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On Solutions of Nonlinear Integral Equations in the Space of Functions of Shiba-Bounded Variation

Jurancy Ereú[1], *José Giménez*[2] *and Liliana Pérez*[3,*]

[1]Departamento de Matemáticas. Universidad Centroccidental Lisandro Alvarado. Barquisimeto, Estado Lara, Venezuela
[2]Departamento de Matemáticas. Universidad de los Andes. Mérida, Estado Mérida, Venezuela
[3]Escuela Superior Politécnica del Litoral. ESPOL, FCNM, Campus Gustavo Galindo Km 30.5 Vía Perimetral, P.O. Box 09-01-5863, Guayaquil, Ecuador

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Abstract: In this paper we discuss the existence and uniqueness of solutions of the nonlinear Hammerstein integral equation, Volterra–Hammerstein integral equation and Volterra integral equation in the space of functions of bounded variation in the sense of Shiba, $(\Lambda_p BV)$.

Keywords: Integral equations, Λ_p bounded variation, Hammerstein integral equation, Volterra-Hammerstein integral equation , Volterra integral equations, existence and uniqueness.

1 Introduction

In this paper, we investigate solutions in some spaces of functions of bounded variation to the so-called Volterra-Hammerstein integral equations. Functions of bounded variation often appear as solutions of non-linear integral equations. Integral equations play an important role in mathematical analysis and its applications to real-world problems. See [1], [2], [3], [4], [5], [6], [7].

Many problems arising from physics, engineering, biology, economics, the relationship with vehicular traffic, the theory of optimal control, modern computing, lead to nonlinear mathematical models described by integral equations, (See [1], [4], [7]). The beginnings of the integral equations can be traced back when Pierre-Simon Laplace, in 1782, used what is now known as the Laplace transform to solve problems of linear difference and differential equations. Later, in 1826, Niels Henrik Abel solved the integral equation

$$u(t) = \int_a^t (t-s)^{-\alpha}\phi(s)ds,$$

where $u(t)$ is a continuous function such that $u(a) = 0$, ϕ is an unknown function, and $0 < \alpha < 1$. In the same year, Siméon Denis Poisson in a memory on the theory of magnetism explored the integral equation

$$\phi(t) = u(t) + \int_0^t k(t-s)\phi(s)ds,$$

in which ϕ is an unknown function. An important type of integral equation is that of Volterra, which was introduced by the Italian mathematician Vito Volterra [8] and his studies published at the end of the 19th century. Volterra equations are classified into two groups. An equation of first type is:

$$f(t) = \int_a^t K(t,s)x(s)ds.$$

A linear Volterra equation of the second type is:

$$x(t) = f(t) + \int_a^t K(t,s)x(s)ds,$$

under the following conditions:

1. Both t and s take values from a to certain $t > 0$; and
2. $K(t,s) = 0$, if at least one of the values of s is greater than the corresponding one of point t,

where one of the limits of integration of the integral is variable. Functions $f(t)$ and $K(t,s)$ are known functions, and $K(t,s)$ is called the kernel of the equation.

* Corresponding author e-mail: lilireperez@gmail.com, lrperez@espol.edu.ec

Volterra integral equations appear in several applications, such as demography, the study of visco-elastic materials, evolutionary problems in biology, epidemic propagation, neurophysiology, the control theory, the study of the behavior of nuclear reactors, mathematics of insurance through the equation of renewal.

On the other hand, the Hammerstein integral equation appears in the nonlinear physical phenomena, such as the dynamics of electromagnetic fluids, and the reformulation of boundary problems with a nonlinear boundary condition of Hammerstein type, see [6], [9].

Once the integral equations have been studied in the space of continuous or derived functions using Riemann integration, it was natural to consider these equations in other classes of function spaces and with other types of integrals. Recently, the existence and uniqueness of solutions of certain nonlinear integral equations in spaces of functions of widespread limited variation have been deeply investigated. For some examples we refer the reader to [10], [11], [12], [13], [14], and the references given therein.

For example, in [14] Bugajewska and O'Regan studied the existence and uniqueness of local and global solutions for nonlinear Hammerstein integral equation as well as Volterra-Hammerstein equation in the space of functions of bounded variation in the sense of Waterman, i.e. the space of functions of bounded Λ-variation, denoted by ΛBV. This was conducted by applying classical methods of nonlinear analysis, namely the Banach fixed point theorem and the Leray-Schauder nonlinear alternative Theorem. Specifically they conducted the Hammerstein integral equation, which is defined for $t\in I=[0,b]$ as

$$x(t)=g(t)+\lambda\int_I K(t,s)f(x(s))\,ds, \quad \lambda\in I, \tag{1.1}$$

and the Volterra–Hammerstein integral equation

$$x(t)=g(t)+\int_0^t K(t,s)f(x(s))\,ds,, \tag{1.2}$$

where integration is considered the Lebesgue sense, and the following hypotheses are assumed:

$(H_1) g: I\to\mathbb{R}$ is a function of Λ-bounded variation.
$(H_2) f:\mathbb{R}\to\mathbb{R}$ is a locally Lipschitz function.
$(H_3) K: I\times I\to\mathbb{R}$ is a function such that

$$V_\Lambda(K(\cdot,s):I)\le M(s), \quad \text{for a.e.}\, s\in I,$$

where $M: I\to\mathbb{R}$ is a Lebesgue integrable functions and $K(t,\cdot)$ is Lebesgue integrable for every $t\in I$.

With some additional conditions, in [14] the following result about the solutions of equation (1.2) was proved:*There exists an interval $J\subset I$ such that the equation (1.2) has a unique solution in ΛBV defined on J.*

In [15] Matute handled the solutions of the Volterra equation

$$x(t)=g(t)+\int_a^t K(t,s)f(x(s))ds, \quad t\in I=[a,b], \tag{1.3}$$

in the space of functions of bounded variation defined on the interval $I=[a,b]$. Also, under the following hypotheses

$(\widetilde{H}_1) g: I\to\mathbb{R}$ is a function of bounded variation.
$\widetilde{H}_2) f:\mathbb{R}\to\mathbb{R}$ is globally Lipschitz with the Lipschitz constant $L>0$. Furthemore, there is a real number

$$\alpha\ge 0 \text{ such that } \max_{s\in[-r,r]}|f(s)|\le(r+\alpha) \text{ for each } r\ge 0.$$

$(\widetilde{H}_3)$The function $K:\{(t,s)\in[a,b]\times[a,b]: s\le t\}\to\mathbb{R}$, is Lebesgue integrable on $[a,t]$ for each $t\in I=[a,b]$ and

$$V(K(\cdot,s):[s,b])\le h(s), \quad \text{for a.e. } s\in I,$$

where $h: I\to\mathbb{R}_+$ is a bounded Lebesgue integrable function,

Thus Matute proved that *If $\widetilde{H}_1$, $\widetilde{H}_2$ and $\widetilde{H}_3$ hold, then there exists a unique solution $x\in BV$, defined on I, for the Volterra equation (1.3).*

Our work is motivated by [14] and [15]. Here, we establish some hypotheses to characterize solutions of the Hammerstein equation (1.1), the Volterra-Hammerstein integral equation (1.2), and the Volterra equation (1.3), in the space of functions of bounded variation in the sense of Shiba, $(\Lambda_p BV)$, because this space is a broader class of functions whose structure is quite well known from the analytical point of view, as well as its characterization through one of the linear and nonlinear composition operators as studied in [16].

This paper is organized as follows: In section Two, we present some basic results which are necessary for the proofs of the main Theorems given later on. In section Three, we use the Banach fixed point Theorem to study the existence and uniqueness of solutions of the Hammerstein equation and the Volterra-Hammerstein integral equation. In section Four, we prove the existence and uniqueness of solutions for the Volterra equation whose proof is based on the Leray-Schauder alternative Theorem. Section Five, illustrates the subject with some applications. The final section is dedicated to conclusion.

2 Preliminaries

In this section we present some results essential for the development of the remainder of this article.

To study the solutions of the nonlinear integral equations (1.1), (1.2) and (1.3), we consider the following hypotheses

$(\widehat{H}_1) g: I\to\mathbb{R}$ is a function of Λ_p bounded variation.
$(\widehat{H}_2) f:\mathbb{R}\to\mathbb{R}$ is a locally Lipschitz function.

$(\widehat{H}_3) K : I\times I \to \mathbb{R}$ is a function such that

$$V_{\Lambda_p}(K(\cdot,s),I)\le M(s),\quad \text{for a.e. } s\in I,$$

where $M : I \to \mathbb{R}$ is a function L_p integrable and $K(t,\cdot)$ is Lebesgue integrable for each $t\in I=[0,b]$, and $V_{\Lambda_p}(\cdot,I)$ it's like in the definition 21.

The following theorems are fundamental basis in the proofs of the main theorems.

Theorem 21*(Banach's Contraction Principle)*
Let $f : X \to X$ be a contraction in a complete metric space X and let $B\subseteq X$ be a closed subset such that $f(B)\subseteq B$. Then f has a unique fixed point in B.

Theorem 22*(Leray-Schauder alternative)*
Let U be an open subset of a Banach's space $(X,\|\cdot\|)$ with $0\in U$, suppose that there is a continuous non-decreasing function $\phi : [0,+\infty)\to[0,+\infty)$ such that $\phi(z)<z$ for $z>0$, the function $H : \overline{U}\to X$ fulfills

$$\|H(x)-H(y)\|\le\phi\,\|x-y\|\ for\ x,y\in\overline{U},$$

where $\overline{U}$ is the closure of U in X. Also $H(U)$ is bounded and $x\ne\lambda H(x)$ for $x\in\partial U$ (where ∂U denotes the boundary of U) and $\lambda\in(0,1]$. Then H has a unique fixed point in U.

Proof. See [17]. □

The following Lemma states a classical inequality widely used in measure and probability Theorems.

Lemma 21 ***(Integral Jensen inequality)***
Let $I\subset\mathbb{R}$ be an open interval and let $f : I\to\mathbb{R}$ be a continuous convex function. Then for each normalized measure space $(\Omega,\sum,\mu)$ and for all μ integral functions $\phi : \Omega\to I$,

$$f\left(\int_\Omega \phi\, d\mu\right)\le\int_\Omega (f_\circ\phi)\, d\mu.$$

□

The next section covers a number of necessary preliminary notions, remarks and background to set the context for our main results.

Definition 21*(see [18]) Given an interval I, and a sequence of non-decreasing positive real numbers $\Lambda=\{\lambda_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{m}\frac{1}{\lambda_n}$ diverges and $1\le p<\infty$, we say that $f\in\Lambda_pBV(I)$* (i.e. f is a function of Λ_p-bounded variation over I, in the sense of Shiba) *if*

$$V_{\Lambda_p}(f,I)=\sup_{\{I_n\}}\left\{\sum_{n=1}^{N}\frac{|f(I_n)|^p}{\lambda_n}\right\}^{\frac{1}{p}}<\infty,$$

where $\{I_n\}$ is a sequence of non-overlapping intervals $I_n=[a_n,b_n]\subset[a,b]$ and $f(I_n)=f(b_n)-f(a_n)$.

Remark 21 $\Lambda_pBV(I)$ equipped with the norm $\|x\|_{\Lambda_p}:=|x(a)|+V_{\Lambda_p}(x)$ is a Banach space.

Remark 22 It is easy to verify that if f is a function locally Lipschitz and $x\in\Lambda_pBV(I)$, then $f\circ x\in\Lambda_pBV(I)$. Because

$$\sum_{n=1}^{N}\frac{|f(x(b_n))-f(x(a_n))|^p}{\lambda_n}\le|L_r|^p\sum_{n=1}^{N}\frac{|(x(b_n)-x(a_n))|^p}{\lambda_n}.$$

Lemma 22 *There exists a constant C such that $\sup_{t\in I}|x(t)|\le C\|x\|_{\Lambda_p}$ for any $x\in\Lambda_pBV(I)$.*

Proof. The proof is analogous to the one given in [14]. □

Lemma 23 *If $f\in\Lambda_pBV(I)$, then f has both left-hand and right-hand limits at every point of I.*

Proof. See in [18]. □

Lemma 24 *Suppose that hypotheses $\widehat{H}_2$, $\widehat{H}_3$ hold and let $F(x)(t):=\int_I K(t,s)f(x(s))ds$ for all $x\in\Lambda_pBV(I)$, $I=[0,b]$. Then*

$$V_{\Lambda_p}(F(x))\le\sup_{s\in I}|f(x(s))|\left(\int_I (M(s))^p\, ds\right)^{\frac{1}{p}}<+\infty.$$

Proof. By remark 22, we have that $f(x)\in\Lambda_pBV(I)$, so it is bounded and by Lemma 23 f has both left-hand and right-hand limits at every point of I. The set of discontinuities of a regulated function is at most countable. Thus, $f(x)$ is measuarable in the Lebesgue sense. As $K(t,\cdot)$ is Lebesgue integrable for every $t\in I$, we have that $K(t,\cdot)f(x(.))$ is Lebesgue integrable for every $t\in I$. Therefore the function $F(x)$ is well-defined. Let $\{I_n\}$ be a sequence of non-overlapping intervals $I_n=[a_n,b_n]\subset[0,b]=I$ with $n=1,...,N$ and let $r>0$. Then

$$\begin{aligned}
&\sum_{n=1}^{N}\frac{|F(x)(I_n)|^p}{\lambda_n}\\
&=\sum_{n=1}^{N}\frac{|F(x)(b_n)-F(x)(a_n)|^p}{\lambda_n}\\
&=\sum_{n=1}^{N}\frac{|\int_I K(b_n,s)f(x(s))ds-\int_I K(a_n,s)f(x(s))ds|^p}{\lambda_n}\\
&=\sum_{n=1}^{N}\frac{|\int_I[K(b_n,s)-K(a_n,s)]f(x(s))ds|^p}{\lambda_n}.
\end{aligned}$$

Since $p\ge1$, the function x^p is convex on $\mathbb{R}$ and, by the earlier arguments, we also have that $[K(b_n,s)-K(a_n,s)]f(x(s))$ is Lebesgue integrable. Thus, the hypotheses of Lemma 21 are satisfied, and consequently

$$\begin{aligned}
&\sum_{n=1}^{N} \frac{|F(x)(b_n)-F(x)(a_n)|^p}{\lambda_n} \\
&\leq \sum_{n=1}^{N} \frac{\int_I |[K(b_n,s)-K(a_n,s)]\, f(x(s))|^p\, ds}{\lambda_n} \\
&\leq \sum_{n=1}^{N} \sup_{s\in I} |f(x(s))|^p \int_I \frac{|K(b_n,s)-K(a_n,s)|^p}{\lambda_n} ds \\
&= \sup_{s\in I} |f(x(s))|^p \int_I \sum_{n=1}^{N} \frac{|K(b_n,s)-K(a_n,s)|^p}{\lambda_n} ds \\
&\leq \sup_{s\in I} |f(x(s))|^p \int_I V^p_{\Lambda_p}(K(\cdot,s))ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left(\sum_{n=1}^{N} \frac{|F(x)(b_n)-F(x)(a_n)|^p}{\lambda_n}\right)^{\frac{1}{p}} \\
&\leq \sup_{s\in I} |f(x(s))| \left(\int_I V^p_{\Lambda_p}(K(\cdot,s))ds\right)^{\frac{1}{p}},
\end{aligned}$$

taking supremum in this inequality and using hypothesis $\widehat{H}_3$ we conclude that

$$\sup_{s\in I} |f(x(s))| \left(\int_I (M(s))^p\, ds\right)^{\frac{1}{p}} < +\infty.$$

□

Lemma 25 *Suppose that hypotheses $\widehat{H}_2$, $\widehat{H}_3$ hold and let $F(x)$ be an integral function defined as in the previous Lemma. Then for all $x,y \in \Lambda_p BV(I)$ and $\lambda \in I$,*

$$V_{\Lambda_p}(\lambda\,(F(x)-F(y))) \leq C\,|\lambda|\,L_r\,\|x-y\|_{\Lambda_p} \left(\int_I (M(s))^p\, ds\right)^{\frac{1}{p}},$$

where C is as in Lemma 22 and L_r is the Lipschitz constant associated with f when it is restricted to the interval I.

Proof. Consider $x,y \in \Lambda_p BV(I)$ and let $\{I_n\}$ be a sequence of non-overlapping intervals $I_n = [a_n,b_n] \subset [0,b] = I$ with $n = 1,\ldots,N$. Then

$$\begin{aligned}
&\sum_{n=1}^{N} \frac{|\lambda F(x)(I_n)-\lambda F(y)(I_n)|^p}{\lambda_n} \\
&= \sum_{n=1}^{N} \frac{|\lambda \int_I [K(b_n,s)-K(a_n,s)]\,[f(x(s))-f(y(s))]\, ds|^p}{\lambda_n}.
\end{aligned}$$

Applying Lemma 21 and using the fact that f locally Lipschitz we have

$$\begin{aligned}
&\sum_{n=1}^{N} \frac{|\lambda F(x)(I_n)-\lambda F(y)(I_n)|^p}{\lambda_n} \\
&\leq \sum_{n=1}^{N} \frac{\int_I |\lambda|^p\, |K(b_n,s)-K(a_n,s)|^p\, |f(x(s))-f(y(s))|^p\, ds}{\lambda_n} \\
&\leq \left(L_r|\lambda| \sup_{s\in I} |x(s)-y(s)|\right)^p \int_I \sum_{n=1}^{N} \frac{|K(b_n,s)-K(a_n,s)|^p}{\lambda_n} ds.
\end{aligned}$$

Applying Lemma 22, from hypothesis $\widehat{H}_3$ we have

$$\begin{aligned}
&\left(\sum_{n=1}^{N} \frac{|\lambda|\,|F(x)(I_n)-F(y)(I_n)|^p}{\lambda_n}\right)^{\frac{1}{p}} \\
&\leq L_r|\lambda| C\, \|x-y\|_{\Lambda_p} \left(\int_I \sum_{n=1}^{N} \frac{|K(b_n,s)-K(a_n,s)|^p}{\lambda_n} ds\right)^{\frac{1}{p}} \\
&\leq CL_r|\lambda|\, \|x-y\|_{\Lambda_p} \left(\int_I V^p_{\Lambda_p}(K(.s))\, ds\right)^{\frac{1}{p}} \\
&\leq CL_r|\lambda|\, \|x-y\|_{\Lambda_p} \left(\int_I (M(s))^p\, ds\right)^{\frac{1}{p}}.
\end{aligned}$$

Thus, we conclude that

$$V_{\Lambda_p}(\lambda(F(x)-F(y))) \leq CL_r|\lambda|\, \|x-y\|_{\Lambda_p} \left(\int_I (M(s))^p\, ds\right)^{\frac{1}{p}}.$$

□

The following Lemma is a special case of the triangular inequality; however, for the work is self-contained, we prove that this inequality is satisfied in $(\Lambda_p BV)$.

Lemma 26 *Suppose that hypotheses $\widehat{H}_1$, $\widehat{H}_2$ and $\widehat{H}_3$ hold and define $G(x) : \Lambda_p BV(I) \to \Lambda_p BV(I)$ by $G(x)(t) = g(t) + \lambda F(x)(t)$, with $F(x)$ is as in Lemma 24 and $\lambda \in I = [0,b]$. Then*

$$\|G(x)\|_{\Lambda_p} \leq \|g\|_{\Lambda_p} + |\lambda|\, \|F(x)\|_{\Lambda_p}.$$

Proof. In Lemma 24 it was shown that $F(x)$ is well-defined and that g is a bounded variation; consequently we may conclude that $G(x)$ is well-defined. Let $\{I_n\}$ be a sequence of non-overlapping intervals $I_n = [a_n,b_n] \subset [0,b] = I$ with $n = 1,\ldots,N$. Since g is of Λ_p-bounded variation by hypothesis $\widehat{H}_1$, and by Lemma 24 $F(x)$ is of Λ_p-bounded variation, the function $G(x)$ is of Λ_p-bounded variation. Hence

$$\begin{aligned}
\|G(x)\|_{\Lambda_p} &= |G(x)(0)| + V_{\Lambda_p}(G(x)) \\
&= |g(0)+\lambda F(x)(0)| + V_{\Lambda_p}(g+\lambda F(x)) \\
&\leq |g(0)| + |\lambda|\,|F(x)(0)| + V_{\Lambda_p}(g+\lambda F(x)). \qquad (2.4)
\end{aligned}$$

Let us compute $V_{\Lambda_p}(g+\lambda F(x))$. Observe that

$$\left(\sum_{n=1}^{N}\frac{|(g(b_n)+\lambda F(x)(b_n))-(g(a_n)+\lambda F(x)(a_n))|^p}{\lambda_n}\right)^{\frac{1}{p}}=$$

$$\left(\sum_{n=1}^{N}\frac{|(g(b_n)-g(a_n))+\lambda\int_I(K(b_n,s)-K(a_n,s))f(x(s))ds|^p}{\lambda_n}\right)^{\frac{1}{p}},$$

so from Minkowski inequality and Lemma 21 we have

$$\begin{aligned}
&\left(\sum_{n=1}^{N}\frac{|(g(b_n)+\lambda F(x)(b_n))-(g(a_n)+\lambda F(x)(a_n))|^p}{\lambda_n}\right)^{\frac{1}{p}}\\
&\leq\left(\sum_{n=1}^{N}\frac{|g(b_n)-g(a_n)|^p}{\lambda_n}\right)^{\frac{1}{p}}+\\
&\quad\left(\sum_{n=1}^{N}\frac{|\int_I\lambda(K(b_n,s)-K(a_n,s))f(x(s))ds|^p}{\lambda_n}\right)^{\frac{1}{p}}\\
&\leq\left(\sum_{n=1}^{N}\frac{|g(b_n)-g(a_n)|^p}{\lambda_n}\right)^{\frac{1}{p}}+\\
&\quad\left(\sum_{n=1}^{N}\frac{\int_I|\lambda|^p|K(b_n,s)-K(a_n,s)|^p|f(x(s))|^p ds}{\lambda_n}\right)^{\frac{1}{p}}\\
&=\left(\sum_{n=1}^{N}\frac{|g(b_n)-g(a_n)|^p}{\lambda_n}\right)^{\frac{1}{p}}+\\
&\quad|\lambda|\left(\sum_{n=1}^{N}\frac{\int_I|K(b_n,s)-K(a_n,s)|^p|f(x(s))|^p ds}{\lambda_n}\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence, taking supremum on both sides of these inequalities, we conclude that

$$V_{\Lambda_p}(g+\lambda F(x))\leq V_{\Lambda_p}(g)+|\lambda|V_{\Lambda_p}(F(x)).$$

Finally, taking into account this estimate in inequality (2.4), we obtain

$$\begin{aligned}
\|G(x)\|_{\Lambda_p}&\leq|g(0)|+|\lambda|\,|F(x)(0)|+V_{\Lambda_p}(g)+|\lambda|V_{\Lambda_p}(F(x))\\
&=\left[|g(0)|+V_{\Lambda_p}(g)\right]+\left[|\lambda|\,|F(x)(0)|+|\lambda|V_{\Lambda_p}(F(x))\right]\\
&=\|g\|_{\Lambda_p}+|\lambda|\,\|F(x)\|_{\Lambda_p}.
\end{aligned}$$

□

Lemma 27 *Let $T=\{(t,s):0\leq t\leq b,0\leq s\leq t\}$ and suppose that $K:T\to\mathbb{R}$ is a function of Λ_p-bounded variation $(p\geq 1)$. Set*

$$\widehat{K}(t,s)=\begin{cases}K(t,s), & 0\leq s\leq t\\ 0, & t<s\leq b.\end{cases}$$

Then

$$V_{\Lambda_p}\left(\widehat{K}(.,s),[0,b]\right)\leq 2\left(\frac{|K(s,s)|}{(\lambda_1)^{\frac{1}{p}}}+V_{\Lambda_p}(K(\cdot,s),[s,b])\right).$$

Proof. Let $\{I_n\}$ be a sequence of non-overlapping intervals $I_n=[a_n,b_n]\subset[0,b]=I$ with $n=1,...,N$, and choose $s\in[a_i,b_i]$ for some i, $1\leq i\leq N$. Calculating $V_{\Lambda_p}\left(\widehat{K}(.,s),[0,b]\right)$, we have

$$\begin{aligned}
&\sum_{n=1}^{N}\frac{|\widehat{K}(b_n,s)-\widehat{K}(a_n,s)|^p}{\lambda_n}\\
&=\frac{|\widehat{K}(b_1,s)-\widehat{K}(a_1,s)|^p}{\lambda_1}+\cdots+\frac{|\widehat{K}(b_i,s)-\widehat{K}(a_i,s)|^p}{\lambda_i}\\
&\quad+\cdots+\frac{|\widehat{K}(b_N,s)-\widehat{K}(a_N,s)|^p}{\lambda_N}\\
&=\frac{|K(b_i,s)|^p}{\lambda_i}+\frac{|K(b_{i+1},s)-K(a_{i+1},s)|^p}{\lambda_{i+1}}+\cdots+\\
&\quad\frac{|K(b_N,s)-K(a_N,s)|^p}{\lambda_N}\\
&\leq\frac{(|K(b_i,s)-K(s,s)|+|K(s,s)|)^p}{\lambda_i}+\frac{|K(b_{i+1},s)-K(a_{i+1},s)|^p}{\lambda_{i+1}}\\
&\quad+\cdots+\frac{|K(b_N,s)-K(a_N,s)|^p}{\lambda_N}\\
&\leq\frac{2^p|K(s,s)|^p}{\lambda_i}+\frac{2^p|K(b_i,s)-K(s,s)|^p}{\lambda_i}+\\
&\quad\frac{|K(b_{i+1},s)-K(a_{i+1},s)|^p}{\lambda_{i+1}}+\cdots+\frac{|K(b_N,s)-K(a_N,s)|^p}{\lambda_N}\\
&\leq\frac{2^p|K(s,s)|^p}{\lambda_1}+2^pV_{\Lambda_p}^p(K(\cdot,s),[s,b]).
\end{aligned}$$

Therefore

$$V_{\Lambda_p}\left(\widehat{K}(.,s),[0,b]\right)\leq\left(\frac{2^p|K(s,s)|^p}{\lambda_1}+2^pV_{\Lambda_p}^p(K(\cdot,s),[s,b])\right)^{\frac{1}{p}}.$$

Because $\frac{1}{p}<1$, it follows that $(a+b)^{\frac{1}{p}}\leq a^{\frac{1}{p}}+b^{\frac{1}{p}}$. Hence,

$$V_{\Lambda_p}\left(\widehat{K}(.,s),[0,b]\right)\leq 2\left(\frac{|K(s,s)|}{(\lambda_1)^{\frac{1}{p}}}+V_{\Lambda_p}(K(\cdot,s),[s,b])\right).$$

□

3 Existence and uniqueness of solutions for the Hammerstein integral equation and the Volterra-Hammerstein integral equation

In this section, we will prove two main theorems of this work that guarantee the existence and uniqueness of the solutions of the equations (1.1) and (1.2), respectively, in the space of function of Shiba bunded variation. In addition to the hypotheses raised in section2, we consider the additional hypothesis

$(\widehat{H}_4)$Let $T=\{(t,s):0\leq t\leq b,0\leq s\leq t\}$ and $K:T\to\mathbb{R}$ such that

$$\frac{|K(s,s)|}{(\lambda_1)^{\frac{1}{p}}}+V_{\Lambda_p}(K(\cdot,s):[s,b])\leq m(s),\text{ for a.e. } s\in I,$$

where $m : I \to \mathbb{R}_+$ is a function L_p integrable and $K(t,\cdot)$ is Lebesgue integrable on $[0,t]$ for every $t \in [0,b]$.

Theorem 31 *Suppose that hypotheses $\widehat{H}_1$, $\widehat{H}_2$ and $\widehat{H}_3$ hold, then there exists a number $\tau > 0$ such that for every λ with $|\lambda| < \tau$, the equation (1.1) has a unique $\Lambda_p BV$-solution, defined on $I = [0,b]$.*

Proof. Let $\{I_n\}$ be a sequence of non-overlapping intervals $I_n = [a_n,b_n] \subset I = [0,b]$ with $n = 1,\ldots,N$.
Define $G(x) : \Lambda_p BV(I) \to \Lambda_p BV(I)$ by

$$G(x)(t) = g(t) + \lambda \int_I K(t,s) f(x(s))ds.$$

Let $r > 0$ such that $\|g\|_{\Lambda_p} < r$ and choose a real number $\tau > 0$ such that

$$\|g\|_{\Lambda_p} + \sup_{s\in I} \tau |f(x(s))| \left[\int_I |K(0,s)|\,ds + \left(\int_I (M(s))^p \right)^{\frac{1}{p}} \right] < r \tag{3.5}$$

and

$$C\tau L_r \left[\int_I |K(0,s)|\,ds + \left(\int_I (M(s))^p \right)^{\frac{1}{p}} \right] < 1, \tag{3.6}$$

where C is guaranteed by Lemma 22 and L_r is the Lipschitz constant associated with f when restricted to the interval I. In order to prove this Theorem we will use Banach's Theorem 21. Denote by $\overline{B}_r$, the closed ball of center zero and radius r in the space $\Lambda_p BV(I)$; that is, $\overline{B}_r := \overline{\{x \in \Lambda_p BV(I) / \|x\|_{\Lambda_p} < r\}}$.
Let us start by proving the inclusion $G(\overline{B}_r) \subset \overline{B}_r$.
Indeed, by Lemma 26, for any $x \in \Lambda_p BV(I)$ we have

$$\|G(x)\|_{\Lambda_p} \le \|g\|_{\Lambda_p} + |\lambda|\, \|F(x)\|_{\Lambda_p}. \tag{3.7}$$

On the other hand,

$$\begin{aligned}\|F(x)\|_{\Lambda_p} &= |F(x)(0)| + V_{\Lambda_p}(F(x)) \\ &= \left| \int_I K(0,s) f(x(s))ds \right| + V_{\Lambda_p}(F(x)) \\ &\le \sup_{s\in I} |f(x(s))| \int_I |K(0,s)|\,ds + V_{\Lambda_p}(F(x)),\end{aligned} \tag{3.8}$$

so by Lemma 24,

$$\|F(x)\|_{\Lambda_p} \le \sup_{s\in I} |f(x(s))| \left[\int_I |K(0,s)|\,ds + \left(\int_I (M(s))^p\,ds \right)^{\frac{1}{p}} \right].$$

Thus, from inequalities (3.7) and (3.5) we get

$$\begin{aligned}&\|G(x)\|_{\Lambda_p} \\ &\le \|g\|_{\Lambda_p} + |\lambda| \sup_{s\in I} |f(x(s))| \left[\int_I |K(0,s)|\,ds + \left(\int_I (M(s))^p\,ds \right)^{\frac{1}{p}} \right] \\ &\le \|g\|_{\Lambda_p} + \sup_{s\in I} \tau |f(x(s))| \left[\int_I |K(0,s)|\,ds + \left(\int_I (M(s))^p\,ds \right)^{\frac{1}{p}} \right] \\ &< r.\end{aligned}$$

Hence, $G(\overline{B}_r) \subset \overline{B}_r$.
Now we proceed to show that G is a contraction. Indeed, for any $x, y \in \overline{B}_r$ we have

$$\begin{aligned}&\|G(x) - G(y)\|_{\Lambda_p} \\ &= |G(x)(0) - G(y)(0)| + V_{\Lambda_p}(G(x) - G(y)) \\ &= |\lambda|\, |F(x)(0) - F(y)(0)| + V_{\Lambda_p}(\lambda (F(x) - F(y))).\end{aligned}$$

But

$$\begin{aligned}|F(x)(0) - F(y)(0)| &= \left| \int_I K(0,s)[f(x(s)) - f(y(s))]ds \right| \\ &\le \int_I |K(0,s)|\, |[f(x(s)) - f(y(s))]|\,ds,\end{aligned}$$

since f is locally Lipschitz, so using Lemma 22 we get

$$\begin{aligned}|F(x)(0) - F(y)(0)| &\le L_r \sup_{s\in I} |x(s) - y(s)| \int_I |K(0,s)|\,ds \\ &\le L_r C \|x - y\|_{\Lambda_p} \int_I |K(0,s)|\,ds.\end{aligned} \tag{3.9}$$

Now, by Lemma 25,

$$V_{\Lambda_p}(\lambda(F(x) - F(y))) \le C L_r |\lambda|\, \|x - y\|_{\Lambda_p} \left(\int_I (M(s))^p\,ds \right)^{\frac{1}{p}};$$

thus, from inequality (3.9) it follows that

$$\begin{aligned}&\|G(x) - G(y)\|_{\Lambda_p} \le \\ &C L_r \tau \left[\int_I |K(0,s)|\,ds + \left(\int_I (M(s))^p\,ds \right)^{\frac{1}{p}} \right] \|x - y\|_{\Lambda_p}.\end{aligned}$$

We conclude, by inequality (3.6), that $G(x)$ is a contraction. Consequently, by Theorem 21, $G(x)$ has a unique fixed point in $\overline{B}_r$, i.e. there exists $x \in \overline{B}_r$ such that

$$g(t) + \lambda \int_I K(t,s) f(x(s))ds = x(t).$$

Therefore, x is a unique solution of equation (1.1). □

Theorem 32 *Suppose that hypotheses $\widehat{H}_1$, $\widehat{H}_2$ and $\widehat{H}_4$ hold, then equation (1.2) has a unique $\Lambda_p BV$-solution, defined on $I = [0,b]$.*

Proof. Working as in the proof of Theorem 31, let $r > 0$ and L_r is the Lipschitz constant associated with f when restricted to the interval I, that

$$\|g\|_{\Lambda_p} + 2 \sup_{s\in I} |f(x(s))| \left[\left(\int_0^b (m(s))^p \right)^{\frac{1}{p}} \right] < r \tag{3.10}$$

$$\text{and} \quad 2CL_r \left[\left(\int_0^b (m(s))^p \right)^{\frac{1}{p}} \right] < 1, \tag{3.11}$$

where C is guaranteed by Lemma 22.
Let $\{I_n\}$ be a sequence of non-overlapping intervals $I_n = [a_n,b_n] \subset I = [0,b]$ with $n = 1,\ldots,N$. Define the function

$$\widetilde{G}(x)(t) = g(t) + \widetilde{F}(x)(t),$$

where

$$\widetilde{F}(x)(t) := \int_0^t K(t,s)f(x(s))ds \quad \text{for } t \in [0,b],$$

$x \in \overline{B}_r = \overline{\{x \in \Lambda_p BV(I)/\|x\|_{\Lambda_p} < r\}}$ and set $\widehat{K}(t,s) = \begin{cases} K(t,s), & 0 \le s \le t \\ 0, & t < s \le b. \end{cases}$

Let us prove that $\widetilde{G}(\overline{B}_r) \subset \overline{B}_r$. Working as in the proof of Lemma 26 we have that

$$\left\|\widetilde{G}(x)\right\|_{\Lambda_p} \le \|g\|_{\Lambda_p} + \left\|\widetilde{F}(x)\right\|_{\Lambda_p}, \tag{3.12}$$

but

$$\begin{aligned}\left\|\widetilde{F}(x)\right\|_{\Lambda_p} &= \left|\widetilde{F}(x)(0)\right| + V_{\Lambda_p}(\widetilde{F}(x)) \\ &= \int_0^0 K(t,s)f(x(s))ds + V_{\Lambda_p}(\widetilde{F}(x)) \\ &= V_{\Lambda_p}(\widetilde{F}(x)).\end{aligned}$$

In order to estimate $V_{\Lambda_p}(\widetilde{F}(x))$, we proceed as in the proof of Lemma 24 to obtain

$$\begin{aligned}&\sum_{n=1}^N \frac{\left|\widetilde{F}(x)(b_n) - \widetilde{F}(x)(a_n)\right|^p}{\lambda_n} \\ &= \sum_{n=1}^N \frac{\left|\int_0^{b_n} K(b_n,s)f(x(s))ds - \int_0^{a_n} K(a_n,s)f(x(s))ds\right|^p}{\lambda_n} \\ &= \sum_{n=1}^N \frac{\left|\int_0^b \widehat{K}(b_n,s)f(x(s))ds - \int_0^b \widehat{K}(a_n,s)f(x(s))ds\right|^p}{\lambda_n} \\ &= \sum_{n=1}^N \frac{\left|\int_0^b \left[\widehat{K}(b_n,s) - \widehat{K}(a_n,s)\right] f(x(s))ds\right|^p}{\lambda_n} \\ &\le \sum_{n=1}^N \sup_{s\in I}|f(x(s))|^p \int_0^b \frac{\left|\widehat{K}(b_n,s) - \widehat{K}(a_n,s)\right|^p}{\lambda_n} ds \\ &= \sup_{s\in I}|f(x(s))|^p \int_0^b \sum_{n=1}^N \frac{\left|\widehat{K}(b_n,s) - \widehat{K}(a_n,s)\right|^p}{\lambda_n} ds.\end{aligned}$$

Thus

$$\begin{aligned}&\left(\sum_{n=1}^N \frac{\left|\widetilde{F}(x)(b_n) - \widetilde{F}(x)(a_n)\right|^p}{\lambda_n}\right)^{\frac{1}{p}} \\ &\le \sup_{s\in I}|f(x(s))| \left(\int_0^b V^p_{\Lambda_p}(\widehat{K}(.,s),[0,b])ds\right)^{\frac{1}{p}}\end{aligned}$$

and by Lemma 27, we have

$$\begin{aligned}&V_{\Lambda_p}(\widetilde{F}(x)) \le \\ &\le \sup_{s\in I}|f(x(s))| \left(\int_0^b 2^p \left[\frac{|K(s,s)|}{(\lambda_1)^{\frac{1}{p}}} + V_{\Lambda_p}(K(\cdot,s),[s,b])\right]^p ds\right)^{\frac{1}{p}}.\end{aligned}$$

Now, by hypothesis $\widehat{H}_4$ we have that

$$\frac{|K(s,s)|}{(\lambda_1)^{\frac{1}{p}}} + V_\Lambda(K(\cdot,s):[s,b]) \le m(s),$$

where m is an L_p integrable function. Hence,

$$V_{\Lambda_p}(\widetilde{F}(x)) \le \sup_{s\in I} 2|f(x(s))| \left(\int_0^b (m(s))^p ds\right)^{\frac{1}{p}}.$$

Thus, using this last estimate in (3.12), we obtain

$$\begin{aligned}\left\|\widetilde{G}(x)\right\|_{\Lambda_p} &\le \|g\|_{\Lambda_p} + \sup_{s\in I} 2|f(x(s))| \left(\int_0^b (m(s))^p ds\right)^{\frac{1}{p}} \\ &< r.\end{aligned}$$

Now, we will show that $\widetilde{G}(x)$ is a contraction. For any $x, y \in \overline{B}_r$, we have

$$\left\|\widetilde{G}(x) - \widetilde{G}(y)\right\|_{\Lambda_p} = V_{\Lambda_p}\left(\widetilde{F}(x) - \widetilde{F}(y)\right).$$

But

$$\begin{aligned}&\sum_{n=1}^N \frac{\left|\widetilde{F}(x)(I_n) - \widetilde{F}(y)(I_n)\right|^p}{\lambda_n} \\ &= \sum_{n=1}^N \frac{\left|\int_0^b \left[\widehat{K}(b_n,s) - \widehat{K}(a_n,s)\right][f(x(s)) - f(y(s))]ds\right|^p}{\lambda_n} \\ &\le \sum_{n=1}^N \sup_{s\in I}|f(x(s)) - f(y(s))|^p \int_0^b \frac{\left|\widehat{K}(b_n,s) - \widehat{K}(a_n,s)\right|^p}{\lambda_n} ds \\ &\le \left(L_r \sup_{s\in I}|x(s) - y(s)|\right)^p \int_0^b V^p_{\Lambda_p}(\widehat{K}(.,s),[0,b])ds.\end{aligned}$$

Hence, applying a similar technique as above, we have

$$V_{\Lambda_p}(\widetilde{F}(x) - \widetilde{F}(x) \le 2L_r C\|x-y\|_{\Lambda_p} \left(\int_0^b (m(s))^p ds\right)^{\frac{1}{p}}.$$

Moreover,

$$\left\|\widetilde{G}(x) - \widetilde{G}(y)\right\|_{\Lambda_p} \le 2L_r C \left(\int_0^b (m(s))^p ds\right)^{\frac{1}{p}} \|x-y\|_{\Lambda_p}.$$

Therefore, by inequality (3.11), $\widetilde{G}(x)$ is a contraction and by Theorem 21, we have that there exists a unique $\widetilde{x} \in \overline{B}_r$ such that

$$g(t) + \int_0^t K(t,s)f(\widetilde{x}(s))ds = \widetilde{x}(t).$$

Hence, $\widetilde{x}$ is the unique solution of equation (1.2). □

4 Existence and uniqueness of solutions for the Volterra equation

In this section we shall prove existence and uniqueness of solutions for the equation (1.3) in the space of functions of Shiba bounded variation. To prove these results two Theorems are established: one guarantees existence and the other guarantees uniqueness of solutions. In addition to the hypotheses raised in section2, we consider the additional hypothesis

$(\widehat{H}_5)$Let $K:\{(t,s)\in[a,b]\times[a,b]:s\le t\}\to\mathbb{R}$ the function such that

$$\frac{|K(s,s)|}{(\lambda_1)^{\frac{1}{p}}}+V_{\Lambda_p}(K(\cdot,s):[s,b])\le h(s),\text{ for a.e. } s\in[a,b],$$

where $h:I\to\mathbb{R}_+$ is a function L_p integrable and $K(t,\cdot)$ is Lebesgue integrable on $[a,t]$ for each $t\in[a,b]$.

Theorem 41 *Suppose that hypotheses $\widehat{H}_1$, $\widehat{H}_2$ and $\widehat{H}_5$ hold, then there exists a solution $\widehat{x}\in\Lambda_pBV$ for the equation (1.3).*

Proof. The proof of this Theorem is based on verifying that the hypotheses of Theorem22, the Leray-Shauder alternative, are satisfied. As in the proof of Theorem 32, let $r>0$ and let L_r be the Lipschitz constant such that

$$\|g\|_{\Lambda_p}+2\sup_{s\in[a,b]}|f(x(s))|\left[\left(\int_a^b(h(s))^p\right)^{\dfrac{1}{p}}\right]<r\quad\text{and}\tag{4.13}$$

$$2CL_r\left[\left(\int_a^b(h(s))^p\right)^{\dfrac{1}{p}}\right]<1.\tag{4.14}$$

Let $\{I_n\}$ be a sequence of non-overlapping intervals $I_n=[a_n,b_n]\subset[a,b]=I$ with $n=1,\dots,N$. Let us define the functions

$$H(x)(t)=g(t)+\widehat{F}(x)(t),\qquad\text{where}$$

$$\widehat{F}(x)(t)=\int_a^t K(t,s)f(x(s))ds\quad\text{with}\quad t\in[a,b],$$

$x\in\overline{B}_r=\overline{\{x\in\Lambda_pBV(I)/\|x\|_{\Lambda_p}<r\}}$ and

$$\widehat{K}(t,s)=\begin{cases}K(t,s), & a\le s\le t\\ 0, & t<s\le b;\end{cases}\tag{4.15}$$

By the same reasoning used in the proof of Theorem 32, we have that

$$\|H(x)\|_{\Lambda_p}\le\|g\|_{\Lambda_p}+\sup_{s\in[a,b]}2|f(x(s))|\left(\int_a^b(h(s))^p\,ds\right)^{\frac{1}{p}}<r.\tag{4.16}$$

$$\|H(x)-H(y)\|_{\Lambda_p}\le 2L_rC\left(\int_a^b(h(s))^p\,ds\right)^{\frac{1}{p}}\|x-y\|_{\Lambda_p}.\tag{4.17}$$

Now, we define the function $\phi:[0,+\infty)\to[0,+\infty)$ by

$$\phi(z)=\left[2L_rC\left(\int_a^b(h(s))^p\,ds\right)^{\frac{1}{p}}\right]z,$$

clearly $\phi(z)<z$ by inequality (4.14). Let $x\in\overline{B}_r$ be such that $x=\lambda H(x)$ for some $\lambda\in(0,1]$. Then

$$\|x\|_{\Lambda_p}=\lambda\|H(x)\|_{\Lambda_p}\le\|H(x)\|_{\Lambda_p}<r,\tag{4.18}$$

by inequality (4.16). In particular, $x\notin\partial B_r$. On the other hand, $H(x)\in\Lambda_pBV$ since $g,\widehat{F}(x)\in\Lambda_pBV$, so $H(x)$ is bounded. By Theorem 22, the Leray-Shauder Alternative, $H(x)$ has a single fixed point, that is, there exists $\widehat{x}\in\Lambda_pBV$ such that $H(\widehat{x})(t)=\widehat{x}(t)$. Thus, $\widehat{x}$ is solution of the Volterra equation (1.3). □

Theorem 42 *Suppose the hypotheses $\widehat{H}_1$, $\widehat{H}_2$ and $\widehat{H}_5$ hold, then there exists a unique solution $\widehat{x}\in\Lambda_pBV$ for the equation (1.3).*

Proof. Working as in the proofs of the previous Theorems. Let $\{I_n\}$ be a sequence of non-overlapping intervals $I_n=[a_n,b_n]\subset[a,b]=I$ with $n=1,\dots,N$ and let us take $s\in[a_i,b_i]$ for some i, $1\le i\le N$. Suppose without loss of generality that

$$L_r\left[2(b-a)^{\frac{1}{p}}\sup_{s\in[a,b]}|K(s,s)|+4(\lambda_1)^{\frac{1}{p}}\left(\int_a^b(h(s))^pds\right)^{\dfrac{1}{p}}\right]<1,\tag{4.19}$$

where L_r is the Lipschitz constant of f and $\sup_{s\in[a,b]}|K(s,s)|<+\infty$. By Theorem 41, there exists a solution of the Volterra equation (1.3). To prove uniqueness, we proceed by reduction to the absurd. Suppose that there are two different solutions of the equation (1.3), $\widehat{x}$, $\widehat{y}$. Then

$$\begin{aligned}|\widehat{y}(t)-\widehat{x}(t)|^p&=\left|\int_a^t K(t,s)[f(\widehat{y}(t))-f(\widehat{x}(t))]ds\right|^p\\&\le\int_a^t|K(t,s)|^p\,|f(\widehat{y}(t))-f(\widehat{x}(t))|^p\,ds\\&\le\int_a^t|K(t,s)|^p\,(L_r)^p\,|\widehat{y}(t)-\widehat{x}(t)|^p\,ds\end{aligned}$$

and so

$$|\widehat{y}(t)-\widehat{x}(t)|\le L_r\left(\int_a^t|K(t,s)|^p\,|\widehat{y}(t)-\widehat{x}(t)|^p\,ds\right)^{\frac{1}{p}}.$$

Hence

$$\sup_{s\in[a,b]}|\widehat{y}(s)-\widehat{x}(s)|\le L_r\sup_{s\in[a,b]}|\widehat{y}(s)-\widehat{x}(s)|\sup_{t\in[a,b]}\left(\int_a^t|K(t,s)|^p\,ds\right)^{\frac{1}{p}}$$

or

$$1 \leq L_r \sup_{t\in[a,b]} \left(\int_a^t |K(t,s)|^p \, ds \right)^{\frac{1}{p}}. \tag{4.20}$$

On the other hand, for $\tau \in [s,b]$

$$\frac{\left|\widehat{K}(\tau,s)-\widehat{K}(s,s)\right|^p}{\lambda_1} \leq \sum_{n=1}^{N} \frac{\left|\widehat{K}(b_n,s)-\widehat{K}(a_n,s)\right|^p}{\lambda_n} \leq V^p_{\Lambda_p}\left(\widehat{K}(.,s),[a,b]\right).$$

Raising to the $\frac{1}{p}$ power, applying basic absolute value inequality, Lemma 27 and the hypothesis $\widehat{H}_5$, we have

$$\begin{aligned}\frac{|K(\tau,s)|}{(\lambda_1)^{\frac{1}{p}}} &\leq \frac{|K(s,s)|}{(\lambda_1)^{\frac{1}{p}}} + 2\left[\frac{|K(s,s)|}{(\lambda_1)^{\frac{1}{p}}} + V_{\Lambda_p}\left(K(\cdot,s),[s,b]\right)\right] \\ &\leq \frac{|K(s,s)|}{(\lambda_1)^{\frac{1}{p}}} + 2h(s).\end{aligned}$$

Therefore,

$$|K(\tau,s)|^p \leq \lambda_1 2^p \left[\frac{|K(s,s)|^p}{\lambda_1} + (2h(s))^p\right].$$

Consequently

$$\begin{aligned}&\int_a^t |K(t,s)|^p \, ds \\ &\leq \int_a^t \sup_{\tau\in[s,b]} |K(\tau,s)|^p \, ds \\ &\leq \int_a^t \lambda_1 2^p \left[\frac{\sup_{s\in[a,b]} |K(s,s)|^p}{\lambda_1} + (2h(s))^p\right] ds \\ &\leq \int_a^b \lambda_1 2^p \left[\frac{\sup_{s\in[a,b]} |K(s,s)|^p}{\lambda_1} + (2h(s))^p\right] ds \\ &= 2^p(b-a) \sup_{s\in[a,b]} |K(s,s)|^p + \lambda_1 4^p \int_a^b (h(s))^p ds,\end{aligned}$$

from which we deduce that

$$\begin{aligned}&L_r \sup_{t\in[a,b]} \left(\int_a^t |K(t,s)|^p \, ds\right)^{\frac{1}{p}} \\ &\leq L_r \left[2(b-a)^{\frac{1}{p}} \sup_{s\in[a,b]} |K(s,s)| + 4(\lambda_1)^{\frac{1}{p}} \left(\int_a^b (h(s))^p ds\right)^{\frac{1}{p}}\right].\end{aligned}$$

From this and inequalities (4.19) and (4.20), we get a contradiction. Hence, the solution of equation (1.3) is unique. □

5 Applications

This section shows an application, where the non-linear integral equation of Hammerstein-Volterra is solved by numerical methods and it guarantees that the solution is considered unique in the space of functions Shiba bounded variation. In addition, the examples 52 and 53 are given to illustrate the conclusions of Theorems 31 and 32, respectively.

Example 51 (Aplication) Because dynamic models of chemical reactors or stationary study in a chemical reactor can be described by a Hammerstein-Volterra mixed integral equation, or by the fact that fluid dynamics can also be modeled by these, many researchers study the solutions of these types of nonlinear integral equations using numerical methods to find the approximate solution, for example the use of the hybrid function with some matrix properties, the Sinc placement based on the exponential double transformation or the Adomian decomposition method. All these methods convert these integral equations into an algebraic equation see [19], [20], and [21]. However, it must be noted that in most of these works, they start from the fact that there is a solution. In [21] consider the Hammerstein-Volterra integral equation

$$\begin{aligned}&x(t) = e^t - (t+1)\sin t + \int_{-1}^{t} e^{-2s} \sin t x^2(s) ds, \\ &\textit{with } -1 \leq t \leq 1,\end{aligned} \tag{5.21}$$

which has $x(t) = e^t$ as an exact solution. Conclude that for large values of M, the approximate solution is indistinguishable from the exact solution and that the method considered is quite powerful. However, to complement these approximate methods we will demonstrate that the considered solution is unique in the space of function of Shiba-bounded variation.

Let's verify that the hypotheses of the Theorem 32 are satisfied

1. Let us prove that $g(t) = e^t - (t+1)\sin t$ is a function of Λ_p bounded variation. Consider a partition $\{t_N\}$ of I, $t_0 = 0 < t_1 < \ldots < t_{N-1} < 1 = t_N$, then $0 < t_k - t_{k-1} \leq 1$; since λ_n is a nondecreasing sequence, we have

$$\begin{aligned}&\sum_{n=1}^{N} \frac{|e^{t_n} - (t_n+1)\sin(t_n) - e^{t_{n-1}} + (t_{n-1}+1)\sin(t_{n-1})|^p}{\lambda_n} \\ &\leq \sum_{n=1}^{N} \frac{|e^{t_n} - (t_n+1)\sin(t_n) - e^{t_{n-1}} + (t_{n-1}+1)\sin(t_{n-1})|^p}{\lambda_1},\end{aligned}$$

Applying triangular inequality and

$$|a+b|^p \leq 2^p\left(|a|^p + |b|^p\right),$$

we have that

$$\begin{aligned}&\sum_{n=1}^{N} \frac{|e^{t_n} - (t_n+1)\sin(t_n) - e^{t_{n-1}} + (t_{n-1}+1)\sin(t_{n-1})|^p}{\lambda_n} \\ &\leq \sum_{n=1}^{N} \frac{H(t_n,t_{n-1})}{\lambda_1},\end{aligned}$$

where

$$H(t_n,t_{n-1}) = 4^p\left|e^{t_n}-e^{t_{n-1}}\right|^p + 4^p\left|t_n\sin(t_n)-t_{n-1}\sin(t_{n-1})\right|^p + 2^p\left|\sin(t_n)-\sin(t_{n-1})\right|^p,$$

by the Mean Value Theorem, for $t\in[-1,1]$

$$\left|e^{t_n}-e^{t_{n-1}}\right| \le e\left|t_n-t_{n-1}\right|,$$
$$\left|t_n\sin(t_n)-t_{n-1}\sin(t_{n-1})\right| \le 2\left|t_n-t_{n-1}\right|,$$
$$\left|\sin(t_n)-\sin(t_{n-1})\right| \le \left|t_n-t_{n-1}\right|$$

and by Lemma 2.5 *in [16],*

$$x^p \le pb^{p-1}x \quad \textit{for} \quad x\in[0,b] \textit{ with } 1\le p<+\infty,$$

it follows that

$$\begin{aligned}
&\sum_{n=1}^{N}\frac{\left|e^{t_n}-(t_n+1)\sin(t_n)-e^{t_{n-1}}+(t_{n-1}+1)\sin(t_{n-1})\right|^p}{\lambda_1}\\
&\le \sum_{n=1}^{N}\frac{\left(e2^{2p}+2^{2p+1}+2^p\right)p}{\lambda_1}\left|t_n-t_{n-1}\right|\\
&\le 2p\frac{\left(e2^{2p}+2^{2p+1}+2^p\right)}{\lambda_1} = M<+\infty,
\end{aligned}$$

from here it follows, $g\in\Lambda_pBV$.
2. It is clear that $f(x)=x^2$ *is locally Lipschitz.*
3. Let us define

$$\widehat{K}(t,s) = \begin{cases} e^{-2s}\sin t, & -1\le s\le t\\ 0, & t<s\le 1. \end{cases} \tag{5.22}$$

Hence

$$\begin{aligned}
\int_{-1}^{1}\widehat{K}(t,s)ds &= \int_{-1}^{t}e^{-2s}\sin t ds\\
&= \frac{1}{2}\sin t(e^2-e^{-2t}).
\end{aligned}$$

It follows that $\widehat{K}(t,.)$ *is Lebesgue integrable, so* $K(t,\cdot)$ *is Lebesgue integrable. On the other hand, by Lemma27*

$$\begin{aligned}
V_{\Lambda_p}\left(\widehat{K}(.,s),[0,1]\right) &\le 2\left(\frac{|K(s,s)|}{(\lambda_1)^{\frac{1}{p}}}+V_{\Lambda_p}\left(K(\cdot,s),[s,1]\right)\right)\\
&= 2\left(\frac{\left|e^{-2s}\sin s\right|}{(\lambda_1)^{\frac{1}{p}}}+V_{\Lambda_p}\left(e^{-2s}\sin t,[s,1]\right)\right)\\
&= 2\left|e^{-2s}\right|\left(\frac{|\sin s|}{(\lambda_1)^{\frac{1}{p}}}+V_{\lambda_p}\left[\sin t,[s,1]\right]\right)\\
&\le 2\left|e^{-2s}\right|\left(\frac{1}{(\lambda_1)^{\frac{1}{p}}}+V_{\lambda_p}\left[\sin t,[s,1]\right]\right)\\
&= M(s).
\end{aligned}$$

It is evident that $M(s)$ *is* Lp *integrable since* $\frac{1}{(\lambda_1)^{\frac{1}{p}}}+V_{\lambda_p}[\sin t,I]<+\infty$. *Therefore all conditions of Theorem 32 are satisfied. Thus, the equation (5.21) has a unique* Λ_pBV*-solution, defined on* $[-1,1]$. □

Example 52 *Consider the equation*

$$x(t) = \sin t + \lambda\int_I K(t,s)(x(s))^2ds,\ \lambda\in I, t\in I=[0,1], \tag{5.23}$$

where $K:[0,1]\times[0,1]\to\mathbb{R}$ *is given by* $K(t,s)=\cos tK_1(s)$ *with*

$$K_1(s) = \begin{cases} \frac{1}{\sqrt[2p]{s}}, & s\in(0,1];\\ 0, & s=0. \end{cases}$$

We started to verify that the hypothesis of Theorem 31 is satisfied:
1. Let us prove that $g(t)=\sin t$ *is a function of* Λ_p *bounded variation. Consider a partition* $\{t_N\}$ *of* I, $t_0=0<t_1<\ldots<t_{N-1}<1=t_N$, *then* $0<t_k-t_{k-1}\le 1$; *since* λ_n *is a nondecreasing sequence, we have*

$$\sum_{n=1}^{N}\frac{\left|\sin(t_n)-\sin(t_{n-1})\right|^p}{\lambda_n} \le \sum_{n=1}^{N}\frac{\left|\sin(t_n)-\sin(t_{n-1})\right|^p}{\lambda_1}$$

Applying triangular inequality, the inequalities

$$|a+b|^p \le 2^p\left(|a|^p+|b|^p\right), \quad x^p\le pb^{p-1}x \quad \textit{for} \quad x\in[0,b]$$

and by the Mean Value Theorem on the interval $[0,1]$*, we have*

$$\begin{aligned}
\sum_{n=1}^{N}\frac{\left|\sin(t_n)-\sin(t_{n-1})\right|^p}{\lambda_1} &\le \frac{pb^{p-1}}{\lambda_1}\sum_{n=1}^{N}\left|t_n-t_{n-1}\right|\\
&\le \frac{pb^{p-1}}{\lambda_1}(1-0) = M<+\infty,
\end{aligned}$$

from here it follows, $g\in\Lambda_pBV$.
2. By example 51 you have $f(x)=x^2$ *is locally Lipschitz.*
3. $\int_0^1 K_1(s)ds = \int_0^1\frac{1}{\sqrt[2p]{s}}ds$ *is convergent, therefore* $K_1(s)$ *is Lebesgue integrable, even more* $K_1(s)$ *is* L_p *integrable. Now*

$$\begin{aligned}
\int_I K(t,s)ds &= \int_I \cos tK_1(s)ds\\
&= \cos t\int_I K_1(s)ds.
\end{aligned}$$

As $K_1(s)$ *is Lebesgue integrable,* $K(t,\cdot)$ *is Lebesgue integrable. On the other hand*

$$\begin{aligned}
V_{\Lambda_p}\left[K(\cdot,s),I\right] &= V_{\Lambda_p}\left[\cos tK_1(s),[0,1]\right]\\
&= |K_1(s)|V_{\Lambda_p}\left[\cos t,[0,1]\right]\\
&= M(s).
\end{aligned}$$

Evidently $M(s)$ *is* L_p *integrable, since* $K_1(s)$ *is* L_p *integrable and* $V_{\lambda_p}[\cos t,I]<+\infty$ *(the bounded variation proof is analogous to that of the function* $\sin t$ *given above). Therefore all conditions of Theorem31 are satisfied. Hence, there exists a number* $\tau>0$ *such that for every* λ *with* $|\lambda|<\tau$*, the equation (5.23) has a unique* Λ_pBV*-solution, defined on* $I=[0,1]$. □

Example 53 *Consider the equation*

$$x(t) = \cos t + \int_o^t K(t,s)\sin(x(s))ds, \ \ t \in I = [0,1], \quad (5.24)$$

where the kernel $K : T \to \mathbb{R}$*, with* $T = \{(t,s) : 0 \le t \le 1, 0 \le s \le t\}$*, is given as in the previous example.*
1. $g(t) = \cos t$ *is a function of* Λ_p *bounded variation from the previous example.*
2. Evidently $f(x) = \sin x$ *is locally Lipschitz.*
3. Just like the example 51 let us define

$$\widehat{K}(t,s) = \begin{cases} K(t,s), & 0 \le s \le t \\ 0, & t < s \le 1. \end{cases} \quad (5.25)$$

For $t \in [0,1]$*,*

$$\begin{aligned} \int_0^1 \widehat{K}(t,s)ds &= \int_0^t K(t,s)ds \\ &= \int_0^t \cos(t)K_1(s)ds \\ &= \cos(t)\int_0^t K_1(s)ds. \end{aligned}$$

As $K_1(s)$ *is Lebesgue integrable, it follows that* $\widehat{K}(t,.)$ *is Lebesgue integrable, so* $K(t,\cdot)$ *is Lebesgue integrable. Working analogously to example 51 it follows that*

$$\begin{aligned} V_{\Lambda_p}\left(\widehat{K}(.,s),[0,1]\right) &\le 2|K_1(s)|\left(\frac{|cos(s)|}{(\lambda_1)^{\frac{1}{p}}} + V_{\lambda_p}[\cos t,[s,1]]\right) \\ &\le 2|K_1(s)|\left(\frac{1}{(\lambda_1)^{\frac{1}{p}}} + V_{\lambda_p}[\cos t,[s,1]]\right) \\ &= M(s). \end{aligned}$$

It is evident that $M(s)$ *is Lp integrable, for* $K_1(s)$ *is* L_p *integrable and* $\dfrac{1}{(\lambda_1)^{\frac{1}{p}}} + V_{\lambda_p}[\cos t, I] < +\infty$*. Therefore all conditions of Theorem 32 are satisfied. Hence, the equation (5.23) has a unique* $\Lambda_p BV$*-solution defined on* $[0,1]$*.* □

6 Conclusion

In this paper, we demonstrate the existence and uniqueness of solutions of the nonlinear integral equations of Hammerstein, Hammerstein-Volterra and Volterra in the space of functions of Shiba-bounded variation. As the main tool for the proof of the main theorems of the Hammerstein and Hammerstein-Volterra nonlinear equations, the Banach fixed point Theorem is used. However, for the Volterra nonlinear equation, the Leray-Schauder Theorem is used. We also give some examples of some nonlinear integral equations, where the existence and uniqueness of the solutions are guaranteed by verifying the hypotheses of the theorems, and an application is shown. We hope that the ideas and techniques used in this article can inspire readers interested in studying these different nonlinear integral equations as well as some new spaces of generalized bounded variation. Moreover, these results are a contribution to the different areas which apply this type of integral equations.

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Jurancy Ereú Received her Ph.D. degree from the Universidad Central de Venezuela (UCV), Venezuela (2012). She currently works as teacher in the Department of Mathematics of the Universidad Centroccidental Lisandro Alvarado, Barquisimeto, Lara, Venezuela. Her fields of interest are: Real and Complex Analysis, Functional Analysis and Operator Theory.

José Giménez Received his Ph.D. degree from the University of Iowa in 2000. He is full professor at the Math Department, School of Sciences of Universidad de Los Andes, Mérida, Venezuela. His fields ofinterest are Real and Complex Analysis, Functional Analysis and Operator Theory.

Liliana Pérez Received her Ph.D. degree from Universidad Central de Venezuela (UCV), Venezuela (2012). She worked as Professor in the Department Mathematics of the Universidad Centro Occidental Lisandro Alvarado, Barquisimeto, Lara, Venezuela, from 2002 to 2017. Currently she works at the Escuela Superior Politecnica del litoral (ESPOL), Guayaquil, Ecuador from 2017. Her fields of interest include differential equations with infinite delay.