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Numerical Treatment of Special Types of Odd-Order Boundary Value Problems Using Nonsymmetric Cases of Jacobi Polynomials

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Abstract: In this article, two new dual Petrov-Galerkin algorithms for solving high odd-order boundary value problems (BVPs) are presented and implemented. The philosophy of applying the Petrov-Galerkin method is built on choosing the trial and test functions such that they satisfy the underlying boundary and dual boundary conditions, respectively. The presented approaches are based on employing the shifted Chebyshev polynomials of third and fourth kinds, respectively, as basis functions. Several numerical experiments are included to ascertain the validity and efficiency of the proposed algorithms. Moreover, comparisons with some other numerical methods in the literature are given.

Keywords: Spectral dual-Petrov-Galerkin method, shifted Chebyshev polynomials of third and fourth kinds, high odd-order boundary value problems.

1 Introduction

Spectral approximations seek to obtain numerical solutions of differential equations by utilizing expansions of orthogonal functions. Spectral methods have the characteristic that various orthogonal systems of infinitely differentiable global functions are chosen as trial functions. Of course, various choices of basis functions lead to various spectral numerical solutions, see, [1,2,3,4].

Due to their high accuracy, spectral methods have been widely used and fruitfully applied to numerical simulations in numerous fields. They are extensively used for solving various physical problems that appear in fluid and heat flow. It is well-known that there are three main types of spectral methods, namely, Galerkin, tau and collocation methods. The collocation method is often employed for treating nonlinear problems, see, [5,6,7]. Galerkin method is basically built on choosing suitable combinations satisfying the underlying boundary conditions, see, [8,9,10,11,12]. Regarding the tau method, it is utilized if the underlying boundary conditions are complicated, see for example, [13,14,15,16,17].

Chebyshev polynomials are crucial in analysis and its applications, and in particular in numerical analysis and approximation theory. It is well-known that there are four kinds of Chebyshev polynomials which are considered as special cases of Jacobi polynomials. Great interests were devoted to employing first and seconds kinds of Chebyshev polynomials $T_i(t)$ and $U_i(t)$ in various numerical applications, see for example, [18,19,20,21], while some other interests ware confined to utilizing third and fourth kinds $V_i(t)$ and $W_i(t)$, see for instance, [22].

Because of their significance in different applications, high even-order BVPs were extensively studied by several researchers. Doha et al. in [23] and [24] employed Chebyshev polynomials of the third and fourth kinds (CPs3 and CPs4) for treating multi-dimensional even-order BVPs. Furthermore, the investigation of odd-order BVPs is crucial in physical

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applications. For example, some draining or coating fluid flow problems can be described by third-order ordinary differential equations, see, [25]. Again, the third-order differential equation involves an operator that arises in some important partial differential equations such as the Kortweg-de Vries equation. Fifth-order BVPs appear in the mathematical modeling of viscoelastic flows, see, [26]. Abd-Elhameed et al. [27] presented a numerical algorithm for treating third and fifth-order BVPs. This algorithm was built on applying the Petrov-Galerkin method using two families of generalized Jacobi polynomials.

The main purpose of the current article is to develop some efficient spectral algorithms based on the shifted Chebyshev polynomials of the third and fourth kinds (SCPs3 and SCPs4) for the numerical treatment of special types of odd-order BVPs.

The rest of the paper is as follows. Section 2 displays some fundamental properties and formulas concerned with CPs3 and CPs4 and their shifted polynomials. In Sections 3 and 4, we present two numerical algorithms for treating high odd-order BVPs based on Petrov-Galerkin methods in terms of SCPs3 and SCPs4, respectively. Section 5 presents some numerical examples accompanied by comparisons with some other techniques in the literature aiming to demonstrate the accuracy and applicability of the two proposed methods. We end the paper with some conclusions in Section 6.

2 Shifted Chebyshev polynomials of third and fourth kinds

The classical CPs3 and CPs4 are defined, respectively, by [28]:

$$V_i(t) = \frac{\cos(i+\frac{1}{2})\vartheta}{\cos\left(\frac{\vartheta}{2}\right)} = \frac{2^{2i}}{\binom{2i}{i}} P_i^{(-\frac{1}{2},\frac{1}{2})}(t), \tag{1}$$

and

$$W_i(t) = \frac{\sin(i+\frac{1}{2})\vartheta}{\sin\left(\frac{\vartheta}{2}\right)} = \frac{2^{2i}}{\binom{2i}{i}} P_i^{(\frac{1}{2},-\frac{1}{2})}(t), \tag{2}$$

where $t = \cos\vartheta$ and $P_i^{(\alpha,\beta)}(t)$ is the classical Jacobi polynomial of degree i.

The orthogonality relations of CPs3 and CPs4 are given as

$$\int_{-1}^{1} \sqrt{\frac{1+t}{1-t}}\, V_i(t)\, V_j(t)\, dt = \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}}\, W_i(t)\, W_j(t)\, dt = \begin{cases} \pi, & i=j, \\ 0, & i \neq j. \end{cases}$$

Using (1) and (2), one can show that

$$W_i(t) = (-1)^i V_i(-t).$$

The polynomials $V_i(t)$ and $W_i(t)$ may be generated by using the recurrence relations:

$$\begin{aligned} V_i(t) &= 2t\, V_{i-1}(t) - V_{i-2}(t), \quad V_0(t)=1, \quad V_1(t) = 2t-1, \quad i \geq 2, \\ W_i(t) &= 2t\, W_{i-1}(t) - W_{i-2}(t), \quad W_0(t)=1, \quad W(t) = 2t+1, \quad i \geq 2. \end{aligned}$$

The following particular values are of interest:

$$V_i(1) = 1, \quad V_i(-1) = (-1)^i\,(2i+1), \quad W_i(1) = (2i+1), \quad W_i(-1) = (-1)^i,$$

$$D^{\upsilon}\, V_i(t) = \frac{\sqrt{\pi}\,(i+\upsilon)!}{2^{\upsilon}\,(i-\upsilon)!} \begin{cases} \dfrac{1}{\Gamma(\upsilon+\frac{1}{2})}, & \text{if } t=1, \\[2ex] \dfrac{(-1)^{i+\upsilon}(2i+1)}{2\Gamma(\upsilon+\frac{3}{2})}, & \text{if } t=-1, \end{cases}$$

where $D \equiv \frac{d}{dt}$ and $\upsilon \in \mathbb{N}$.

The SCPs3 and SCPs4 may be defined respectively on (a,b), by

$$\mathcal{V}_i(t) = V_i\left(\frac{2t-a-b}{b-a}\right), \quad \mathcal{W}_i(t) = W_i\left(\frac{2t-a-b}{b-a}\right).$$

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It is easy to transform all results and relations of CPs3 and CPs4 to give their counterparts of shifted polynomials. The orthogonality relations of $\mathcal{V}_i(t)$ and $\mathcal{W}_i(t)$ are given by

$$\int_a^b \sqrt{\frac{t-a}{b-t}}\,\mathcal{V}_i(t)\,\mathcal{V}_j(t)\,dt = \int_a^b \sqrt{\frac{b-t}{t-a}}\,\mathcal{W}_i(t)\,\mathcal{W}_j(t)\,dt = \begin{cases} (b-a)\,\dfrac{\pi}{2}, & i=j, \\ 0, & i\neq j. \end{cases} \tag{3}$$

The following theorem and its corollary are needed hereafter.

Theorem 1. *For $\upsilon \in \mathbb{N}$, the $\upsilon th-$ derivative of the SCPs3 can be expressed in the form*

$$D^{\upsilon}\mathcal{V}_i(t) = \sum_{m=0}^{i-\upsilon} \zeta_{i,m,\upsilon}\,\mathcal{V}_m(t), \quad \upsilon \geq 1, \tag{4}$$

where

$$\zeta_{i,m,\upsilon} = \frac{(b-a)^{\upsilon}}{(\upsilon-1)!}\begin{cases} \dfrac{\left(\frac{i-m+\upsilon-2}{2}\right)!\,\left(\frac{i+m+\upsilon}{2}\right)!}{\left(\frac{i-m-\upsilon}{2}\right)!\,\left(\frac{i+m-\upsilon}{2}\right)!}, & (i-\upsilon-m)\ \text{even}, \\ \dfrac{\left(\frac{i-m+\upsilon-1}{2}\right)!\,\left(\frac{i+m+\upsilon-1}{2}\right)!}{\left(\frac{i-m-\upsilon-1}{2}\right)!\,\left(\frac{i+m-\upsilon+1}{2}\right)!}, & (i-\upsilon-m)\ \text{odd}. \end{cases} \tag{5}$$

Proof. For the proof of Theorem 1, see, [8].

Corollary 1. *For $\upsilon \in \mathbb{N}$, the $\upsilon th-$ derivative of the SCPs4 can be expressed in the form*

$$D^{\upsilon}\mathcal{W}_i(t) = \sum_{m=0}^{i-\upsilon} \xi_{i,m,\upsilon}\,\mathcal{W}_m(t), \quad \upsilon \geq 1,$$

with

$$\xi_{i,m,\upsilon} = (-1)^{i+m+\upsilon}\,\zeta_{i,m,\upsilon}, \tag{6}$$

where $\zeta_{i,m,\upsilon}$ are as given in (5).

3 Shifted Chebyshev third-kind Petrov-Galerkin method

In this section, we implement a numerical method namely, shifted Chebyshev third-kind Petrov-Galerkin method (SC3-PGM) to treat numerically the high odd-order differential equation:

$$D^{2n+1}y(t) + \sum_{m=0}^{2n} \mu_m\, D^m y(t) = g(t), \quad t\in(a,b),\ n\geq 1, \tag{7}$$

subject to the homogeneous boundary conditions

$$y^{(r)}(a) = y^{(r)}(b) = y^{(n)}(a) = 0, \quad 0\leq r\leq n-1, \tag{8}$$

with $\mu_m, 0\leq m\leq 2n$, are real coefficients. We define the following spaces

$$\begin{aligned} \Omega_N =& \text{span}\{\mathcal{V}_0(t),\mathcal{V}_1(t),\mathcal{V}_2(t),\ldots,\mathcal{V}_{N-2n-1}(t)\}, \\ \Phi_N =& \{\phi(t)\in\Omega_N \ :\ \phi^{(r)}(a)=\phi^{(r)}(b)=\phi^{(n)}(a)=0,\ 0\leq r\leq n-1\}, \\ \Psi_N =& \{\psi(t)\in\Omega_N \ :\ \psi^{(r)}(a)=\psi^{(r)}(b)=\psi^{(n)}(b)=0,\ 0\leq r\leq n-1\}. \end{aligned}$$

The SC3-PGM is to find $y_N^n \in \Phi_N$, such that

$$\left(D^{2n+1}y_N^n(t),\psi(t)\right)_w + \sum_{m=0}^{2n}\mu_m\,(D^m y_N^n(t),\psi(t))_w = (g(t),\psi(t))_w,\ \forall\ \psi(t)\in\Psi_N, \tag{9}$$

where $(y(t),\psi(t))_w = \int\limits_a^b w(t)\,y(t)\,\psi(t)\,dt$ and $w(t) = \sqrt{\dfrac{t-a}{b-t}}$.

3.1 Selection of trial and test basis

Without loss of generality, we consider $(a,b) \equiv (-1,1)$. We are going to select suitable trial and test functions.
For this end, for $0 \le i \le N-2n-1, n \ge 1$, let

$$\phi_{i,n}(t) = V_i(t) + \sum_{m=1}^{2n+1} \rho_{m,i} V_{i+m}(t), \quad t \in (-1,1), \tag{10}$$

$$\psi_{i,n}(t) = V_i(t) + \sum_{m=1}^{2n+1} \sigma_{m,i} V_{i+m}(t), \quad t \in (-1,1), \tag{11}$$

where the coefficients $\{\rho_{m,i}\}$ and $\{\sigma_{m,i}\}$ are chosen such that $\phi_{i,n}(t) \in \Phi_{i+2n+1}$ and $\psi_{i,n}(t) \in \Psi_{i+2n+1}$, respectively.
The conditions $\phi_{i,n}^{(r)}(-1) = 0,\ 0 \le r \le n$, lead to:

$$\sum_{m=1}^{2n+1} \frac{(-1)^m (2i+2m+1)(i+m+r)!}{(i+m-r)!} \rho_{m,i} = -\frac{(2i+1)(i+r)!}{(i-r)!}, \tag{12}$$

while, the conditions $\phi_{i,n}^{(r)}(1) = 0,\ 0 \le r \le n-1$, lead to:

$$\sum_{m=1}^{2n+1} \frac{(i+m+r)!}{(i+m-r)!} \rho_{m,i} = -\frac{(i+r)!}{(i-r)!}. \tag{13}$$

Hence, Eqs. (12) and (13) constitute a system of $(2n+1)$ equations whose determinant differs from zero. The coefficients $\{\rho_{m,i}\}$ are given by

$$\rho_{2m,i} = \frac{(-1)^m \binom{n}{m} (i+n+1)!\,(i+m)!(2i+4m+2n+3)}{i!\,(i+n+m+1)!(2i+2n+3)}, \qquad 1 \le m \le n, \qquad 0 \le i \le N-2n-1,$$
$$\rho_{2m+1,i} = \frac{(-1)^m \binom{n}{m} (i+n+1)!\,(i+m)!(2i+4m+2n+1)}{i!\,(i+n+m+1)!(2i+2n+3)}, \qquad 0 \le m \le n, \qquad 0 \le i \le N-2n-1.$$

Similarly, it can be verified that the coefficients $\{\sigma_{m,i}\}$ take the form

$$\sigma_{2m,i} = \frac{(-1)^m \binom{n}{m} (i+m)!\,(i+n+1)!}{i!\,(i+n+m+1)!}, \qquad 1 \le m \le n, \qquad 0 \le i \le N-2n-1,$$
$$\sigma_{2m+1,i} = \frac{(-1)^{m+1} \binom{n}{m} (i+m)!\,(i+n+1)!}{i!\,(i+n+m+1)!}, \qquad 0 \le m \le n, \qquad 0 \le i \le N-2n-1.$$

Now, if we set $\left(\dfrac{2t-a-b}{b-a}\right)$ instead of t in (10) and (11), then, it is easy to see that the basis functions and their dual basis given as

$$\phi_{i,n}(t) = \mathcal{V}_i(t) + \sum_{m=1}^{2n+1} \rho_{m,i} \mathcal{V}_{i+m}(t),$$

and

$$\psi_{i,n}(t) = \mathcal{V}_i(t) + \sum_{m=1}^{2n+1} \sigma_{m,i} \mathcal{V}_{i+m}(t),$$

fulfill (8), i.e. $\phi_{i,n}(t) \in \Phi_{i+2n+1}$ and $\psi_{i,n}(t) \in \Psi_{i+2n+1}$, respectively.

For subsequent computations, we write $\phi_{i,n}(t)$ and $\psi_{i,n}(t)$ as

$$\phi_{i,n}(t) = \sum_{m=0}^{2n+1} \rho_{m,i} \mathcal{V}_{i+m}(t), \tag{14}$$

$$\psi_{i,n}(t) = \sum_{m=0}^{2n+1} \sigma_{m,i} \mathcal{V}_{i+m}(t), \tag{15}$$

where $\rho_{m,i}$ are given by

$$\rho_{m,i} = \begin{cases} \dfrac{(-1)^{\frac{m}{2}} \binom{n}{\frac{m}{2}} (i+\frac{m}{2})!\,(i+n+1)!\,(2i+2m+2n+3)}{i!\,(i+n+\frac{m}{2}+1)!\,(2i+2n+3)}, & m \text{ even}, \\ \dfrac{(-1)^{\frac{m-1}{2}} \binom{n}{\frac{m-1}{2}} (i+\frac{m-1}{2})!\,(i+n+1)!\,(2i+2m+2n+3)}{i!\,(i+n+\frac{m+1}{2})!\,(2i+2n+3)}, & m \text{ odd}, \end{cases} \tag{16}$$

and $\sigma_{m,i}$ are given by

$$\sigma_{m,i} = \begin{cases} \dfrac{(-1)^{\frac{m}{2}} \binom{n}{\frac{m}{2}} (i+\frac{m}{2})!\,(i+n+1)!}{i!\,(i+n+\frac{m}{2}+1)!}, & m \text{ even}, \\ \dfrac{(-1)^{\frac{m-1}{2}} \binom{n}{\frac{m-1}{2}} (i+\frac{m-1}{2})!\,(i+n+1)!}{i!\,(i+n+\frac{m+1}{2})!}, & m \text{ odd}. \end{cases} \tag{17}$$

Note that the linear independence of the basis $\phi_{i,n}(t)$ and $\psi_{i,n}(t)$ yields

$$\Phi_N = \text{span}\{\phi_{i,n}(t) : 0 \le i \le N-2n-1\},$$

and

$$\Psi_N = \text{span}\{\psi_{i,n}(t) : 0 \le i \le N-2n-1\}.$$

Now, using (9), the Petrov-Galerkin approximation leads to the following equation

$$\left(D^{2n+1}y_N^n(t), \psi_{j,n}(t)\right)_w + \sum_{m=0}^{2n} \mu_m \left(D^m y_N^n(t), \psi_{j,n}(t)\right)_w = (g(t), \psi_{j,n}(t))_w,\ \psi_{j,n}(t) \in \Psi_N. \tag{18}$$

Let us denote

$$\mathbf{A}_n = \left(a_{ij}^n\right)_{0\le i,j\le N-2n-1}, \qquad a_{ij}^n = \left(D^{2n+1}\phi_{i,n}(t), \psi_{j,n}(t)\right)_w,$$

$$\mathbf{B}_{mn} = \left(b_{ij}^{mn}\right)_{0\le i,j\le N-2n-1}, \qquad b_{ij}^{mn} = \left(D^m \phi_{i,n}(t), \psi_{j,n}(t)\right)_w, \quad 0 \le m \le 2n,$$

$$\mathbf{G}_n = \left(g_0^n, g_1^n, \dots, g_{N-2n-1}^n\right)^T, \qquad g_j^n = \left(g(t), \psi_{j,n}(t)\right)_w,$$

$$y_N^n(t) = \sum_{i=0}^{N-2n-1} c_i^n \phi_{i,n}(t), \qquad \mathbf{C}_n = \left(c_0^n, c_1^n, \dots, c_{N-2n-1}^n\right)^T.$$

Then (18) is equivalent to the following linear system

$$\left(\mathbf{A}_n + \sum_{m=0}^{2n} \mu_m \mathbf{B}_{mn}\right) \mathbf{C}_n = \mathbf{G}_n, \tag{19}$$

and the nonzero entries of $\mathbf{A}_n$ and $\mathbf{B}_{mn}$ $(0 \le m \le 2n)$ are given as follows.

Theorem 2. *Let $\phi_{i,n}(t)$ and $\psi_{i,n}(t)$ be as selected in (14) and (15), respectively. In addition, let $a_{ij}^n = \left(D^{2n+1}\phi_{i,n}(t), \psi_{j,n}(t)\right)_w$ and $b_{ij}^{mn} = \left(D^m\phi_{i,n}(t), \psi_{j,n}(t)\right)_w$, $0 \le m \le 2n$, then*

$$\Phi_N = span\{\phi_{0,n}(t), \phi_{1,n}(t), \cdots, \phi_{N-2n-1,n}(t)\},$$

$$\Psi_N = span\{\psi_{0,n}(t), \psi_{1,n}(t), \cdots, \psi_{N-2n-1,n}(t)\},$$

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and the nonzero elements of the matrices A_n and B_{mn} $(0 \leq m \leq 2n)$ are given explicitly by:

$$a^n_{ii} = \frac{(-1)^n\, 2^{4n+1} \pi ((i+1)_n)^2 (i+n+1)(2i+2n+1)}{(b-a)^{2n}(2i+2n+3)}, \tag{20}$$

$$a^n_{ij} = \left(\frac{2}{b-a}\right)^{2n} \pi \sum_{k=0}^{2n+1} \sum_{\ell=0}^{2n+1} \rho_{k,i}\, \sigma_{\ell,j}\, \zeta_{i+k,j+\ell,2n+1}, \quad j = i+s,\, s \geq 1, \tag{21}$$

$$b^{mn}_{ij} = \left(\frac{2}{b-a}\right)^{m-1} \pi \sum_{k=0}^{2n+1} \sum_{\ell=0}^{2n+1} \rho_{k,i}\, \sigma_{\ell,j}\, \zeta_{i+k,j+\ell,m}, \quad j = i+s,\, -2n-1 \leq s \leq 4n+1, \tag{22}$$

where $\zeta_{i,k,m}$, $\rho_{k,i}$ and $\sigma_{k,i}$ are defined as in (5), (16) and (17), respectively.

Proof. Each member of the two families $\{\phi_{i,n}(t)\}_{i \geq 0}$ and $\{\psi_{i,n}(t)\}_{i \geq 0}$ satisfies the homogeneous boundary conditions (8) and their dual conditions, that is $\phi_{i,n}(t) \in \Phi_N$ and $\psi_{i,n}(t) \in \Psi_N$ for $0 \leq i \leq N-2n-1$. Moreover, $\{\phi_{i,n}(t)\}_{0 \leq i \leq N-2n-1}$ and $\{\psi_{i,n}(t)\}_{0 \leq i \leq N-2n-1}$ are linearly independent. Both of them is of dimension $(N-2n)$. Hence

$$\Phi_N = \operatorname{span}\{\phi_{0,n}(t), \phi_{1,n}(t), \ldots, \phi_{N-2n-1,n}(t)\},$$

and

$$\Psi_N = \operatorname{span}\{\psi_{0,n}(t), \psi_{1,n}(t), \ldots, \psi_{N-2n-1,n}(t)\}.$$

To prove (20) and (21), we make use of relation (4) to get

$$D^{2n+1} \mathcal{V}_i(t) = \left(\frac{2}{b-a}\right)^{2n+1} \sum_{k=0}^{i-2n-1} \zeta_{i,k,2n+1}\, \mathcal{V}_k(t), \quad i \geq 2n+1, \tag{23}$$

where $\zeta_{i,k,2n+1}$ is defined as in (5).

Replacing i by $i+k$ in (23) and using the orthogonality relation (3), it can be shown that, for $i+k \geq j+\ell+2n+1$, we have

$$\left(D^{2n+1} \mathcal{V}_{i+k}(t), \mathcal{V}_{j+\ell}(t)\right)_w = \left(\frac{2}{b-a}\right)^{2n} \pi\, \zeta_{i+k,j+\ell,2n+1}. \tag{24}$$

Due to (14), (15) and (24), a^n_{ij} takes the form

$$a^n_{ij} = \left(\frac{2}{b-a}\right)^{2n} \pi \sum_{k=0}^{2n+1} \sum_{\ell=0}^{2n+1} \rho_{k,i}\, \sigma_{\ell,j}\, \zeta_{i+k,j+\ell,2n+1},$$

which proves (21).

If we set $i = j$ in (21) and make use of formulae (5), (16) and (17), then we get

$$a^n_{ii} = \frac{(-1)^n\, 2^{4n+1} \pi ((i+1)_n)^2 (i+n+1)(2i+2n+1)}{(b-a)^{2n}(2i+2n+3)},$$

which proves (20).

Finally to prove (22), for $0 \leq m \leq 2n$, we have

$$b^{mn}_{ij} = \left(D^m \phi_{i,n}(t), \psi_{j,n}(t)\right)_w = \sum_{k=0}^{2n+1} \sum_{\ell=0}^{2n+1} \rho_{k,i}\, \sigma_{\ell,j} \left(D^m \mathcal{V}_{i+k}(t), \mathcal{V}_{j+\ell}(t)\right)_w,$$

and with the aid of the orthogonality relation (3), b^{mn}_{ij} can be computed to give formula (22).

Based on Theorem 2, if $\mu_m = 0,\ 0 \leq m \leq 2n$, a nonsingular upper triangular linear system is obtained. The following corollary exhibits this result.

Corollary 2. *For the cases $\mu_m = 0$, the system (19) reduces to $A_n\, \mathbf{C}_n = \mathbf{G}_n$, where A_n is an upper triangular matrix with the following solution*

$$c^n_i = \left(g^n_i - \sum_{j=i+1}^{N-2n-1} a^n_{ij}\, c^n_j\right) \Big/ a^n_{ii}, \quad 0 \leq i \leq N-2n-1,$$

where a^n_{ii} and a^n_{ij} are given by (20) and (21), respectively.

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3.2 Nonhomogeneous boundary conditions

In this section, we consider the odd-order differential equations (7) subject to the nonhomogeneous boundary conditions

$$y^{(r)}(a) = \alpha_r, \quad y^{(r)}(b) = \beta_r, \quad y^{(n)}(a) = \gamma, \quad 0 \le r \le n-1. \tag{25}$$

With the aid of the following transformation:

$$Y(t) = y(t) + \sum_{s=0}^{2n} \eta_s \left(\frac{2t-a-b}{b-a}\right)^s,$$

where η_s are uniquely determined such that $Y(t)$ satisfies the homogeneous boundary conditions

$$Y^{(r)}(a) = Y^{(r)}(b) = Y^{(n)}(a) = 0, \quad 0 \le r \le n-1, \tag{26}$$

problem (7) governed by (25) is equivalent to the following modified problem:

$$D^{2n+1}Y(t) + \sum_{m=0}^{2n} \mu_m D^m Y(t) = g^*(t), \quad t \in (a,b),\ n \ge 1,$$

governed by (26), where

$$g^*(t) = g(t) + \sum_{s=0}^{2n} \delta_s t^s,$$

and δ_s are some constants to be determined in terms of η_s.

Now, after the application of SC3-PGM, a linear system similar to that given in (19) can be obtained.

4 Shifted Chebyshev fourth-kind Petrov-Galerkin method

In this section, we focus on solving the high odd-order differential equation (7)-(8) using the shifted Chebyshev fourth-kind Petrov-Galerkin method (SC4-PGM).

If we set

$$\mho_N = \text{span}\{\mathcal{W}_0(t), \mathcal{W}_1(t), \mathcal{W}_2(t), \dots, \mathcal{W}_{N-2n-1}(t)\},$$

$$\Phi_N = \{\phi(t) \in \mho_N \ : \ \phi^{(r)}(a) = \phi^{(r)}(b) = \phi^{(n)}(a) = 0,\ 0 \le r \le n-1\},$$

$$\Psi_N = \{\psi(t) \in \mho_N \ : \ \psi^{(r)}(a) = \psi^{(r)}(b) = \psi^{(n)}(b) = 0,\ 0 \le r \le n-1\},$$

then, the SC4-PGM for solving (7)-(8) is to find $\mathbf{y}_N^n \in \Phi_N$ such that

$$\left(D^{2n+1}\,\mathbf{y}_N^n(t), \psi(t)\right)_{\varpi} + \sum_{m=0}^{2n} \mu_m \left(D^m\,\mathbf{y}_N^n(t), \psi(t)\right)_{\varpi} = (g(t), \psi(t))_{\varpi}, \quad \forall\, \psi(t) \in \Psi_N,$$

where $\varpi(t) = \sqrt{\dfrac{b-t}{t-a}}$.

We choose the basis functions in terms of SCPs4 to be as follows:

$$\phi_{i,n}(t) = \sum_{m=0}^{2n+1} \mathrm{p}_{m,i} \mathcal{W}_{i+m}(t), \tag{27}$$

$$\psi_{i,n}(t) = \sum_{m=0}^{2n+1} \mathrm{q}_{m,i} \mathcal{W}_{i+m}(t), \tag{28}$$

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with

$$\mathfrak{p}_{m,i}=\begin{cases}\sigma_{m,i}, & m \text{ even},\\ -\sigma_{m,i}, & m \text{ odd},\end{cases} \tag{29}$$

$$\mathfrak{q}_{m,i}=\begin{cases}\rho_{m,i}, & m \text{ even},\\ -\rho_{m,i}, & m \text{ odd},\end{cases} \tag{30}$$

and the coefficients $\rho_{m,i}$ and $\sigma_{m,i}$ are given as in (16) and (17), respectively.

Now, we are ready to state the main theorem of the current section.

Theorem 3. *If we take the basis functions and their dual basis $\phi_{i,n}(t)$ and $\psi_{i,n}(t)$ as given in (27) and (28), respectively, and if we assume $\mathbf{y}_N^n(t)=\sum_{i=0}^{N-2n-1}\mathfrak{c}_i^n\,\phi_{i,n}(t)$ is the Petrov-Galerkin approximation to (7)-(8), then the expansion coefficients $\{\mathfrak{c}_i^n,\ 0\le i\le N-2n-1\}$ satisfy the matrix system:*

$$\left(\mathcal{A}_n+\sum_{m=0}^{2n}\mu_m\,\mathcal{B}_{mn}\right)\mathcal{C}_n=\mathcal{G}_n, \tag{31}$$

where the nonzero elements of the matrices $\mathcal{A}_n$ and $\mathcal{B}_{mn}$ $(0\le m\le 2n)$ are given as follows:

$$\mathfrak{a}_{ii}^n=\frac{(-1)^n\,2^{4n+1}\,\pi\,((i+1)_n)^2(i+n+1)}{(b-a)^{2n}}, \tag{32}$$

$$\mathfrak{a}_{ij}^n=\left(\frac{2}{b-a}\right)^{2n}\pi\sum_{k=0}^{2n+1}\sum_{\ell=0}^{2n+1}\mathfrak{p}_{k,i}\,\mathfrak{q}_{\ell,j}\,\xi_{i+k,j+\ell,2n+1},\quad j=i+s,\ s\ge 1, \tag{33}$$

$$\mathfrak{b}_{ij}^{mn}=\left(\frac{2}{b-a}\right)^{m-1}\pi\sum_{k=0}^{2n+1}\sum_{\ell=0}^{2n+1}\mathfrak{p}_{k,i}\,\mathfrak{q}_{\ell,j}\,\xi_{i+k,j+\ell,m},\quad j=i+s,\ -2n-1\le s\le 4n+1, \tag{34}$$

where $\xi_{i,k,m},\mathfrak{p}_{k,i}$ and $\mathfrak{q}_{k,i}$ are defined as in (6), (29) and (30), respectively.

Proof. Each of the two families individual $\{\phi_{i,n}(t)\}_{i\ge 0}$ and $\{\psi_{i,n}(t)\}_{i\ge 0}$ satisfies the homogeneous boundary conditions (8) and their dual conditions, that is $\phi_{i,n}(t)\in\Phi_N$ and $\psi_{i,n}(t)\in\Psi_N$ for $0\le i\le N-2n-1$. Moreover, $\{\phi_{i,n}(t)\}_{0\le i\le N-2n-1}$ and $\{\psi_{i,n}(t)\}_{0\le i\le N-2n-1}$ are linearly independent. Both of them is of dimension $(N-2n)$. Hence

$$\Phi_N=\operatorname{span}\{\phi_{0,n}(t),\phi_{1,n}(t),\dots,\phi_{N-2n-1,n}(t)\},$$

and

$$\Psi_N=\operatorname{span}\{\psi_{0,n}(t),\psi_{1,n}(t),\dots,\psi_{N-2n-1,n}(t)\}.$$

Now, Applying the same technique introduced in Theorem 2, we can get the elements (32)-(34).
The following corollary treats the case in which $\mu_m=0,\ 0\le m\le 2n$.

Corollary 3. *If $\mu_m=0,\ 0\le m\le 2n,$ then the system (31) reduces to $\mathcal{A}_n\,\mathcal{C}_n=\mathcal{G}_n$, where $\mathcal{A}_n$ is an upper triangular matrix and its solution is given explicitly as*

$$\mathfrak{c}_i^n=\left(g_i^n-\sum_{j=i+1}^{N-2n-1}\mathfrak{a}_{ij}^n\,\mathfrak{c}_j^n\right)\Big/\mathfrak{a}_{ii}^n,\quad 0\le i\le N-2n-1,$$

where $\mathfrak{a}_{ii}^n$ and $\mathfrak{a}_{ij}^n$ are given by (32) and (33), respectively.

5 Numerical results

In this section, some numerical results are presented aiming to exhibit the efficiency and applicability of the proposed algorithms in Sections 3 and 4.

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Example 1. Consider the following singulary perturbed linear third-order BVP [29]:

$$-\varepsilon y^{(3)}(t)+y(t)=g(t), \quad 0\leq t\leq 1,$$

subject to

$$y(0)=0, \quad y(1)=0, \quad y^{(1)}(0)=0,$$

where $g(t)$ is taken to be compatible with the exact solution: $y(t)=6\varepsilon t^3(1-t)^5$.

Akram [29] introduced this problem and applied the quartic spline method (QSM) for its numerical solution. In Table 1, we list the L^∞-error using the SC3-PGM and SC4-PGM with $N=6,8,10$ and $\varepsilon=\frac{1}{16},\frac{1}{32},\frac{1}{64}$. In addition, Table 2 displays a comparison between the best absolute errors resulted using the application of SC3-PGM and SC4-PGM, in case of $N=10$, with those obtained using QSM [29]. This table demonstrates the accuracy of our algorithms comparable with the method developed in [29]. Furthermore, Figure 1 displays the maximum various absolute errors if our two algorithms SC3-PGM and SC4-PGM are applied for $N=8$ and various values of $\varepsilon=\frac{1}{32},\frac{1}{64},\frac{1}{128},\frac{1}{256},\frac{1}{512},\frac{1}{1024}$.

Table 1: L^∞-error for $N=6,8,10$ for Example 1.

N	ε	SC3-PGM	SC4-PGM
6	1/16	$4.0\cdot 10^{-4}$	$5.8\cdot 10^{-4}$
	1/32	$1.7\cdot 10^{-4}$	$2.9\cdot 10^{-4}$
	1/64	$7.0\cdot 10^{-5}$	$1.4\cdot 10^{-4}$
8	1/16	$1.3\cdot 10^{-15}$	$8.3\cdot 10^{-16}$
	1/32	$4.9\cdot 10^{-16}$	$5.0\cdot 10^{-16}$
	1/64	$2.9\cdot 10^{-16}$	$2.2\cdot 10^{-16}$
10	1/16	$1.3\cdot 10^{-15}$	$8.1\cdot 10^{-16}$
	1/32	$3.3\cdot 10^{-16}$	$4.9\cdot 10^{-16}$
	1/64	$1.9\cdot 10^{-16}$	$2.0\cdot 10^{-16}$

Table 2: Comparison of our methods and the QSM [29] for Example 1.

ε	QSM [29]	SC3-PGM	SC4-PGM
1/16	$6.4\cdot 10^{-6}$	$1.3\cdot 10^{-15}$	$8.1\cdot 10^{-16}$
1/32	$2.1\cdot 10^{-6}$	$3.3\cdot 10^{-16}$	$4.9\cdot 10^{-16}$
1/64	$4.6\cdot 10^{-7}$	$1.9\cdot 10^{-16}$	$2.0\cdot 10^{-16}$

Example 2. Consider the following BVP [30,31]:

$$y^{(3)}(t)+y(t)=(t-4)\sin(t)+(1-t)\cos(t), \quad 0\leq t\leq 1,$$

subject to

$$y(0)=0, \quad y^{(1)}(0)=-1, \quad y(1)=0,$$

with the analytical solution: $y(t)=(t-1)\sin(t)$.

For solving the above problem, Khan and Sultana [31] used the second, fourth and sixth order parametric quintic spline methods (PQSMs), while Abd El-Salam et al. [30] applied a nonpolynomial spline technique (NST). In Table 3, we list the L^∞-error using the SC3-PGM and SC4-PGM with $N=6,8,10,12,14,16,18$. Moreover, Table 4 exhibits a comparison between the best absolute errors if SC3-PGM and SC4-PGM are applied with those obtained using the NST [30] and second, fourth and sixth order PQSM [31]. Figure 2 illustrates L^2 and L^∞ errors if our two algorithms are applies for $N=6,8,10,12,14,16,18$.

Example 3. Consider the following fifth-order BVP [32,33,34,35,36,37]:

$$y^{(5)}(t)=y(t)-15e^t-10te^t, \qquad 0\leq t\leq 1,$$

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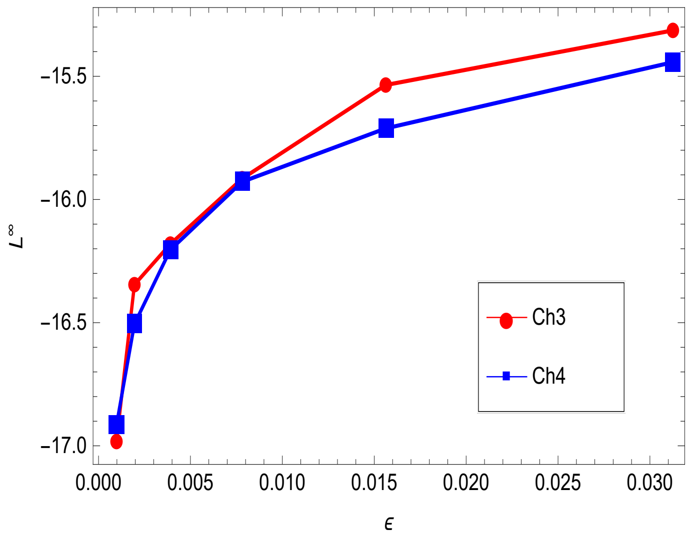

Fig. 1: $L^\infty - error$ of our algorithms for $N = 8$ and various values of ε for Example 1.

Table 3: L^∞-error for various values of N for Example 2.

N	SC3-PGM	SC4-PGM
6	$6.0 \cdot 10^{-7}$	$5.7 \cdot 10^{-7}$
8	$8.6 \cdot 10^{-10}$	$6.7 \cdot 10^{-10}$
10	$7.3 \cdot 10^{-13}$	$4.8 \cdot 10^{-13}$
12	$4.5 \cdot 10^{-16}$	$3.3 \cdot 10^{-16}$
14	$8.4 \cdot 10^{-17}$	$4.6 \cdot 10^{-17}$
16	$8.3 \cdot 10^{-17}$	$4.2 \cdot 10^{-17}$
18	$3.7 \cdot 10^{-17}$	$4.0 \cdot 10^{-17}$

Table 4: Comparison between our methods with those obtained in [30] and [31] for Example 2.

NST [30]	Second-order [31]	Fourth-order [31]	Sixth-order [31]	SC3-PGM	SC4-PGM
$6.30 \cdot 10^{-11}$	$2.00 \cdot 10^{-4}$	$9.48 \cdot 10^{-12}$	$7.15 \cdot 10^{-14}$	$3.7 \cdot 10^{-17}$	$4.0 \cdot 10^{-17}$

subject to

$$y(0) = 0, \quad y^{(1)}(0) = 1, \quad y^{(2)}(0) = 0, \quad y(1) = 0, \quad y^{(1)}(1) = -e,$$

with the exact solution: $y(t) = t(1-t)\,e^t$.

This problem is introduced in many articles, see, [32,33,34,35,36,37]. In Table 5, the L^∞-error using SC3-PGM and SC4-PGM are listed for $N = 8,10,12,14,16,18$. Moreover, Table 6 illustrates a comparison between the best absolute errors if SC3-PGM and SC4-PGM are applied with $N = 18$, with the following methods:

– Homotopy perturbation method (HPM) [32].
– Iteration method (IM) [32].
– Sextic spline method (SPM) [33].
– B-spline method in [34].
– Residual correction method (RCM) [36].
– Variational iteration method (VIM) [37].

Table 6 demonstrates the high accuracy of our methods compared with the above methods. In addition, Figure 3 displays the errors in L^2 and L^∞ if our two algorithms are applied for $N = 4,6,8,10,12,14$.

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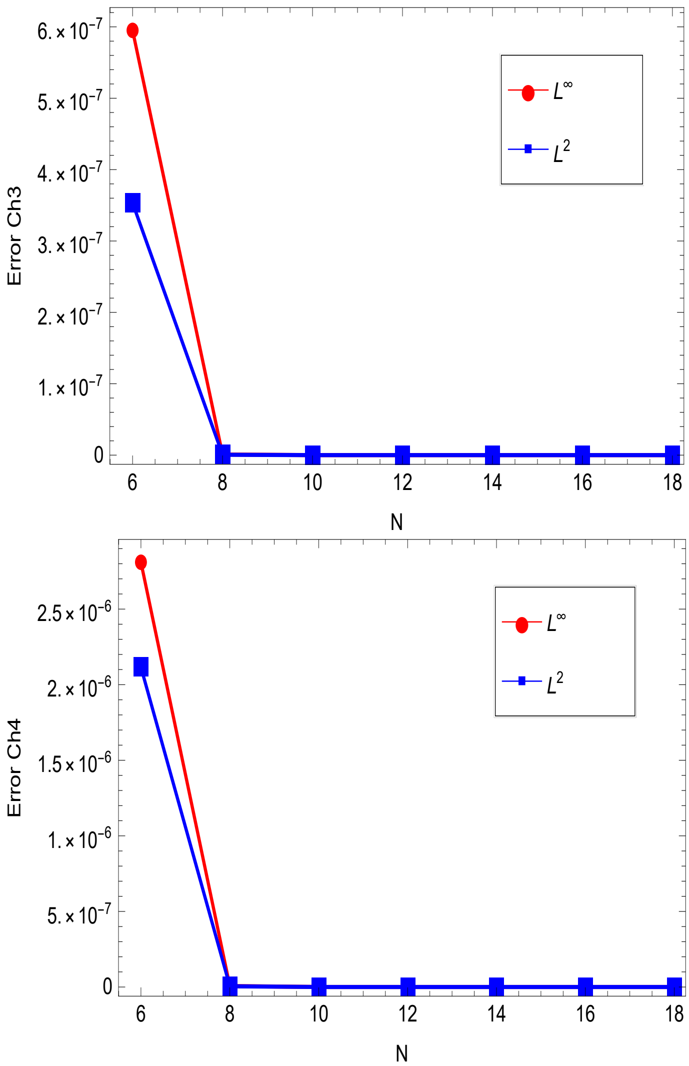

Fig. 2: ($L^2 - error$) and ($L^\infty - error$) for our algorithms for various values of N for Example 2.

Table 5: L^∞-error for various values of N for Example 3.

N	SC3-PGM	SC4-PGM
8	$1.0 \cdot 10^{-7}$	$5.7 \cdot 10^{-8}$
10	$1.3 \cdot 10^{-10}$	$5.6 \cdot 10^{-11}$
12	$1.0 \cdot 10^{-13}$	$3.7 \cdot 10^{-14}$
14	$1.1 \cdot 10^{-16}$	$1.0 \cdot 10^{-16}$
16	$9.5 \cdot 10^{-17}$	$9.6 \cdot 10^{-17}$
18	$8.4 \cdot 10^{-17}$	$8.7 \cdot 10^{-17}$

Example 4. Consider the linear ninth-order BVP [38]:

$$y^{(ix)}(t) = -9e^t + y(t), \qquad 0 < t < 1, \tag{35}$$

subject to the boundary conditions

$$y^{(r)}(0) = 1 - r, \quad 0 \le r \le 4,$$
$$y^{(r)}(1) = -re, \quad 0 \le r \le 3,$$

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Table 6: Comparison between our methods with those obtained in [34,32,33,36,37] for Example 3.

t	B-Spline [34]	HPM [32]	IM [32]	SPM [33]	RCM [36]	VIM [37]	SC3-PGM	SC4-PGM
0.1	$8\cdot10^{-3}$	$3\cdot10^{-11}$	$2\cdot10^{-5}$	$1\cdot10^{-13}$	$1\cdot10^{-8}$	$4\cdot10^{-7}$	$1\cdot10^{-16}$	$1\cdot10^{-16}$
0.2	$1\cdot10^{-3}$	$2\cdot10^{-10}$	$1\cdot10^{-4}$	$1\cdot10^{-13}$	$6\cdot10^{-8}$	$3\cdot10^{-6}$	$1\cdot10^{-16}$	$1\cdot10^{-16}$
0.3	$5\cdot10^{-3}$	$4\cdot10^{-10}$	$2\cdot10^{-4}$	$3\cdot10^{-14}$	$2\cdot10^{-7}$	$9\cdot10^{-6}$	0	0
0.4	$3\cdot10^{-3}$	$8\cdot10^{-10}$	$4\cdot10^{-4}$	$1\cdot10^{-13}$	$3\cdot10^{-7}$	$2\cdot10^{-5}$	0	0
0.5	$8\cdot10^{-3}$	$1\cdot10^{-9}$	$4\cdot10^{-4}$	$3\cdot10^{-13}$	$4\cdot10^{-7}$	$2\cdot10^{-5}$	$6\cdot10^{-17}$	$6\cdot10^{-17}$
0.6	$6\cdot10^{-3}$	$2\cdot10^{-9}$	$4\cdot10^{-4}$	$5\cdot10^{-13}$	$4\cdot10^{-7}$	$3\cdot10^{-5}$	$6\cdot10^{-17}$	0
0.7	$5\cdot10^{-3}$	$2\cdot10^{-9}$	$2\cdot10^{-4}$	$6\cdot10^{-13}$	$4\cdot10^{-7}$	$3\cdot10^{-5}$	0	$1\cdot10^{-16}$
0.8	$9\cdot10^{-3}$	$2\cdot10^{-9}$	$1\cdot10^{-4}$	$6\cdot10^{-13}$	$3\cdot10^{-7}$	$2\cdot10^{-5}$	$1\cdot10^{-16}$	$1\cdot10^{-16}$
0.9	$9\cdot10^{-3}$	$1\cdot10^{-9}$	$9\cdot10^{-5}$	$4\cdot10^{-13}$	$1\cdot10^{-7}$	$7\cdot10^{-6}$	$1\cdot10^{-16}$	$6\cdot10^{-17}$

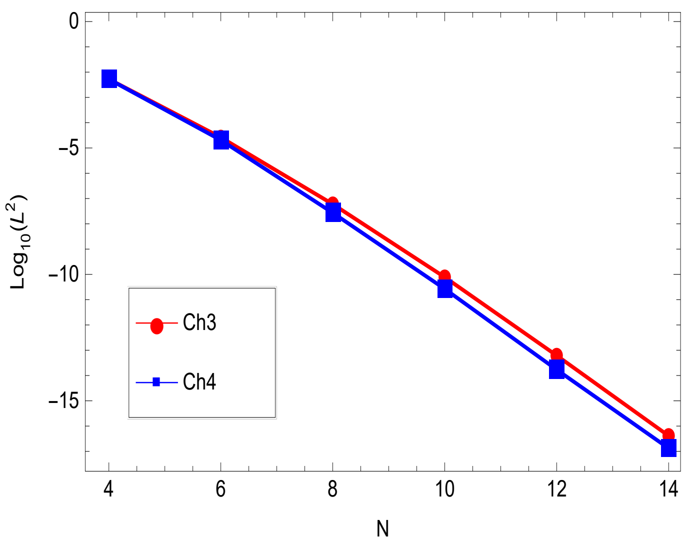

Fig. 3: ($L^2-error$) and ($L^\infty-error$) for our algorithms for various values of N for Example 3.

with the exact solution: $y(t) = (1-t)\,e^t$.

Wazwaz [38] used the modified decomposition method (MDM) for obtaining the approximate solution to (35).

Table 7 lists the maximum absolute error using our two methods SC3-PGM and SC4-PGM, for $N = 10,12,14,16$. Moreover, Table 8 displays a comparison between the best absolute errors resulted if SC3-PGM and SC4-PGM are applied with $N = 16$, with those obtained using the MDM [38]. Furthermore, Figure 4 displays the $\log_{10}$ error in L^2 if our two algorithms are applied for $N = 6,8,10,12,14$.

Table 7: L^∞-error for various values of N for Example 4.

N	SC3-PGM	SC4-PGM
10	$9.5\cdot10^{-11}$	$8.2\cdot10^{-11}$
12	$8.0\cdot10^{-14}$	$4.1\cdot10^{-14}$
14	$1.2\cdot10^{-16}$	$1.2\cdot10^{-16}$
16	$1.1\cdot10^{-16}$	$1.1\cdot10^{-16}$

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Table 8: Comparison between our methods with the MDM for Example 4.

t	Modified Decomp. [38]	SC3-PGM	SC4-PGM
0.0	0.0	0.0	0.0
0.1	$2.0\cdot10^{-10}$	$2.2\cdot10^{-16}$	$2.2\cdot10^{-16}$
0.2	$2.0\cdot10^{-10}$	$1.1\cdot10^{-16}$	$1.1\cdot10^{-16}$
0.3	$2.0\cdot10^{-10}$	$1.1\cdot10^{-16}$	$1.1\cdot10^{-16}$
0.4	$2.0\cdot10^{-10}$	$1.1\cdot10^{-16}$	0.0
0.5	$2.0\cdot10^{-10}$	$1.1\cdot10^{-16}$	0.0
0.6	$6.0\cdot10^{-10}$	$1.1\cdot10^{-16}$	$1.1\cdot10^{-16}$
0.7	$1.0\cdot10^{-9}$	0.0	$1.1\cdot10^{-16}$
0.8	$2.0\cdot10^{-9}$	$1.1\cdot10^{-16}$	$1.1\cdot10^{-16}$
0.9	$3.4\cdot10^{-9}$	$1.1\cdot10^{-16}$	0.0
1.0	0.0	$2.2\cdot10^{-16}$	$2.2\cdot10^{-16}$

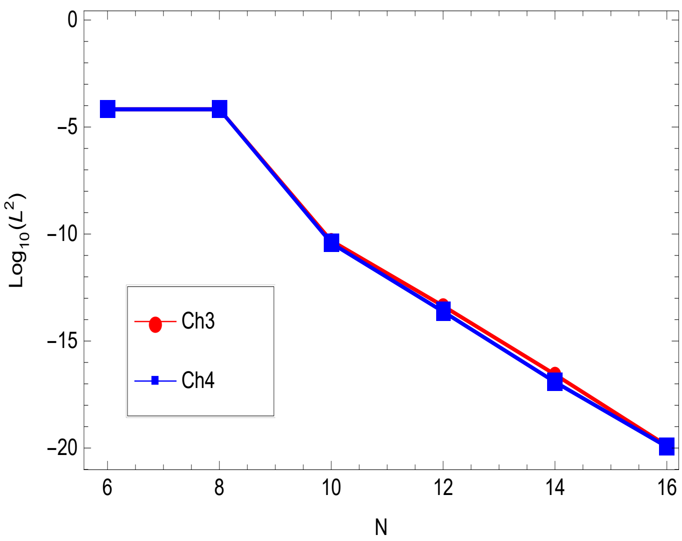

Fig. 4: $\log_{10}(L^2 - error)$ and $\log_{10}(L^\infty - error)$ for our algorithms for various values of N for Example 4.

6 Conclusion

Two efficient direct solvers for $(2n+1)$th-order boundary value problems are presented. Shifted Chebyshev polynomials of third and fourth kinds are employed as basis functions and the Petrov-Galerkin method is applied to transform the odd-order problems governed by their boundary conditions into linear algebraic systems that can be efficiently solved. It was found that for specific kinds of odd-order boundary value problems, the resulting systems are upper triangular and this greatly simplifies the numerical computations required for obtaining the numerical solutions corresponding to such cases. We tested the accuracy of the presented algorithm by means of four examples and we compared the results with those given using some other approaches. It was shown that, high accuracy results may be obtained using a low number of Chebyshev basis, the numerical errors decay rapidly as the number of the basis functions increases and that the new approach is more accurate than the quartic spline, parametric quintic spline, nonpolynomial spline technique, B-spline, homotopy perturbation, variational iteration, residual correction and modified decomposition methods.

Conflicts of Interests
The authors declare that they have no conflicts of interests

References

[1] M. A. Zaky, A Legendre spectral quadrature tau method for the multi-term time-fractional diffusion equations, *Comput. Appl. Math.* **37**(3), 3525–3538 (2018).
[2] M. M. Alsuyuti, E. H. Doha, S. S. Ezz-Eldien, B. I. Bayoumi and D. Baleanu, Modified Galerkin algorithm for solving multitype fractional differential equations, *Math. Meth. Appl. Sci.* **42**(5), 1389-1412 (2019).
[3] C. Canuto, M. Y. Hussaini, A. Quarteroni and Jr. Thomas, *Spectral methods in fluid dynamics*, Springer, 2012.
[4] D. Gottlieb and S. A. Orszag, *Numerical analysis of spectral methods: theory and applications*, Society for Industrial and Applied Mathematics, 1977.
[5] C. S. Chen, A. Karageorghis and F. Dou, A novel RBF collocation method using fictitious centres, *Appl. Math. Lett.* **101**, 106069 (2020).
[6] A. Babaei, H. Jafari and S. Banihashemi, Numerical solution of variable order fractional nonlinear quadratic integro-differential equations based on the sixth-kind Chebyshev collocation method, *J. Comput. Appl. Math.* **377**, 112908 (2020).
[7] S. A. Rakhshan and S. Effati, A generalized Legendre–Gauss collocation method for solving nonlinear fractional differential equations with time varying delays, *Appl. Numer. Math.* **146**, 342–360 (2019).
[8] W. M. Abd-Elhameed and M. A. Bassuony, On the coefficients of differentiated expansions and derivatives of Chebyshev polynomials of the third and fourth kinds, *Acta Math. Sci.* **35**(2), 326–338 (2015).
[9] E. H. Doha, W. M. Abd-Elhameed and A. H. Bhrawy, New spectral-Galerkin algorithms for direct solution of high even-order differential equations using symmetric generalized Jacobi polynomials, *Collect. Math.* **64**(3), 373–394 (2013).
[10] P. Mokhtary, Spectrally accurate and well-posed Jacobi Galerkin method for multiple delay pantograph integral equations, *Iran. J. Sci. Technol. Trans. Sci.* **43**(3), 959–967 (2019).
[11] M. M. Alsuyuti, E. H. Doha, S. S. Ezz-Eldien and I. K. Youssef, Spectral Galerkin schemes for a class of multi-order fractional pantograph equations, *J. Comput. Appl. Math.* **384**, 113157 (2020).
[12] Y. Wang and L. Mei, A conservative spectral Galerkin method for the coupled nonlinear space-fractional Schrödinger equations, *Int. J. Comput. Math.* **96**(12), 2387–2410 (2019).
[13] Y. Qin and H. Ma, Legendre-Tau-Galerkin and spectral collocation method for nonlinear evolution equations, *Appl. Numer. Math.* **153**, 52–65 (2020).
[14] R. M. Hafez and Y. H. Youssri, Jacobi spectral discretization for nonlinear fractional generalized seventh-order KDV equations with convergence analysis, *Tbilisi Math. J.* **13**(2), 129–148 (2020).
[15] W. M. Abd-Elhameed and Y. H. Youssri, Sixth-kind Chebyshev spectral approach for solving fractional differential equations, *Int. J. Nonlinear Sci. Numer. Simul.* **20**(2), 191–203 (2019).
[16] S. Shahmorad, M. H. Ostadzad and D. Baleanu, A Tau–like numerical method for solving fractional delay integro–differential equations, *Appl. Numer. Math.* **151**, 322–336 (2020).
[17] A. Faghih and P. Mokhtary, An efficient formulation of Chebyshev tau method for constant coefficients systems of multi-order FDEs, *J. Sci. Comput.* **82**(1), 6 (2020).
[18] J. P. Boyd, *Chebyshev and Fourier spectral methods*, Courier Corporation, 2001.
[19] E. H. Doha, W. M. Abd-Elhameed and Y. H. Youssri, Second kind Chebyshev operational matrix algorithm for solving differential equations of Lane–Emden type, *New Astron.* **23**, 113–117 (2013).
[20] K. Julien and M. Watson, Efficient multi-dimensional solution of PDEs using Chebyshev spectral methods, *J. Comput. Phys.* **228**(5), 1480–1503 (2009).
[21] H. Habenom and D. L. Suthar, Numerical solution for the time-fractional Fokker–Planck equation via shifted Chebyshev polynomials of the fourth kind, *Adv. Differ. Equ.*, **2020**, 315 (2020).
[22] E. H. Doha, W. M. Abd-Elhameed and M. M. Alsuyuti, On using third and fourth kinds Chebyshev polynomials for solving the integrated forms of high odd-order linear boundary value problems, *J. Egypt. Math. Soc.* **23**(2), 397–405 (2015).
[23] W. M. Abd-Elhameed, E. H. Doha and Y. H. Youssri, New wavelets collocation method for solving second-order multipoint boundary value problems using Chebyshev polynomials of third and fourth kinds, *Abstr. Appl. Anal.* **2013**, (2013).
[24] E. H. Doha, W. M. Abd-Elhameed and M. A. Bassuony, New algorithms for solving high even-order differential equations using third and fourth Chebyshev–Galerkin methods, *J.Comput. Phys.* **236**, 563–579 (2013).
[25] J. Ahmed, Numerical solutions of third-order boundary value problems associated with draining and coating flows, *Kyungpook Math. J.* **57**(4), (2017).
[26] A. R. Davies, A. Karageorghis and T. N. Phillips, Spectral Galerkin methods for the primary two-point boundary value problem in modelling viscoelastic flows, *Int. J. Numer. Methods Eng.* **26**(3), 647–662 (1988).
[27] W. M. Abd-Elhameed, E. H. Doha and Y. H. Youssri, Efficient spectral-Petrov-Galerkin methods for third-and fifth-order differential equations using general parameters generalized Jacobi polynomials, *Quaest. Math.* **36**(1), 15–38 (2013).
[28] J. C. Mason and D. C. Handscomb, *Chebyshev Polynomials*, CRC press, 2002.
[29] G. Akram, Quartic spline solution of a third order singularly perturbed boundary value problem, *ANZIAM J.* **53**, 44–58 (2011).
[30] F. A. Abd El-Salam, A. A. El-Sabbagh and Z. A. Zaki, The numerical solution of linear third order boundary value problems using nonpolynomial spline technique, *J. Amer. Sci.* **6**(12), 303–309 (2010).
[31] A. Khan and T. Sultana, Parametric quintic spline solution of third-order boundary value problems, *Inter. J. Comput. Math.* **89**(12), 1663–1677 (2012).

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[32] M. A. Noor and S. T. Mohyud-Din, A new approach to fifth-order boundary value problems, *Int. J. Nonlin. Sci.* **7**(2), 143–148 (2009).
[33] J. Rashidinia, M. Ghasemi and R. Jalilian, An $O(h^6)$ numerical solution of general nonlinear fifth-order two point boundary value problems, *Numer. Algorith.* **55**(4), 403–428 (2010).
[34] H. N. Caglar, S. H. Caglar and E. H. Twizell, The numerical solution of fifth-order boundary value problems with sixth-degree B-spline functions, *Appl. Math. Lett.* **12**(5), 25–30 (1999).
[35] M. A. Noor and S. T. Mohyud-Din, Variational iteration method for fifth-order boundary value problems using He's polynomials, *Math. Probl. Eng.* **2008**, (2008).
[36] C. C. Wang, Z. Y. Lee and Y. Kuo, Application of residual correction method in calculating upper and lower approximate solutions of fifth-order boundary-value problems, *Appl. Math. Comput.* **199**(2), 677–690 (2008).
[37] J. Zhang, The numerical solution of fifth-order boundary value problems by the variational iteration method, *Comput. Math. Appl.* **58**, 2347–2350 (2009).
[38] A. M. Wazwaz, Approximate solutions to boundary value problems of higher order by the modified decomposition method, *Comput. Math, Appl.* **40**(6-7), 679–691 (2000).

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