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Generalized Schwarz algorithm for a class of variational inequalities

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Abstract: We consider a decomposition to m-subdomains of the obstacle problem, which is modeled by a variational inequality of first species, using the auxiliary sequences, and we have proving a alternating relation between the solutions on each subdomains. We also proving a geommetrical convergence between the nth iteration and the solution of the initial problem, and we obtained a result on the error estimate contains a logarithmic factor with an extra power of $|\log(h)|$.

Keywords: Variational inequalities; Schwarz algorithm; finite element method; L^∞ error estimate.

1 Introduction

The Schwarz alternating method of decomposing the domain, has lately been found to be an effective means for solving elliptic partial differential equations on a multi processing computing system. Pierre-Louis Lions, was the starting point of an intense research activity to develop this tool of calculation, see [1,2,3]. In this paper, we are interested in the Schwarz alternating method which is used to solve a class of elliptic variational inequality in the context of overlapping non-matching grids, precisely in the error analysis in the maximum norm of obstacle type problems. The maximum error analysis of overlapping non-matching grids for the obstacle problem which Ω is the union of two sub-domains has been studied in [4], the same error analysis of a non-matching grids for linear and nonlinear elliptic partial differential equations and elliptic quasi-variational inequalities has studied in [5,6,7,8,9].

In our work we consider a domain Ω which is the union of m overlapping sub-domains where each sub-domain has its own triangulation. To prove the main result, we introduce the m continuous and discrete Schwarz sequences, and prove a main result concerning the error estimate of solution in L^∞-norm, taking into account the combination of geometrical convergence and uniform convergence of finite element approximation.

This paper consists of two parts. In the first, we formulate the problem of continuous and discrete elliptic variational inequality, we show the monotonicity and stability of discrete solution, then we define the Schwarz algorithm for m sub-domains with overlapping matching grids. In the second part, we establish m auxiliary Schwarz sequences, and we prove the main result of this work.

2 The generalized Schwarz alternating method

2.1 Elliptic obstacle problem

Let Ω be a convex domain in $\mathbb{R}^2$ with suffciently smooth boundary $\partial\Omega$.
We consider the bilinear form

$$a(u,v) = \int_\Omega (\nabla u.\nabla v)dx, \qquad (1)$$

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the linear form

$$(f,v)=\int_{\Omega} f(x).v(x)dx, \tag{2}$$

the right hande-side

$$f\in L^{\infty}(\Omega), \tag{3}$$

the obstacle

$$\Psi\in W^{2,\infty}(\Omega)\quad \text{such that}\quad \Psi\geq 0\quad on\quad \partial\Omega, \tag{4}$$

and the nonempty convex set

$$K_g=\{v\in H^1(\Omega): v=g\quad on\quad \partial\Omega, v\leq\Psi\quad on\quad \Omega\}, \tag{5}$$

where g is a regular function defined on $\partial\Omega$.

We consider the obstacle problem: find $u\in K_g$ such that

$$a(u,v-u)\geq(f,v-u),\quad \forall v\in K_g, \tag{6}$$

Let V_h be the space of finite elements consisting of continuous piecewise linear functions. The discrete counterpart of (6) consists of finding $u_h\in K_{gh}$ such that

$$a(u_h,v-u_h)\geq(f,v-u_h),\quad \forall v\in K_{gh}, \tag{7}$$

where

$$K_{gh}=\{\ v\in v_h: v=\pi_h g\quad \text{on}\quad \partial\Omega, v\leq r_h\Psi\quad \text{on}\quad \Omega\}, \tag{8}$$

π_h is an interpolation operator on $\partial\Omega$, and r_h is the usual finite element restriction operator on Ω. The lemma below establishes a monotonicity property of the solution of (6) with respect to the obstacle and the boundary condition.
The lemma below establishes a monotonicity property of the solution of (6) with respect to the obstacle and the boundary condition.

Lemma 1 *Let* $(\Psi,g);(\tilde{\Psi},\tilde{g})$ *be a pair of data, and* $u=\sigma(\Psi,g);\tilde{u}=\tilde{\sigma}(\tilde{\Psi},\tilde{g})$ *the corresponding solutions to* (6). *If* $\Psi\geq\tilde{\Psi}$ *and* $g\geq\tilde{g}$, *then* $\sigma(\Psi,g)\geq\sigma(\tilde{\Psi},\tilde{g})$.

Proof. let $v=min(0,u-\tilde{u})$. In the region where v is negative $(v<0)$,we have

$$u<\tilde{u}\leq\psi\leq\tilde{\psi} \tag{9}$$

which means that the obstacle is not active for u. So, for that v, we have

$$a(u,v)=(f,v) \tag{10}$$

$$\tilde{u}+v\leq\tilde{\psi} \tag{11}$$

so

$$a(\tilde{u},v)=(f,v) \tag{12}$$

Subtracting (10) and (12) from each other, we obtain

$$a(\tilde{u}-u,v)\geq 0 \tag{13}$$

but,

$$a(v,v) = a(u-\tilde{u},v) = -a(\tilde{u}-u,v) \leq 0 \tag{14}$$

so

$$v = 0 \tag{15}$$

and consequently

$$u \geq \tilde{u} \tag{16}$$

which completes the proof.
The proof for the discrete case is similar.

Proposition 1 *Under the notations and conditions of the preceding lemma, we have*

$$\|u-\tilde{u}\|_{L^\infty(\Omega)} \leq \|\Psi-\tilde{\Psi}\|_{L^\infty(\Omega)} + \|g-\tilde{g}\|_{L^\infty(\partial\Omega)}, \tag{17}$$

Proof. Setting

$$\phi \leq \|\psi-\tilde{\psi}\|_{L^\infty(\Omega)} + \|g-\tilde{g}\|_{L^\infty(\partial\Omega)} \tag{18}$$

we have

$$\psi \leq \tilde{\psi}+\psi-\tilde{\psi} \leq \tilde{\psi}+|\psi-\tilde{\psi}| \leq \tilde{\psi}+\|\psi-\tilde{\psi}\|_{L^\infty(\Omega)}$$

$$\tag{19}$$

$$\leq \psi+\|\psi-\tilde{\psi}\|_{L^\infty(\Omega)} + \|g-\tilde{g}\|_{L^\infty(\partial\Omega)}$$

hence

$$\psi \leq \tilde{\psi}+\phi \tag{20}$$

On the other hand, we have

$$g \leq \tilde{g}+g-\tilde{g} \leq \tilde{g}+|g-\tilde{g}| \leq \tilde{g}+\|g-\tilde{g}\|_{L^\infty(\partial\Omega)}$$

$$\tag{21}$$

$$\leq g+\|g-\tilde{g}\|_{L^\infty(\partial\Omega)} + \|\psi-\tilde{\psi}\|_{L^\infty(\Omega)}$$

so

$$g \leq \tilde{g}+\phi \tag{22}$$

Now, making use of **Lemma2.1**, we obtain

$$\sigma(\psi,g) \leq \sigma(\psi+\phi, g+\phi) = \sigma(\tilde{\psi},\tilde{g})+\phi \tag{23}$$

or

$$\sigma(\psi,g)-\sigma(\tilde{\psi},\tilde{g}) \leq \phi \tag{24}$$

Similarly, interchanging the roles of the couples (ψ,g) and $(\tilde{\psi},\tilde{g})$, we obtain

$$\sigma(\tilde{\psi},\tilde{g})-\sigma(\psi,g) \leq \phi \tag{25}$$

The proof for the discrete case is similar.

Remark 1 *if* $\psi = \tilde{\psi}$*, then* (17) *becomes*

$$\|u-\tilde{u}\|_{L^\infty(\Omega)} \leq \|g-\tilde{g}\|_{L^\infty(\partial\Omega)}, \tag{26}$$

Theorem 1 *[10] Under conditions* (3) *and* (4)*, there exists a constant* C *independent of* h *such that*

$$\|u-u_h\|_{L^\infty(\Omega)} \leq ch^2 \log|h|^2, \tag{27}$$

2.2 The continuous Schwarz sequences

Consider the model obstacle problem: find $u \in K_0(g=0)$ such that

$$a(u,v) \geq f(v-u) \quad \forall v \in K_0, \tag{28}$$

We decompose Ω into m overlapping subdomains such that

$$\Omega = \bigcup_{i=1}^{m} \Omega_i, \quad \Omega_i \cap \Omega_j \neq \emptyset, \quad i=\overline{1,m}, \quad j=\overline{1,m}, \quad i \neq j \tag{29}$$

and u satisfies the local regularity condition

$$u/\Omega_i \in W^{2,p}(\Omega_i); \quad 2 < p < \infty, \tag{30}$$

We denote by $\partial\Omega_i$ the boundary of Ω_i, and $\Gamma_{ij} = \partial\Omega_i \cap \Omega_j, i \neq j$. The intersection of $\bar{\Gamma_i}$ and $\bar{\Gamma_j}$; $i \neq j$ is assumed to be empty. Choosing $u_0 = \Psi$, we define the alternating Schwarz sequences (u_i^{n+1}) on Ω_i such that $u_i^{n+1} \in K$ solves

$$\begin{cases} a_i(u_i^{n+1}, v-u_i^{n+1}) \geq (f_i, v-u_i^{n+1}) \quad \text{in} \quad \Omega_i \\ u_i^{n+1} = u_j^{n+1_{ij}} \quad \text{on} \quad \Gamma_{ij} \end{cases} \tag{31}$$

where $i=\overline{1,m}$, $j=\overline{1,m}$, $i \neq j$ and

$$1_{ij} = \begin{cases} 1 & si \quad i > j \\ 0 & si \quad i < j \end{cases}$$

2.3 Geometrical convergence

Theorem 2 *The sequences* (u_1^{n+1}), (u_2^{n+1}),...,(u_m^{n+1}); $n \geq 0$ *produced by the Schwarz alternating method converge geometrically to the solution u, of the obstacle problem* (28). *More precisely, there exist m constants* k_1, k_2 ,..., $k_m \in (0,1)$, $\forall\, i=\overline{1,m-1}$, $j=\overline{2,m}$, $i<j$ *such that for all* $n \geq 0$

$$\begin{cases} \|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} \leq k_i^n k_j^n \|u-u^0\|_{L^\infty(\Gamma_{ij})} \\ \|u_j - u_j^{n+1}\|_{L^\infty(\Omega_j)} \leq k_i^{n+1} k_j^n \|u-u^0\|_{L^\infty(\Gamma_{ji})} \end{cases} \tag{32}$$

we consider a function $w_l \in L^\infty(\Omega_l)$ *continu in* $\overline{\Omega_l} \setminus (\overline{\Gamma_l} \cap \partial\Omega)$
such that

$$\begin{cases} \Delta w_l = 0 & dans \quad \Omega_l, l=\overline{1,m} \\ w_l = & \begin{cases} 0 & sur \quad \partial\Omega_l \backslash \overline{\Gamma_l} \\ 1 & sur \quad \Gamma_l \end{cases} \end{cases}$$

and

$$k_t = sup\{w_s(x) \backslash x \in \partial\Omega_t \cap \Omega, t \neq s\} \in (0,1), \forall t,s = \overline{1,m} \tag{33}$$

Proof. from the principle of the maximum

$$\|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} \leq \|u_i - u_i^{n+1}\|_{L^\infty(\Gamma_{ij})}$$

and

$$\begin{aligned}
\|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} &\leq \|u_j - u_j^{n}\|_{L^\infty(\Gamma_{ij})} \leq \|w_i u_j - w_i u_j^{n}\|_{L^\infty(\Gamma_{ij})} \\
&\leq \|w_i u_j - w_i u_j^{n}\|_{L^\infty(\Omega_j)} \leq \|w_i u_j - w_i u_j^{n}\|_{L^\infty(\Gamma_{ji})} \\
&\leq \|w_i\|_{L^\infty(\Gamma_{ji})} \|u_j - u_j^{n}\|_{L^\infty(\Gamma_{ji})} \leq \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j u_j - w_j u_j^{n}\|_{L^\infty(\Gamma_{ji})} \\
&\leq \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j u_i - w_j u_i^{n}\|_{L^\infty(\Gamma_{ji})} \leq \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j u_i - w_i u_i^{n}\|_{L^\infty(\Omega_i)} \\
&\leq \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j u_i - w_j u_i^{n}\|_{L^\infty(\Gamma_{ij})} \leq \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j\|_{L^\infty(\Gamma_{ij})} \|u_i - u_i^{n}\|_{L^\infty(\Gamma_{ij})}
\end{aligned}$$

using (33), hence

$$\|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} \leq k_i k_j \|u_i - u_i^{n}\|_{L^\infty(\Gamma_{ij})}$$

by induction, we obtain

$$\begin{aligned}
\|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} &\leq k_i^n k_j^n \|u_i - u_i^{1}\|_{L^\infty(\Gamma_{ij})} \\
&\leq k_i^n k_j^n \|u - u^{0}\|_{L^\infty(\Gamma_{ij})}
\end{aligned}$$

where $u_i^1 = u^0$ sur Γ_{ij}, $u = 0$ sur $\partial\Omega_i \cap \partial\Omega$
similarly, we have

$$\begin{aligned}
\|u_j - u_j^{n+1}\|_{L^\infty(\Omega_j)} &\leq \|u_j - u_j^{n+1}\|_{L^\infty(\Gamma_{ji})} \\
&\leq \|u_i - u_i^{n+1}\|_{L^\infty(\Gamma_{ji})} \leq \|w_j u_i - w_j u_i^{n+1}\|_{L^\infty(\Gamma_{ji})} \\
&\leq \|w_j u_i - w_j u_i^{n+1}\|_{L^\infty(\Omega_i)} \leq \|w_j u_i - w_j u_i^{n+1}\|_{L^\infty(\Gamma_{ij})} \\
&\leq \|w_j\|_{L^\infty(\Gamma_{ij})} \|u_i - u_i^{n+1}\|_{L^\infty(\Gamma_{ij})} \leq \|w_j\|_{L^\infty(\Gamma_{ij})} \|w_i u_j - w_i u_j^{n}\|_{L^\infty(\Gamma_{ij})} \\
&\leq \|w_j\|_{L^\infty(\Gamma_{ij})} \|w_i u_j - w_i u_j^{n}\|_{L^\infty(\Omega_j)} \leq \|w_j\|_{L^\infty(\Gamma_{ij})} \|w_i u_j - w_i u_j^{n}\|_{L^\infty(\Gamma_{ji})} \\
&\leq \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j\|_{L^\infty(\Gamma_{ij})} \|u_j - u_j^{n}\|_{L^\infty(\Gamma_{ji})} \leq k_i k_j \|u_i - u_j^{n}\|_{L^\infty(\Gamma_{ji})}
\end{aligned}$$

by induction, we obtain

$$\begin{aligned}
\|u_j - u_j^{n+1}\|_{L^\infty(\Omega_j)} &\leq k_i^n k_j^n \|u_j - u_i^{1}\|_{L^\infty(\Gamma_{ji})} \leq k_i^n k_j^n \|w_j u_i - w_j u_i^{1}\|_{L^\infty(\Gamma_{ji})} \\
&\leq k_i^n k_j^n \|u_i - u_i^{1}\|_{L^\infty(\Omega_i)} \leq k_i^n k_j^n \|u_i - u_i^{1}\|_{L^\infty(\Gamma_{ij})} \\
&\leq k_i^{n+1} k_j^n \|u_i - u_i^{1}\|_{L^\infty(\Gamma_{ij})} \leq k_i^{n+1} k_j^n \|u_j - u_j^{0}\|_{L^\infty(\Gamma_{ij})} \\
&\leq k_i^{n+1} k_j^n \|u_j - u_j^{0}\|_{L^\infty(\Omega_j)} \leq k_i^{n+1} k_j^n \|u - u^{0}\|_{L^\infty(\Gamma_{ji})}
\end{aligned}$$

2.4 The discretization

let τ^{h_i} be a standard regular and quasi-uniform finite element triangulation in Ω_i, h_i being the meshsizes. We assume that every two triangulation are mutually independent on $\Omega_i \cap \Omega_j$, in the sense that a triangle belonging to one triangulation does not necessarily belong to the other, $i = \overline{1,m}$, $j = \overline{1,m}$, $i \neq j$

Let $V_{h_{ij}} = V_{h_{ij}}(\Omega_i)$ be the space of continuous piecewise linear functions on, τ^{h_i} which vanish on $\partial\Omega \cap \Omega_i$. For $w \in C(\bar{\tau}_i)$ we define

$$V_{h_{ij}}^{(w)} = \{v \in V_{h_{ij}} : v = 0 \quad \text{on} \quad \partial\Omega_i \cap \partial\Omega; v = \pi_{h_{ij}}(w) \quad \text{on} \quad \Gamma_{ij}\}, \tag{34}$$

where π_{h_i} denotes a suitable interpolation operator on Γ_{ij}

The discrete Schwarz sequence $u_{ih}^{n+1} \in V_{h_{ij}}^{(u_{jh}^{n+1_{ij}})}$ such that

$$\begin{cases} a_i(u_{ih}^{n+1}, v - u_{ih}^{n+1}) \geq (f_i, v - u_{ih}^{n+1}) \quad \forall v \in V_{h_{ij}}^{(u_{jh}^{n+1_{ij}})} \\ u_{ih}^{n+1} \leq r_h.\psi, \quad v \leq r_h.\psi \end{cases} \tag{35}$$

3 L^∞-error analysis

3.1 Definition of m auxilairy sequences

For $\omega_0^{ih} = u_0^{ih} = r_h\psi; \quad i = \overline{1,m}$, we define the sequences $\omega_{ih}^{n+1} \in V_{h_{ij}}^{(u_j^{n+1_{ij}})}$ such that

$$\begin{cases} a_i(\omega_{ih}^{n+1}, v - \omega_{ih}^{n+1}) \geq (f_i, v - \omega_{ih}^{n+1}) \quad \forall v \in V_{h_{ij}}^{(u_j^{n+1_{ij}})} \\ \omega_{ih}^{n+1} \leq r_h.\psi, \quad v \leq r_h.\psi \end{cases} \tag{36}$$

To simplify the notation, we take

$$|.|_{ij} = \|.\|_{L^\infty(\Gamma_{ij})}$$

$$\|.\|_i = \|.\|_{L^\infty(\Omega_i)} \quad h_{ij} = h \quad \pi_{h_{ij}} = \pi_h \tag{37}$$

Lemma 2 *for* $i = \overline{1, m-1}, \quad j = \overline{2,m}, \quad i < j$

$$\begin{cases} \|u_i^{n+1} - u_{ih}^{n+1}\|_i \leq \sum_{p=1}^{n+1} \|u_i^p - \omega_{ih}^p\|_i + \sum_{p=0}^{n} \|u_j^p - \omega_{jh}^p\|_j \\ \|u_j^{n+1} - u_{jh}^{n+1}\|_j \leq \sum_{p=0}^{n+1} \|u_j^p - \omega_{jh}^p\|_j + \sum_{p=1}^{n+1} \|u_i^p - \omega_{ih}^p\|_i \end{cases} \tag{38}$$

*Proof.*By induction
for $n = 0$, using the discrete version of **Remark** 2.3, we get

$$\begin{aligned}\|u_i^1-u_{ih}^1\|_i &\le \|u_i^1-\omega_{ih}^1\|_i+\|\omega_{ih}^1-u_{ih}^1\|_i\\ &\le \|u_i^1-\omega_{ih}^1\|_i+|\pi_h u_j^0-\pi_h u_{jh}^0|_{ij}\\ &\le \|u_i^1-\omega_{ih}^1\|_i+|u_j^0-u_{jh}^0|_{ij}\\ &\le \|u_i^1-\omega_{ih}^1\|_i+\|u_j^0-u_{jh}^0\|_j\end{aligned}$$

$$\begin{aligned}\|u_j^1-u_{jh}^1\|_j &\le \|u_j^1-\omega_{jh}^1\|_j+\|\omega_{jh}^1-u_{jh}^1\|_j\\ &\le \|u_j^1-\omega_{jh}^1\|_j+|\pi_h u_i^1-\pi_h u_{ih}^1|_{ji}\\ &\le \|u_j^1-\omega_{jh}^1\|_j+|u_i^1-u_{ih}^1|_{ji}\\ &\le \|u_j^1-\omega_{jh}^1\|_j+\|u_i^1-u_{ih}^1\|_i\\ &\le \|u_j^1-\omega_{jh}^1\|_j+\|u_i^1-\omega_{ih}^1\|_i+\|u_j^0-u_{jh}^0\|_j\end{aligned}$$

so

$$\|u_i^1-u_{ih}^1\|_1 \le \sum_{p=1}^{1}\|u_i^p-\omega_{ih}^p\|_i+\sum_{p=0}^{0}\|u_j^p-\omega_{jh}^p\|_j$$

$$\|u_j^1-u_{jh}^1\|_j \le \sum_{p=0}^{1}\|u_j^p-\omega_{jh}^p\|_j+\sum_{p=1}^{1}\|u_i^p-\omega_{ih}^p\|_i$$

for $n=1$, using the discrete version of **Remark** 2.3, we get

$$\begin{aligned}\|u_i^2-u_{ih}^2\|_i &\le \|u_i^2-\omega_{ih}^2\|_i+\|\omega_{ih}^2-u_{ih}^2\|_i\\ &\le \|u_i^2-\omega_{ih}^2\|_i+|\pi_h u_j^1-\pi_h u_{jh}^1|_{ij}\\ &\le \|u_i^2-\omega_{ih}^2\|_i+|u_j^1-u_{jh}^1|_{ij}\\ &\le \|u_i^2-\omega_{ih}^2\|_i+\|u_j^1-u_{jh}^1\|_j\\ &\le \|u_i^2-\omega_{ih}^2\|_i+\|u_j^1-\omega_{jh}^1\|_j+\|u_i^1-\omega_{ih}^1\|_i+\|u_j^0-u_{jh}^0\|_j\end{aligned}$$

$$\begin{aligned}\|u_j^2-u_{jh}^2\|_j &\le \|u_j^2-\omega_{jh}^2\|_j+\|\omega_{jh}^2-u_{jh}^2\|_j\\ &\le \|u_j^2-\omega_{jh}^2\|_j+|\pi_h u_i^2-\pi_h u_{ih}^2|_{ji}\\ &\le \|u_j^2-\omega_{jh}^2\|_{ij}+|u_i^2-u_{ih}^2|_{ji}\\ &\le \|u_j^2-\omega_{jh}^2\|_j+\|u_i^2-u_{ih}^2\|_i\\ &\le \|u_j^2-\omega_{jh}^2\|_j+\|u_i^2-\omega_{ih}^2\|_i+\|u_j^1-\omega_{jh}^1\|_j+\|u_i^1-\omega_{ih}^1\|_i+u_j^0-u_{jh}^0\|_j\end{aligned}$$

so

$$\|u_i^2-u_{ih}^2\|_i \le \sum_{p=1}^{2}\|u_i^p-\omega_{ih}^p\|_i+\sum_{p=0}^{1}\|u_j^p-\omega_{jh}^p\|_j$$

$$\|u_j^2 - u_{jh}^2\|_j \leq \sum_{p=0}^{2} \|u_j^p - \omega_{jh}^p\|_j + \sum_{p=1}^{2} \|u_i^p - \omega_{ih}^p\|_i$$

We suppose that

$$\|u_j^n - u_{jh}^n\|_j \leq \sum_{p=0}^{n} \|u_j^p - \omega_{jh}^p\|_j + \sum_{p=1}^{n} \|u_i^p - \omega_{ih}^p\|_i$$

Then, using the discrete version of **Remark** 2.3 again, we get

$$\begin{aligned}
\|u_i^{n+1} - u_{ih}^{n+1}\|_i &\leq \|u_i^{n+1} - \omega_{ih}^{n+1}\|_i + \|\omega_{ih}^{n+1} - u_{ih}^{n+1}\|_i \\
&\leq \|u_i^{n+1} - \omega_{ih}^{n+1}\|_i + |\pi_h u_j^n - \pi_h u_{jh}^n|_{ij} \\
&\leq \|u_i^{n+1} - \omega_{ih}^{n+1}\|_i + |u_j^n - u_{jh}^n|_{ij} \\
&\leq \|u_i^{n+1} - \omega_{ih}^{n+1}\|_i + \|u_j^n - u_{jh}^n\|_j \\
&\leq \|u_i^{n+1} - u_{ih}^{n+1}\|_i + \sum_{p=0}^{n} \|u_j^p - \omega_{jh}^p\|_j + \sum_{p=1}^{n} \|u_i^p - \omega_{ih}^p\|_i
\end{aligned}$$

then

$$\|u_i^{n+1} - u_{ih}^{n+1}\|_i \leq \sum_{p=1}^{n+1} \|u_i^p - \omega_{ih}^p\|_i + \sum_{p=0}^{n} \|u_j^p - \omega_{jh}^p\|_j$$

$$\begin{aligned}
\|u_j^{n+1} - u_{jh}^{n+1}\|_j &\leq \|u_j^{n+1} - \omega_{jh}^{n+1}\|_j + \|\omega_{jh}^{n+1} - u_{jh}^{n+1}\|_j \\
&\leq \|u_j^{n+1} - \omega_{jh}^{n+1}\|_j + |\pi_h u_i^{n+1} - \pi_h u_{jh}^{n+1}|_{ji} \\
&\leq \|u_j^{n+1} - \omega_{jh}^{n+1}\|_j + |u_i^{n+1} - u_{ih}^{n+1}|_{ji} \\
&\leq \|u_j^{n+1} - \omega_{jh}^{n+1}\|_j + \|u_i^{n+1} - u_{ih}^{n+1}\|_i \\
&\leq \|u_j^{n+1} - u_{jh}^{n+1}\|_j + \sum_{p=1}^{n+1} \|u_i^p - \omega_{ih}^p\|_i + \sum_{p=0}^{n} \|u_j^p - \omega_{jh}^p\|_j
\end{aligned}$$

then

$$\|u_j^{n+1} - u_{jh}^{n+1}\|_j \leq \sum_{p=0}^{n+1} \|u_j^p - \omega_{jh}^p\|_j + \sum_{p=1}^{n+1} \|u_i^p - \omega_{ih}^p\|_i$$

Lemma 3 $\forall\ i = \overline{1, m-1},\ j = \overline{2, m},\ i < j$.*Then there exists a constantc independent of h and n such that*

$$\|u_i^{n+1} - u_{ih}^{n+1}\|_i \leq 2(n+1)Ch^2|\log h|^3$$

$$\|u_j^{n+1} - u_{jh}^{n+1}\|_j \leq (2n+3)Ch^2|\log h|^3 \tag{39}$$

Proof. By induction
for $n = 0$, using **Theorm 2.4**

$$\begin{aligned}\|u_i^1 - u_{ih}^1\|_i &\leq \|u_i^1 - \omega_{ih}^i\|_i + \|\omega_{ih}^1 - u_{ih}^1\|_i \\ &\leq \|u_i^1 - \omega_{ih}^1\|_i + \|u_j^0 - u_{jh}^0\|_j \\ &\leq ch^2 \log|h|^2 + ch^2 \log|h|^2 \\ &\leq 2ch^2 \log|h|^2\end{aligned}$$

$$\begin{aligned}\|u_j^1 - u_{jh}^1\|_j &\leq \|u_j^1 - \omega_{jh}^1\|_j + \|\omega_{jh}^1 - u_{jh}^1\|_j \\ &\leq \|u_j^1 - \omega_{jh}^1\|_j + \|u_i^1 - u_{ih}^1\|_i \\ &\leq ch^2 \log|h|^2 + 2ch^2 \log|h|^2 \\ &\leq 3ch^2 \log|h|^2\end{aligned}$$

now we suppose that

$$\|u_j^n - u_{jh}^n\|_j \leq (2n+1)Ch^2|\log h|^3$$

$$\begin{aligned}\|u_i^{n+1} - u_{ih}^{n+1}\|_i &\leq \|u_i^{n+1} - \omega_{ih}^{n+1}\|_i + \|\omega_{ih}^{n+1} - u_{ih}^{n+1}\|_i \\ &\leq \|u_i^{n+1} - \omega_{ih}^{n+1}\|_i + \|u_j^n - u_{jh}^n\|_j \\ &\leq ch^2 \log|h|^2 + (2n+1)ch^2 \log|h|^2 \\ &\leq 2(n+1)ch^2 \log|h|^2\end{aligned}$$

$$\begin{aligned}\|u_j^{n+1} - u_{jh}^{n+1}\|_j &\leq \|u_j^{n+1} - \omega_{jh}^{n+1}\|_j + \|\omega_{jh}^{n+1} - u_{jh}^{n+1}\|_j \\ &\leq \|u_j^{n+1} - \omega_{jh}^{n+1}\|_j + \|u_i^{n+1} - u_{ih}^{n+1}\|_i \\ &\leq ch^2 \log|h|^2 + 2(n+1)ch^2 \log|h|^2 \\ &\leq (2n+3)ch^2 \log|h|^2\end{aligned}$$

3.2 L^∞ error estimate

Theorem 3 *Let $h = max(h_i, h_j)$, $\quad i = \overline{1, m-1}$; $\quad j = \overline{2, m}$, $\quad i < j$. Then, there exists a constant C independent of both h and n such that*

$$\|u_M - u_{Mh}^{n+1}\|_{L^\infty(\Omega_M)} \leq Ch^2|\log h|^3; \quad M = \overline{i, j} \tag{40}$$

Proof. Let us give the proof for $M = i$. The case $M = j$ is similar.
for $N = i$, let $k = max(k_i, k_j)$
using **Theorem 2.5**, **lemma 3.2**, we obtain

$$\begin{aligned}
\|u_i - u_{ih}^{n+1}\|_i &\leq \|u_i - u_i^{n+1}\|_i + \|u_i^{n+1} - u_{ih}^{n+1}\|_i \\
&\leq k^{2n}|u - u^0|_{ij} + \|u_i^{n+1} - u_{ih}^{n+1}\|_i \\
&\leq k^{2n}|u - u^0|_{ij} + \sum_{p=1}^{n+1} \|u_i^p - \omega_{ih}^p\|_i + \sum_{p=0}^{n} \|u_j^p - \omega_{jh}^p\|_j \\
&\leq k^{2n}|u - u^0|_{ij} + 2(n+1)Ch^2|\log h|^2
\end{aligned}$$

we suppose that

$$k^{2n} \leq h^2$$

we obtain

$$\|u_i - u_{ih}^{n+1}\|_i \leq Ch^2|\log h|^3$$

4 Conclusion

In this work, we have established a approach of the alternating Schwarz algorithm for m overlaping subdomains with nonmatching grids, for the class of elliptic variational inequality. This type of estimation which we have obtained relies on the geometrical convergence and the error estimate between the continuous and discrete Schwarz iterates. We will see that this result plays an important role in the study of the numerecal analysis for the class of elliptic variational inequality in the context overlapping nonmatching grids, using the parallel Schwarz method.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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