(T) \Big] \\ &+ \mathfrak{I}^{\gamma} g(\tau, u(\tau), v(\tau))(z). \end{aligned}$$

where

$$\kappa_1(z) = \frac{z}{T^2 - (\varpi_1 \varpi_2)}, \;\; \text{where} \;\; T^2 - (\varpi_1 \varpi_2) \neq 0,$$
$$\varpi_1 = \frac{\xi \alpha^2}{2}, \; \varpi_2 = \frac{\zeta \beta^2}{2}.$$

Using the operators $\mathfrak{T}$ and $\widehat{\mathfrak{T}}$ for new problem analog results (1)-(2) can be established for the new problem. We emphasise that the above-mentioned problems are new.

7 Conclusion

In this research paper, we have effectively obtained, for the coupled systems (1) and (2), respectively, the necessary conditions for existence, uniqueness, and stability in the sense of Hyers-Ulam. Use Leray-Schauder alternative and Banach fixed point theorems to obtain the required results. Furthermore, to the solution of the coupled systems (1) and (2), we have established some proper conditions for Hyers-Ulam stability. With the help of examples, we also illustrated our main theoretical results.

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