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Bispectral Density Estimation of Continuous Time Series with Missed Observations

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Abstract: In this paper, we study the estimation of the bispectral density function of a strictly stationary r-vector valued continuous time series. The case of interest is when some of observations are mising due to some random failure. Bispectral density function is devoleped in case of $L-$joint segments of observations. The modified biperiodogram is defined and smoothed to estimate the bispectral density matrix. The theoriotical properties of the proposed estimator are explored.

Keywords: Missed observations, joint segments of observations, modified biperiodograms, bispectral density matrix.

1 Introduction

Second order spectra plays an important role in signal processing and the need for power spectral analysis arises in a variety of contexts, such as the design of optimal filters, detection of narrow band signals, the estimation of finite parameter linear models, ... etc. Though not explicitly stated, the assumption is that the signal is Gaussian. If the signal is non-Gaussian, there is a need for higher order spectral analysis. The simplest higher order spectrum one can compute is the bispectrum, which is the Fourier transform of the third order lagged moments.

Mostly all estimation methods developed for the analysis of second and higher order spectra in case of all the observations are available as in [1], [2],[3], [4], [5], [6], and [7]. There are also several literature on the estimation of second order spectrum only when some of observations are missing. [8], [9], [10], [11] and [12] studied untapered time series with some randomly missed observations. [13] estimated the spectral estimator of the multidimensional strictly stationary continuous time process. [14] studied the periodogram analysis with missing observations. [15] studied the spectral analysis of timr series in joint segmants of observations. [16] introduced the asymptotic properties of spectral estimates of second-order with missed observations. [17] addressed the spectral analysis of a strictly stationary r-vector valued continuous time series with randomly missing observations in disjoint segments of observations . [18] discussed the spectral analysis of a strictly stationary r-vector valued continuous time series with randomly missing observations in joint segments of observations. [19] studied the periodogram analysis with missed observations between two vector valued stochastic process. [20] proposed a quasi-Whittle spectral estimator for parametric families of time series models in the presence of missing data. It is noticeable that until the late 19th century, no methods are available for the estimation of the bispectrum where observations are missing and the data is irregularly spaced. [21] first introduced the estimation of bispectral density function in the case of randomly missing observations from discete time series. [22] studied the trispectrum deconvolution of linear discrete processes with randomly missing observations. In this paper, we study the estimation of bispectral density function for a zero-mean strictly stationary r-vector valued continuous time series when some observations are missing.

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2 Observed series

Let $X(t)(t\in R)$ be a zero mean r-vector valued third order stationary continuous time series with

$$E(X(t)X(t+t_1)X(t+t_2)) = C_{XXX}(t_1,t_2),(t,t_1,t_2\in R), \tag{1}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|C_{XXX}(t_1,t_2)|dt_1dt_2<\infty, \tag{2}$$

where $C_{XXX}(t_1,t_2)$ is the third order covariance function of $X(t)$, then the bispectral density function $f_{XXX}(\lambda_1,\lambda_2)$ is given as

$$f_{XXX}(\lambda_1,\lambda_2)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}C_{XXX}(t_1,t_2)e^{-it_1\lambda_1-it_2\lambda_2}dt_1dt_2,(\lambda_1,\lambda_2\in R), \tag{3}$$

it exists for all λ_1,λ_2 if Eq.(2) is achieved. Using the assumed stationary, we then set down

Assumption I. $X(t)(t\in R)$ is a strictly stationary continuous series all of whose moments exist. For each $j=1,2,...,k-1$ and any K-tuble $a_1,a_2,...,a_k$, we have (see [18])

$$\int_{R^{k-1}}\left|u_jC_{a_1,a_2,...,a_k}(t_1,t_2,...,t_{k-1})\right|<\infty,\quad k=2,3,... \tag{4}$$

where

$$C_{a_1,a_2,...,a_k}(t_1,t_2,...,t_{k-1})=cum\Big\{X_{a_1}(t+t_1),X_{a_2}(t+t_2),...,X_{a_k}(t)\Big\}, \tag{5}$$

$$(a_1,a_2,...,a_k=1,2,...,r;t_1,t_2,...,t_{k-1},t\in R;k=2,3,...)$$

Because cumulants are measures of the joint dependence of random variables, Eq.(4) is a form of mixing or asymptotic independence requirement for values of $X(t)$ well separated in time. If $X(t)$ satisfies Assumption I, we may define its cumulant bispectral densities by

$$f_{a_1,...,a_k}(\lambda_1,\lambda_2,...\lambda_{k-1})=(2\pi)^{-k+1}\int_{R^{k-1}}C_{a_1,...,a_k}(t_1,...,t_{k-1})\exp\Big(-i\sum_{j=1}^{k-1}\lambda_jt_j\Big)dt_1...dt_{k-1}, \tag{6}$$

($-\infty<\lambda_j<\infty,a_1,a_2,...,a_k=1,2,...,r;k=2,...$). If $k=3$ the bi-spectra $f_{a_1a_2a_3}(\lambda_1,\lambda_2)$ are collected together in the matrix $f_{X_1X_2X_3}(\lambda_1,\lambda_2)$ of (3).

Assumption II. Let $h_a^T(t)=h_a(\frac{t}{T}),t\in[0,T)$ is bounded and vanishes for all t outside the interval $[0,T)$, that is called data window.

3 Modified series

To describe the missed observations in $X(t)$, we introduce the process $D(t)=\{D_a(t),t\in R\}_{a=1,2,...,r}$ (Bernoolli sequance of random vaiables) where

$$D_a(t)=\begin{cases}1 \text{ if } X_a(t) \text{ is observed at time } t;\\ 0 \text{ if } X_a(t) \text{ is missed at time } t.\end{cases}$$

Let $D(t)$ be independant of $X(t)$ such that for every t

$$P\{D_a(t)=1\}=p,P\{D_a(t)=0\}=q,$$

note that

$$E\{D_a(t)\}=p.$$

The modified time series can be represented by

$$Y(t) = D(t)X(t),$$

with components,

$$Y_a(t) = D_a(t)X_a(t).$$

Let $X(t)\big(t \in (0,T)\big)$ be an observed stretch of data with some randomly missing observations, then the third order periodogram $I_{YYY}(\lambda_1,\lambda_2)$ "the so-called biperiodogram" is given by (see [23])

$$I_{YYY}(\lambda_1,\lambda_2) = \frac{1}{(2\pi)^2 T}\int_0^T Y(t)e^{-i\lambda_1 t}dt \int_0^T Y(t)e^{-i\lambda_2 t}dt \int_0^T Y(t)e^{i(\lambda_1+\lambda_2)t}dt, \quad -\infty < \lambda_1,\lambda_2 < \infty. \tag{7}$$

Now, let $T = L(N-M)+M$, where L is the number of joint segments, N is the length of each segment and M is the length of joint parts, $0 \le M < N-M$. The given stretch of data has the following tapered biperiodogram (see [6])

$$I_{YYY}^{l(N-M)}(\lambda_1,\lambda_2) = \left[(2\pi)^2 \int_{l(N-M)}^{(l+1)(N-M)+M} \left[h^{(N)}\Big(t-l(N-M)\Big)\right]^2 dt\right]^{-1}$$
$$\times \int_{l(N-M)}^{(l+1)(N-M)+M} h^{l(N-M)}\Big(t-l(N-M)\Big)e^{-i\lambda_1 t}Y(t)dt \int_{l(N-M)}^{(l+1)(N-M)+M} h^{l(N-M)}\Big(t-l(N-M)\Big)e^{-i\lambda_2 t}Y(t)dt$$
$$\times \int_{l(N-M)}^{(l+1)(N-M)+M} h^{l(N-M)}\Big(t-l(N-M)\Big)e^{i(\lambda_1+\lambda_2)t}Y(t)dt \tag{8}$$

where -$\infty < \lambda_1,\lambda_2 < \infty, l = 0,1,...,L-1$ and $h(t)$ is the data window satisfies Assumption II. We can put $I_{YYY}^{l(N-M)}(\lambda_1,\lambda_2)$ as

$$I_{YYY}^{l(N-M)}(\lambda_1,\lambda_2) = (2\pi)^{-1/2}\left[\int_{l(N-M)}^{(l+1)(N-M)+M} \left[h^{(N)}\Big(t-l(N-M)\Big)\right]^2 dt\right]^{1/2} d_Y^{l(N-M)}(\lambda_1)d_Y^{l(N-M)}(\lambda_2)d_Y^{l(N-M)}(\lambda_1+\lambda_2) \tag{9}$$

where

$$d_Y^{l(N-M)}(\lambda) = \left[2\pi \int_{l(N-M)}^{(l+1)(N-M)+M} \left[h^{(N)}\Big(t-l(N-M)\Big)\right]^2 dt\right]^{-1/2} \int_{l(N-M)}^{(l+1)(N-M)+M} h^{l(N-M)}\Big(t-l(N-M)\Big)e^{-i\lambda t}Y(t)dt \tag{10}$$

is the expanded finite Fourier transform of a given stretch of data. Now, for $a = 1,2,...,r$, then $d_a^{l(N-M)}(\lambda)$ has the following properties (see [18])

$$E\Big\{d_a^{l(N-M)}(\lambda)\Big\} = 0 \tag{11}$$

$$Cov\Big\{d_a^{l(N-M)}(\lambda_1), d_b^{l(N-M)}(-\lambda_2)\Big\} = p^2 e^{-i(\lambda_1-\lambda_2)l(N-M)}\int_R f_{ab}(v)\phi_{ab}(\lambda_1-v,\lambda_2-v)dv, \tag{12}$$

where

$$\phi_{ab}^{(N)}(\lambda_1,\lambda_2) = (2\pi)^{-1}\left[\int_0^N\int_0^N \Big(h_a^{(N)}(t_1)\Big)^2\Big(h_b^{(N)}(t_2)\Big)^2 dt_1dt_2\right]^{-\frac{1}{2}} H_a^{(N-M)}(\lambda_1)\overline{H_b^{(N-M)}(\lambda_2)}, \tag{13}$$

and
$$H_a^{(N)}(\lambda) = \int_0^N h_a^{(N)}(t)e^{-i\lambda t}dt.$$

$$Cum\Big\{d_{a_1}^{l_1(N-M)}(\lambda_1),...,d_{a_k}^{l_k(N-M)}(\lambda_k)\Big\} =$$
$$(2\pi)^{\frac{k}{2}-1}p^k\left(\prod_{j=1}^k\int_o^N \Big(h_{a_j}^{(N)}(t_j)\Big)^2 dt_j\right)^{-\frac{1}{2}} f_{a_1a_2...a_k}(\lambda_1,\lambda_2,...\lambda_{k-1})G_{a_1...a_k}^{(N)}\left(\sum_{j=1}^k\lambda_j\right) + O\left(N^{-\frac{k}{2}}\right) \tag{14}$$

where $O\left(N^{-\frac{k}{2}}\right)$ is uniform in $\lambda_1, \lambda_2, ..., \lambda_{k-1}$ as $N \to \infty$, $k \geq 2$ and

$$G^{(N)}_{a_1...a_k} = \int_0^N \left(\prod_{j=1}^{k} h^{(N)}_{a_j}(t_j)\right) e^{-i\lambda t} dt, \lambda \neq 0, \lambda, t \in R. \tag{15}$$

Now, let $h^{(N)}\left(t - l(N-M)\right)$ be a set of orthonormal taperes, i.e.

$$\int_{l(N-M)}^{(l+1)(N-M)+M} \left[h^{(N)}\left(t - l(N-M)\right)\right]^2 dt = 1 \tag{16}$$

(see [24]). From this orthonormality and by Eqs.(9) and (10), one can write

$$I^{l(N-M)}_{abc}(\lambda_1, \lambda_2) = (2\pi)^{-\frac{1}{2}} d^{l(N-M)}_{a}(\lambda_1) d^{l(N-M)}_{b}(\lambda_2) d^{l(N-M)}_{c}(\lambda_1 + \lambda_2) \tag{17}$$

4 Statistical properties of the proposed esimator

Using the expanded finite Fourier transform (10), we construct the modified biperiodogram as

$$I^{l(N-M)}_{abc}(\lambda_1, \lambda_2) = \left[(2\pi)^2 p^3 \int_{l(N-M)}^{(l+1)(N-M)+M} h^{(N)}_a(t) h^{(N)}_b(t) h^{(N)}_c(t) dt\right]^{-1} \alpha^{l(N-M)}_a(\lambda_1) \alpha^{l(N-M)}_b(\lambda_2) \alpha^{l(N-M)}_c(\lambda_1 + \lambda_2), \tag{18}$$

such that

$$\alpha^{l(N-M)}_a(\lambda) = \sqrt{2\pi \int_{l(N-M)}^{(l+1)(N-M)+M} \left[h^{(N)}_a(t)\right]^2 dt d^{l(N-M)}_a(\lambda)}.$$

The smoothed bispectral density estimate is constructed as:

$$f^{(T)}_{abc}(\lambda_1, \lambda_2) = \frac{1}{L} \int_0^L I^{l(N-M)}_{abc}(\lambda_1, \lambda_2) du, a, b, c = 1, 2, ..., r. \tag{19}$$

Theorem 4.1. Let $X(t)(t \in R)$ be a strictly stationary r-vector valued continuous time series with mean zero, and satisfy Assumption I. Let $I^{(T)}_{YYY}(\mu, \nu) = \left\{I^{(T)}_{abc}(\mu, \nu)\right\}_{a,b,c=1,2,...,r}$ be given by (18), and $\phi_a(t)$ satisfy Assumption II for $a = 1, 2, ..., r$, then

$$E\left\{I^{l(N-M)}_{abc}(\mu, \nu)\right\} = f_{abc}(\mu, \nu) + O(N^{-\frac{3}{2}}), p \to 1 \tag{20}$$

$$
\begin{aligned}
&Cov\left\{I_{a_1b_1c_1}^{l(N-M)}(\mu_1,\nu_1),I_{a_2b_2c_2}^{l(N-M)}(\mu_2,\nu_2)\right\}\\
&\qquad=\left(2\pi G_{a_1b_1c_1}G_{a_2b_2c_2}\phi_{a_1a_2}^{(N)}(\mu_1-\mu_2)\phi_{b_1b_2}^{(N)}(\nu_1-\nu_2)\phi_{c_1c_2}^{(N)}(\gamma_1-\gamma_2)\right)^{-1}\\
&\times\Big[G_{a_1b_1}G_{c_1a_2}G_{b_2c_2}\phi_{a_1b_1}^{(N)}(\mu_1-\nu_1)\phi_{c_1a_2}^{(N)}(\gamma_1-\mu_2)\phi_{b_2c_2}^{(N)}(-\mu_2)f_{a_1b_1}(\mu_1)f_{c_1a_2}(\gamma_1)f_{b_2c_2}(\nu_2)\\
&+G_{a_1b_1}G_{c_1b_2}G_{a_2c_2}\phi_{a_1b_1}^{(N)}(\mu_1-\nu_1)\phi_{c_1b_2}^{(N)}(\gamma_1-\nu_2)\phi_{a_2c_2}^{(N)}(-\nu_2)f_{a_1b_1}(\mu_1)f_{c_1b_2}(\gamma_1)f_{a_2c_2}(\mu_2)\\
&+G_{a_1b_1}G_{c_1c_2}G_{a_2b_2}\phi_{a_1b_1}^{(N)}(\mu_1-\nu_1)\phi_{c_1c_2}^{(N)}(\gamma_1-\gamma_2)\phi_{a_2b_2}^{(N)}(\mu_2-\nu_2)f_{a_1b_1}(\mu_1)f_{c_1c_2}(\gamma_1)f_{a_2b_2}(\mu_2)\\
&+G_{a_1c_1}G_{b_1a_2}G_{b_2c_2}\phi_{a_1c_1}^{(N)}(-\nu_1)\phi_{b_1a_2}^{(N)}(\nu_1-\mu_2)\phi_{b_2c_2}^{(N)}(-\mu_2)f_{a_1c_1}(\mu_1)f_{b_1a_2}(\nu_1)f_{b_2c_2}(\nu_2)\\
&+G_{a_1c_1}G_{b_1b_2}G_{a_2c_2}\phi_{a_1c_1}^{(N)}(-\nu_1)\phi_{b_1b_2}^{(N)}(\nu_1-\nu_2)\phi_{a_2c_2}^{(N)}(-\nu_2)f_{a_1c_1}(\mu_1)f_{b_1b_2}(\nu_1)f_{a_2c_2}(\mu_2)\\
&+G_{a_1c_1}G_{b_1c_2}G_{a_2b_2}\phi_{a_1c_1}^{(N)}(-\nu_1)\phi_{b_1c_2}^{(N)}(\nu_1-\gamma_2)\phi_{a_2b_2}^{(N)}(\mu_2-\nu_2)f_{a_1c_1}(\mu_1)f_{b_1c_2}(\nu_1)f_{a_2b_2}(\mu_2)\\
&+G_{a_1a_2}G_{b_1c_1}G_{b_2c_2}\phi_{a_1a_2}^{(N)}(\mu_1-\mu_2)\phi_{b_1c_1}^{(N)}(-\mu_1)\phi_{b_2c_2}^{(N)}(-\mu_2)f_{a_1a_2}(\mu_1)f_{b_1c_1}(\nu_1)f_{b_2c_2}(\nu_2)\\
&+G_{a_1a_2}G_{b_1b_2}G_{c_1c_2}\phi_{a_1a_2}^{(N)}(\mu_1-\mu_2)\phi_{b_1b_2}^{(N)}(\nu_1-\nu_2)\phi_{c_2c_2}^{(N)}(-\gamma_1-\gamma_2)f_{a_1a_2}(\mu_1)f_{b_1b_2}(\nu_1)f_{c_1c_2}(\gamma_1)\\
&+G_{a_1a_2}G_{b_1c_2}G_{c_1b_2}\phi_{a_1a_2}^{(N)}(\mu_1-\mu_2)\phi_{b_1c_2}^{(N)}(\nu_1-\gamma_2)\phi_{c_1b_2}^{(N)}(\gamma_1-\nu_2)f_{a_1a_2}(\mu_1)f_{b_1c_2}(\nu_1)f_{c_1b_2}(\gamma_1)\\
&+G_{a_1b_2}G_{b_1c_1}G_{a_2c_2}\phi_{a_1b_2}^{(N)}(\mu_1-\nu_2)\phi_{b_1c_1}^{(N)}(-\nu_2)\phi_{a_2c_2}^{(N)}(\mu_1)f_{a_1b_2}(\mu_1)f_{b_1c_1}(\nu_1)f_{a_2c_2}(\mu_2)\\
&+G_{a_1b_2}G_{b_1a_2}G_{c_1c_2}\phi_{a_1b_2}^{(N)}(\mu_1-\nu_2)\phi_{b_1a_2}^{(N)}(\nu_1-\mu_2)\phi_{c_1c_2}^{(N)}(\gamma_1-\gamma_2)f_{a_1b_2}(\mu_1)f_{b_1a_1}(\nu_1)f_{c_2c_2}(\gamma_1)\\
&+G_{a_1b_2}G_{b_1c_2}G_{c_1a_2}\phi_{a_1b_2}^{(N)}(\mu_1-\nu_2)\phi_{b_1c_2}^{(N)}(\nu_1-\gamma_2)\phi_{c_1a_2}^{(N)}(-\gamma_1-\mu_2)f_{a_1b_2}(\mu_1)f_{b_1c_2}(\nu_1)f_{c_1a_2}(\gamma_1)\\
&+G_{a_1c_2}G_{b_1c_1}G_{a_2b_2}\phi_{a_1c_2}^{(N)}(\mu_1-\gamma_2)\phi_{b_1c_1}^{(N)}(-\mu_1)\phi_{a_2b_2}^{(N)}(\mu_2-\nu_2)f_{a_1c_2}(\mu_1)f_{b_1c_1}(\nu_1)f_{a_2b_2}(\mu_2)\\
&+G_{a_1c_2}G_{b_1a_2}G_{c_1b_2}\phi_{a_1c_2}^{(N)}(\mu_1-\gamma_2)\phi_{b_1a_2}^{(N)}(\nu_1-\mu_2)\phi_{c_1b_2}^{(N)}(\gamma_1-\nu_2)f_{a_1c_2}(\mu_1)f_{b_1a_2}(\nu_1)f_{c_1b_2}(\gamma_1)\\
&+G_{a_1c_2}G_{b_1b_2}G_{c_1a_2}\phi_{a_1c_2}^{(N)}(\mu_1-\gamma_2)\phi_{b_1b_2}^{(N)}(\nu_1-\nu_2)\phi_{c_1a_2}^{(N)}(\gamma_1-\mu_2)f_{a_1c_2}(\mu_1)f_{b_1b_2}(\nu_1)f_{c_1a_2}(\gamma_1)\Big]\\
&+O\left(N^{-1}\right)
\end{aligned}
\tag{21}
$$

where $\gamma_1=\mu_1+\nu_1$ and $\gamma_2=\mu_2+\nu_2$.

$$
\begin{aligned}
&Cum\left\{I_{a_1b_1c_1}^{l_1(N-M)}(\mu_1,\nu_1),I_{a_2b_2c_2}^{l_2(N-M)}(\mu_2,\nu_2),\ldots,I_{a_kb_kc_k}^{l_k(N-M)}(\mu_k,\nu_k)\right\}\\
&\qquad=\left[\prod_{i=1}^{k}G_{a_ib_ic_i}\prod_{i=1,3,\ldots,k-1}\Phi_{a_pa_q}^{(N)}(\mu_i+\mu_{i+1})\Phi_{b_pb_q}^{(N)}(\nu_i+\nu_{i+1})\Phi_{c_pc_q}^{(N)}(\gamma_i+\gamma_{i+1})\right]^{-1}\\
&\times\sum_{v}\left\{\prod_{j=1}^{k}G_{a_jb_jc_j}f_{d_je_js_j}(\xi_j,\delta_j)\right\}\left\{\prod_{j=1,3,\ldots,k-1}\Phi_{d_pd_q}^{(N)}(\xi_j+\xi_{j+1})\Phi_{e_pe_q}^{(N)}(\delta_j+\delta_{j+1})\Phi_{s_ps_q}^{(N)}(\beta_j+\beta_{j+1})\right\}\\
&\qquad+O(N^{-1})
\end{aligned}
\tag{22}
$$

where k is even number and the summation extends over all partitions $\left\{(d_1,\xi_1),(e_1,\delta_1),(s_1,\beta_1)\right\},\ldots,\left\{(d_k,\xi_k),(e_k,\delta_k),(s_k,\beta_k)\right\}$ into pairs of the quantities $\left\{(a_1,\mu_1),(b_1,\nu_1),(c_1,\gamma_1)\right\},\ldots,\left\{(a_k,\mu_k),(b_k,\nu_k),(c_k,\gamma_k)\right\}$ with note that $(\mu_i+\nu_i)=\gamma_i\;\forall\;i=1,2,\ldots,k$.

Proof. From (18), we get:

$$I_{abc}^{l(N-M)}(\mu,\nu) = \left[(2\pi)^2 p^3 \int\limits_{l(N-M)}^{(l+1)(N-M)+M} h_a^{(N)}(t)h_b^{(N)}(t)h_c^{(N)}(t)dt\right]^{-1} \times \left[\iiint\limits_{l(N-M)}^{(l+1)(N-M)+M} \left(h_a^{(N)}(t_1)\right)^2 \left(h_b^{(N)}(t_2)\right)^2 \left(h_c^{(N)}(t_3)\right)^2 dt_1dt_2dt_3\right]^{\frac{1}{2}} \times d_a^{l(N-M)}(\mu)d_b^{l(N-M)}(\nu)d_c^{l(N-M)}(\mu+\nu)$$

Take the expectation

$$E\left\{I_{abc}^{l(N-M)}(\mu,\nu)\right\} = (2\pi)^{-\frac{1}{2}} p^{-3} \left[\int\limits_{l(N-M)}^{(l+1)(N-M)+M} h_a^{(N)}(t)h_b^{(N)}(t)h_c^{(N)}(t)dt\right]^{-1} \times \left[\iiint\limits_{l(N-M)}^{(l+1)(N-M)+M} \left(h_a^{(N)}(t_1)\right)^2 \left(h_b^{(N)}(t_2)\right)^2 \left(h_c^{(N)}(t_3)\right)^2 dt_1dt_2dt_3\right]^{\frac{1}{2}} \times E\left\{d_a^{l(N-M)}(\mu)d_b^{l(N-M)}(\nu)d_c^{l(N-M)}(\mu+\nu)\right\}$$

from stationarity of time series and cumulants properties, then

$$E\left\{I_{abc}^{l(N-M)}(\mu,\nu)\right\} = (2\pi)^{-\frac{1}{2}} p^{-3} \left[\int\limits_{l(N-M)}^{(l+1)(N-M)+M} h_a^{(N)}(t)h_b^{(N)}(t)h_c^{(N)}(t)dt\right]^{-1} \times \left[\iiint\limits_{l(N-M)}^{(l+1)(N-M)+M} \left(h_a^{(N)}(t_1)\right)^2 \left(h_b^{(N)}(t_2)\right)^2 \left(h_c^{(N)}(t_3)\right)^2 dt_1dt_2dt_3\right]^{\frac{1}{2}} \times Cum\left\{d_a^{l(N-M)}(\mu),d_b^{l(N-M)}(\nu),d_c^{l(N-M)}(\mu+\nu)\right\}$$

then by (14), the proof of (20) is complete. Take $\gamma_1=\mu_1+\nu_1$ and $\gamma_2=\mu_2+\nu_2$, then

$$Cov\left\{I_{a_1b_1c_1}^{l(N-M)}(\mu_1,\nu_1),I_{a_2b_2c_2}^{l(N-M)}(\mu_2,\nu_2)\right\} = (2\pi)^{-1}\left(G_{a_1b_1c_1}G_{a_2b_2c_2}\right)^{-1}\left[\iint\limits_{l(N-M)}^{(l+1)(N-M)+M} \left(h_{a_1}^{(N)}(t_1)\right)^2 \left(h_{a_2}^{(N)}(t_1)\right)^2 dt_1^2\right]^{\frac{1}{2}} \times \left[\iint\limits_{l(N-M)}^{(l+1)(N-M)+M} \left(h_{b_1}^{(N)}(t_2)\right)^2 \left(h_{b_2}^{(N)}(t_2)\right)^2 dt_2^2\right]^{\frac{1}{2}} \times \left[\iint\limits_{l(N-M)}^{(l+1)(N-M)+M} \left(h_{c_1}^{(N)}(t_3)\right)^2 \left(h_{c_2}^{(N)}(t_3)\right)^2 dt_3^2\right]^{\frac{1}{2}} \times Cov\left\{d_{a_1}^{l(N-M)}(\mu_1)d_{b_1}^{l(N-M)}(\nu_1)d_{c_1}^{l(N-M)}(\gamma_1),d_{a_2}^{l(N-M)}(\mu_2)d_{b_2}^{l(N-M)}(\nu_2)d_{c_2}^{l(N-M)}(\gamma_2)\right\}$$

from (13), then

$$Cov\left\{I_{a_1b_1c_1}^{l(N-M)}(\mu_1,\nu_1),I_{a_2b_2c_2}^{l(N-M)}(\mu_2,\nu_2)\right\} = \left[2\pi G_{a_1b_1c_1}G_{a_2b_2c_2}\phi_{a_1a_2}^{(N)}(\mu_1-\mu_2)\phi_{b_1b_2}^{(N)}(\nu_1-\nu_2)\phi_{c_1c_2}^{(N)}(\gamma_1-\gamma_2)\right]^{-1} \times Cov\left\{d_{a_1}^{l(N-M)}(\mu_1)d_{b_1}^{l(N-M)}(\nu_1)d_{c_1}^{l(N-M)}(\gamma_1),d_{a_2}^{l(N-M)}(\mu_2)d_{b_2}^{l(N-M)}(\nu_2)d_{c_2}^{l(N-M)}(\gamma_2)\right\}$$

since

$$Cov\Big\{d_{a_1}^{l(N-M)}(\mu_1)d_{b_1}^{l(N-M)}(\nu_1)d_{c_1}^{l(N-M)}(\gamma_1),d_{a_2}^{l(N-M)}(\mu_2)d_{b_2}^{l(N-M)}(\nu_2)d_{c_2}^{l(N-M)}(\gamma_2)\Big\}$$
$$=E\Big\{d_{a_1}^{l(N-M)}(\mu_1)d_{b_1}^{l(N-M)}(\nu_1)d_{c_1}^{l(N-M)}(\gamma_1)d_{a_2}^{l(N-M)}(\mu_2)d_{b_2}^{l(N-M)}(\nu_2)d_{c_2}^{l(N-M)}(\gamma_2)\Big\}$$
$$-E\Big\{d_{a_1}^{l(N-M)}(\mu_1)d_{b_1}^{l(N-M)}(\nu_1)d_{c_1}^{l(N-M)}(\gamma_1)\Big\}E\Big\{d_{a_2}^{l(N-M)}(\mu_2)d_{b_2}^{l(N-M)}(\nu_2)d_{c_2}^{l(N-M)}(\gamma_2)\Big\}$$

and from Isserlis theorem (see [25]), then

$$\begin{aligned}
Cov\Big\{I_{a_1b_1c_1}^{l(N-M)}(\mu_1,\nu_1),I_{a_2b_2c_2}^{l(N-M)}(\mu_2,\nu_2)\Big\} &= \Big[2\pi G_{a_1b_1c_1}G_{a_2b_2c_2}\phi_{a_1a_2}^{(N)}(\mu_1-\mu_2)\phi_{b_1b_2}^{(N)}(\nu_1-\nu_2)\phi_{c_1c_2}^{(N)}(\gamma_1-\gamma_2)\Big]^{-1}\\
&\times\Big[Cov\Big\{d_{a_1}^{l(N-M)}(\mu_1),d_{b_1}^{l(N-M)}(\nu_1)\Big\}Cov\Big\{d_{c_1}^{l(N-M)}(\gamma_1),d_{a_2}^{l(N-M)}(\mu_2)\Big\}\times Cov\Big\{d_{b_2}^{l(N-M)}(\nu_2),d_{c_2}^{l(N-M)}(\gamma_2)\Big\}\\
&+Cov\Big\{d_{a_1}^{l(N-M)}(\mu_1),d_{b_1}^{l(N-M)}(\nu_1)\Big\}\times Cov\Big\{d_{c_1}^{l(N-M)}(\gamma_1),d_{b_2}^{l(N-M)}(\nu_2)\Big\}Cov\Big\{d_{a_2}^{l(N-M)}(\mu_2),d_{c_2}^{l(N-M)}(\gamma_2)\Big\}\\
&+Cov\Big\{d_{a_1}^{l(N-M)}(\mu_1),d_{b_1}^{l(N-M)}(\nu_1)\Big\}Cov\Big\{d_{c_1}^{l(N-M)}(\gamma_1),d_{c_2}^{l(N-M)}(\gamma_2)\Big\}\times Cov\Big\{d_{a_2}^{l(N-M)}(\mu_2),d_{b_2}^{l(N-M)}(\nu_2)\Big\}\\
&+Cov\Big\{d_{a_1}^{l(N-M)}(\mu_1),d_{c_1}^{l(N-M)}(\gamma_1)\Big\}\times Cov\Big\{d_{b_1}^{l(N-M)}(\nu_1),d_{a_2}^{l(N-M)}(\mu_2)\Big\}Cov\Big\{d_{b_2}^{l(N-M)}(\nu_2),d_{c_2}^{l(N-M)}(\gamma_2)\Big\}\\
&+Cov\Big\{d_{a_1}^{l(N-M)}(\mu_1),d_{c_1}^{l(N-M)}(\gamma_1)\Big\}Cov\Big\{d_{b_1}^{l(N-M)}(\nu_1),d_{b_2}^{l(N-M)}(\nu_2)\Big\}\times Cov\Big\{d_{a_2}^{l(N-M)}(\mu_2),d_{c_2}^{l(N-M)}(\gamma_2)\Big\}\\
&+Cov\Big\{d_{a_1}^{l(N-M)}(\mu_1),d_{c_1}^{l(N-M)}(\gamma_1)\Big\}\times Cov\Big\{d_{b_1}^{l(N-M)}(\nu_1),d_{c_2}^{l(N-M)}(\gamma_2)\Big\}Cov\Big\{d_{a_2}^{l(N-M)}(\mu_2),d_{b_2}^{l(N-M)}(\nu_2)\Big\}\\
&+Cov\Big\{d_{a_1}^{l(N-M)}(\mu_1),d_{a_2}^{l(N-M)}(\mu_2)\Big\}Cov\Big\{d_{b_1}^{l(N-M)}(\nu_1),d_{c_1}^{l(N-M)}(\gamma_1)\Big\}\times Cov\Big\{d_{b_2}^{l(N-M)}(\nu_2),d_{c_2}^{l(N-M)}(\gamma_2)\Big\}\\
&+Cov\Big\{d_{a_1}^{l(N-M)}(\mu_1),d_{a_2}^{l(N-M)}(\mu_2)\Big\}\times Cov\Big\{d_{b_1}^{l(N-M)}(\nu_1),d_{b_2}^{l(N-M)}(\nu_2)\Big\}Cov\Big\{d_{c_1}^{l(N-M)}(\gamma_1),d_{c_2}^{l(N-M)}(\gamma_2)\Big\}\\
&+Cov\Big\{d_{a_1}^{l(N-M)}(\mu_1),d_{a_2}^{l(N-M)}(\mu_2)\Big\}Cov\Big\{d_{b_1}^{l(N-M)}(\nu_1),d_{c_2}^{l(N-M)}(\gamma_2)\Big\}\times Cov\Big\{d_{c_1}^{l(N-M)}(\gamma_1),d_{b_2}^{l(N-M)}(\nu_2)\Big\}\\
&+Cov\Big\{d_{a_1}^{l(N-M)}(\mu_1),d_{b_2}^{l(N-M)}(\nu_2)\Big\}\times Cov\Big\{d_{b_1}^{l(N-M)}(\nu_1),d_{c_1}^{l(N-M)}(\gamma_1)\Big\}Cov\Big\{d_{a_2}^{l(N-M)}(\mu_2),d_{c_2}^{l(N-M)}(\gamma_2)\Big\}\\
&+Cov\Big\{d_{a_1}^{l(N-M)}(\mu_1),d_{b_2}^{l(N-M)}(\nu_2)\Big\}Cov\Big\{d_{b_1}^{l(N-M)}(\nu_1),d_{a_2}^{l(N-M)}(\mu_2)\Big\}\times Cov\Big\{d_{c_1}^{l(N-M)}(\gamma_1),d_{c_2}^{l(N-M)}(\gamma_2)\Big\}\\
&+Cov\Big\{d_{a_1}^{l(N-M)}(\mu_1),d_{b_2}^{l(N-M)}(\nu_2)\Big\}\times Cov\Big\{d_{b_1}^{l(N-M)}(\nu_1),d_{c_2}^{l(N-M)}(\gamma_2)\Big\}Cov\Big\{d_{c_1}^{l(N-M)}(\gamma_1),d_{a_2}^{l(N-M)}(\mu_2)\Big\}\\
&+Cov\Big\{d_{a_1}^{l(N-M)}(\mu_1),d_{c_2}^{l(N-M)}(\gamma_2)\Big\}Cov\Big\{d_{b_1}^{l(N-M)}(\nu_1),d_{c_1}^{l(N-M)}(\gamma_1)\Big\}\times Cov\Big\{d_{a_2}^{l(N-M)}(\mu_2),d_{b_2}^{l(N-M)}(\nu_2)\Big\}\\
&+Cov\Big\{d_{a_1}^{l(N-M)}(\mu_1),d_{c_2}^{l(N-M)}(\gamma_2)\Big\}\times Cov\Big\{d_{b_1}^{l(N-M)}(\nu_1),d_{a_2}^{l(N-M)}(\mu_2)\Big\}Cov\Big\{d_{c_1}^{l(N-M)}(\gamma_1),d_{b_2}^{l(N-M)}(\nu_2)\Big\}\\
&+Cov\Big\{d_{a_1}^{l(N-M)}(\mu_1),d_{c_2}^{l(N-M)}(\gamma_2)\Big\}Cov\Big\{d_{b_1}^{l(N-M)}(\nu_1),d_{b_2}^{l(N-M)}(\nu_2)\Big\}\times Cov\Big\{d_{c_1}^{l(N-M)}(\gamma_1),d_{a_2}^{l(N-M)}(\mu_2)\Big\}\Big]
\end{aligned} \tag{23}$$

then by (12), the proof of (21) is completed. From (18), we have

$$\begin{aligned}
&Cum\Big\{I_{a_1b_1c_1}^{l_1(N-M)}(\mu_1,\nu_1),\ldots,I_{a_kb_kc_k}^{l_k(N-M)}(\mu_k,\nu_k)\Big\}\\
&\quad=(2\pi)^{-\frac{k}{2}}p^{-3k}\Big[\prod_{i=1}^{k}G_{a_ib_ic_i}\Big]^{-1}\Big[\prod_{i=1}^{k}\int\limits_{l_i(N-M)}^{(l_i+1)(N-M)+M}\Big[h_{a_i}(t_i)\Big]^2dt_{(i)}\\
&\quad\times\prod_{i=1}^{k}\int\limits_{l_i(N-M)}^{(l_i+1)(N-M)+M}\Big[h_{b_i}(t_i)\Big]^2dt_{(i)}\prod_{i=1}^{k}\int\limits_{l_i(N-M)}^{(l_i+1)(N-M)+M}\Big[h_{c_i}(t_i)\Big]^2dt_{(i)}\Big]^{\frac{1}{2}}\\
&\quad\times Cum\Big\{d_{a_1}^{l_1(N-M)}(\mu_1)d_{b_1}^{l_1(N-M)}(\nu_1)d_{c_1}^{l_1(N-M)}(\mu_1+\nu_1),\ldots,d_{a_k}^{l_k(N-M)}(\mu_k)d_{b_k}^{l_k(N-M)}(\nu_k)d_{c_k}^{l_k(N-M)}(\mu_k+\nu_k)\Big\}
\end{aligned}$$

By theorem(2.3.2) in [4] p.21, we get

$$Cum\Big\{d_{a_1}^{l_1(N-M)}(\mu_1)d_{b_1}^{l_1(N-M)}(\nu_1)d_{c_1}^{l_1(N-M)}(\mu_1+\nu_1),...,d_{a_k}^{l_k(N-M)}(\mu_k)d_{b_k}^{l_k(N-M)}(\nu_k)d_{c_k}^{l_k(N-M)}(\mu_k+\nu_k)\Big\}$$
$$=\sum_V Cum\Big\{d^{l_i(N-M)}(\mu_i), i\in V_1\Big\}...Cum\Big\{d^{l_i(N-M)}(\mu_i), i\in V_s\Big\},$$

where the summation extens over all indecomposable partitions $V=\Big[\bigcup_{j=1}^{s}V_j\Big]\in I$, $I=(a_1,...,a_k;b_1,...,b_k;c_1,...,c_k)$, $1\le s\le k$ of the transformed table

$$\begin{array}{ccc}(a_1,\mu_1),(b_1,\nu_1),(c_1,\mu_1+\nu_1) & & \Big\{(d_1,\xi_1),(e_1,\delta_1),(s_1,\beta_1)\Big\}\\ (a_2,\mu_2),(b_2,\nu_2),(c_2,\mu_2+\nu_2) & \rightarrow & \Big\{(d_2,\xi_2),(e_2,\delta_2),(s_2,\beta_2)\Big\}\\ \vdots & & \vdots\\ (a_k,\mu_k),(b_k,\nu_k),(c_k,\mu_k+\nu_k) & & \Big\{(d_k,\xi_k),(e_k,\delta_k),(s_k,\beta_k)\Big\}\end{array},$$

$$Cum\Big\{d_{a_1}^{l_1(N-M)}(\mu_1)d_{b_1}^{l_1(N-M)}(\nu_1)d_{c_1}^{l_1(N-M)}(\mu_1+\nu_1),...,d_{a_k}^{l_k(N-M)}(\mu_k)d_{b_k}^{l_k(N-M)}(\nu_k)d_{c_k}^{l_k(N-M)}(\mu_k+\nu_k)\Big\}$$
$$=\sum_V (2\pi)^{\frac{k}{2}}p^{3k}\left[\prod_{j=1}^{k}\left(\int_0^N\Big[h_{d_j}(t_j)\Big]^2dt_j\int_0^N\Big[h_{e_j}(t_j)\Big]^2dt_j\int_0^N\Big[h_{s_j}(t_j)\Big]^2dt_j\right)\right]^{-\frac{1}{2}}$$
$$\times\Big\{\prod_{j=1}^{k}G_{a_jb_jc_j}f_{d_je_js_j}(\xi_j,\delta_j)\Big\}+O(N^{-1}).$$

Then,

$$Cum\Big\{I_{a_1b_1c_1}^{l_1(N-M)}(\mu_1,\nu_1),...,I_{a_kb_kc_k}^{l_k(N-M)}(\mu_k,\nu_k)\Big\}=$$
$$\left[\prod_{i=1}^{k}G_{a_ib_ic_i}\right]^{-1}\left[\prod_{i=1}^{k}\int_{l_i(N-M)}^{(l_i+1)(N-M)+M}\Big[h_{a_i}(t_i)\Big]^2dt_i\prod_{i=1}^{k}\int_{l_i(N-M)}^{(l_i+1)(N-M)+M}\Big[h_{b_i}(t_i)\Big]^2dt_i\right.$$
$$\left.\times\prod_{i=1}^{k}\int_{l_i(N-M)}^{(l_i+1)(N-M)+M}\Big[h_{c_i}(t_i)\Big]^2dt_i\right]^{\frac{1}{2}}\sum_V\Big\{\prod_{j=1}^{k}G_{a_jb_jc_j}f_{d_je_js_j}(\mu_j,\delta_j)$$
$$\times\left[\prod_{j=1}^{k}\left(\int_0^N\Big[h_{d_j}(t_j)\Big]^2dt_j\int_0^N\Big[h_{e_j}(t_j)\Big]^2dt_j\int_0^N\Big[h_{s_j}(t_j)\Big]^2dt_j\right)\right]^{-\frac{1}{2}}\Big\}+O(N^{-1}). \quad (24)$$

if k is even number and by (13), we get the proof of (22).

Theorem 4.3. Let $X(t)(t\in R)$ be a strictly stationary r-vector valued continuous time series with mean zero, and satisfy Assumption I. Let $f_{abc}^{(T)}(\mu,\nu)$ be given by (19), $a,b,c=1,2,...,r$, then:

$$E\Big\{f_{abc}^{T}(\mu,\nu)\Big\}=f_{abc}(\mu,\nu)+O(N^{-\frac{3}{2}}), \quad (25)$$

$$
\begin{aligned}
& Cov\left\{f^{(T)}_{a_1b_1c_1}(\mu_1,\nu_1),f^{(T)}_{a_2b_2c_2}(\mu_2,\nu_2)\right\} \\
&= \left(2\pi L^2\phi^{(N)}_{a_1a_2}(\mu_1-\mu_2)\phi^{(N)}_{b_1b_2}(\nu_1-\nu_2)\phi^{(N)}_{c_1c_2}(\gamma_1-\gamma_2)\right)^{-1}\int_0^L\int_0^L\left(G_{a_1b_1c_1}(l_1,l_1,l_1)G_{a_2b_2c_2}(l_2,l_2,l_2)\right)^{-1} \\
&\times\Big[G_{a_1b_1}(l_1,l_1)G_{c_1a_2}(l_1,l_2)G_{b_2c_2}(l_2,l_2)\phi^{(N)}_{a_1b_1}(\mu_1-\nu_1)\phi^{(N)}_{c_1a_2}(\gamma_1-\mu_2)\phi^{(N)}_{b_2c_2}(-\mu_2)f_{a_1b_1}(\mu_1)f_{c_1a_2}(\gamma_1)f_{b_2c_2}(\nu_2) \\
&+G_{a_1b_1}(l_1,l_1)G_{c_1b_2}(l_1,l_2)G_{a_2c_2}(l_2,l_2)\phi^{(N)}_{a_1b_1}(\mu_1-\nu_1)\phi^{(N)}_{c_1b_2}(\gamma_1-\nu_2)\phi^{(N)}_{a_2c_2}(-\nu_2)f_{a_1b_1}(\mu_1)f_{c_1b_2}(\gamma_1)f_{a_2c_2}(\mu_2) \\
&+G_{a_1b_1}(l_1,l_1)G_{c_1c_2}(l_1,l_2)G_{a_2b_2}(l_2,l_2)\phi^{(N)}_{a_1b_1}(\mu_1-\nu_1)\phi^{(N)}_{c_1c_2}(\gamma_1-\gamma_2)\phi^{(N)}_{a_2b_2}(\mu_2-\nu_2)f_{a_1b_1}(\mu_1)f_{c_1c_2}(\gamma_1)f_{a_2b_2}(\mu_2) \\
&+G_{a_1c_1}(l_1,l_1)G_{b_1a_2}(l_1,l_2)G_{b_2c_2}(l_2,l_2)\phi^{(N)}_{a_1c_1}(-\nu_1)\phi^{(N)}_{b_1a_2}(\nu_1-\mu_2)\phi^{(N)}_{b_2c_2}(-\mu_2)f_{a_1c_1}(\mu_1)f_{b_1a_2}(\nu_1)f_{b_2c_2}(\nu_2) \\
&+G_{a_1c_1}(l_1,l_1)G_{b_1b_2}(l_1,l_2)G_{a_2c_2}(l_2,l_2)\phi^{(N)}_{a_1c_1}(-\nu_1)\phi^{(N)}_{b_1b_2}(\nu_1-\nu_2)\phi^{(N)}_{a_2c_2}(-\nu_2)f_{a_1c_1}(\mu_1)f_{b_1b_2}(\nu_1)f_{a_2c_2}(\mu_2) \\
&+G_{a_1c_1}(l_1,l_1)G_{b_1c_2}(l_1,l_2)G_{a_2b_2}(l_2,l_2)\phi^{(N)}_{a_1c_1}(-\nu_1)\phi^{(N)}_{b_1c_2}(\nu_1-\gamma_2)\phi^{(N)}_{a_2b_2}(\mu_2-\nu_2)f_{a_1c_1}(\mu_1)f_{b_1c_2}(\nu_1)f_{a_2b_2}(\mu_2) \\
&+G_{a_1a_2}(l_1,l_2)G_{b_1c_1}(l_1,l_1)G_{b_2c_2}(l_2,l_2)\phi^{(N)}_{a_1a_2}(\mu_1-\mu_2)\phi^{(N)}_{b_1c_1}(-\mu_1)\phi^{(N)}_{b_2c_2}(-\mu_2)f_{a_1a_2}(\mu_1)f_{b_1c_1}(\nu_1)f_{b_2c_2}(\nu_2) \\
&+G_{a_1a_2}(l_1,l_2)G_{b_1b_2}(l_1,l_2)G_{c_1c_2}(l_1,l_2)\phi^{(N)}_{a_1a_2}(\mu_1-\mu_2)\phi^{(N)}_{b_1b_2}(\nu_1-\nu_2)\phi^{(N)}_{c_1c_2}(\gamma_1-\gamma_2)f_{a_1a_2}(\mu_1)f_{b_1b_2}(\nu_1)f_{c_1c_2}(\gamma_1) \\
&+G_{a_1a_2}(l_1,l_2)G_{b_1c_2}(l_1,l_2)G_{c_1b_2}(l_1,l_2)\phi^{(N)}_{a_1a_2}(\mu_1-\mu_2)\phi^{(N)}_{b_1c_2}(\nu_1-\gamma_2)\phi^{(N)}_{c_1b_2}(\gamma_1-\nu_2)f_{a_1a_2}(\mu_1)f_{b_1c_2}(\nu_1)f_{c_1b_2}(\gamma_1) \\
&+G_{a_1b_2}(l_1,l_2)G_{b_1c_1}(l_1,l_1)G_{a_2c_2}(l_2,l_2)\phi^{(N)}_{a_1b_2}(\mu_1-\nu_2)\phi^{(N)}_{b_1c_1}(-\nu_2)\phi^{(N)}_{a_2c_2}(\mu_1)f_{a_1b_2}(\mu_1)f_{b_1c_1}(\nu_1)f_{a_2c_2}(\mu_2) \\
&+G_{a_1b_2}(l_1,l_2)G_{b_1a_2}(l_1,l_2)G_{c_1c_2}(l_1,l_2)\phi^{(N)}_{a_1b_2}(\mu_1-\nu_2)\phi^{(N)}_{b_1a_2}(\nu_1-\mu_2)\phi^{(N)}_{c_1c_2}(\gamma_1-\gamma_2)f_{a_1b_2}(\mu_1)f_{b_1a_1}(\nu_1)f_{c_2c_2}(\gamma_1)+ \\
&+G_{a_1b_2}(l_1,l_2)G_{b_1c_2}(l_1,l_2)G_{c_1a_2}(l_1,l_2)\phi^{(N)}_{a_1b_2}(\mu_1-\nu_2)\phi^{(N)}_{b_1c_2}(\nu_1-\gamma_2)\phi^{(N)}_{c_1a_2}(-\gamma_1-\mu_2)f_{a_1b_2}(\mu_1)f_{b_1c_2}(\nu_1)f_{c_1a_2}(\gamma_1) \\
&+G_{a_1c_2}(l_1,l_2)G_{b_1c_1}(l_1,l_1)G_{a_2b_2}(l_2,l_2)\phi^{(N)}_{a_1c_2}(\mu_1-\gamma_2)\phi^{(N)}_{b_1c_1}(-\mu_1)\phi^{(N)}_{a_2b_2}(\mu_2-\nu_2)f_{a_1c_2}(\mu_1)f_{b_1c_1}(\nu_1)f_{a_2b_2}(\mu_2) \\
&+G_{a_1c_2}(l_1,l_2)G_{b_1a_2}(l_1,l_2)G_{c_1b_2}(l_1,l_2)\phi^{(N)}_{a_1c_2}(\mu_1-\gamma_2)\phi^{(N)}_{b_1a_2}(\nu_1-\mu_2)\phi^{(N)}_{c_1b_2}(\gamma_1-\nu_2)f_{a_1c_2}(\mu_1)f_{b_1a_2}(\nu_1)f_{c_1b_2}(\gamma_1) \\
&+G_{a_1c_2}(l_1,l_2)G_{b_1b_2}(l_1,l_2)G_{c_1a_2}(l_1,l_2)\phi^{(N)}_{a_1c_2}(\mu_1-\gamma_2)\phi^{(N)}_{b_1b_2}(\nu_1-\nu_2)\phi^{(N)}_{c_1a_2}(\gamma_1-\mu_2)f_{a_1c_2}(\mu_1)f_{b_1b_2}(\nu_1)f_{c_1a_2}(\gamma_1)\Big] \\
&\times du_1du_2+O\left(N^{-1}\right)
\end{aligned}
\tag{26}
$$

Proof. By (19), we have

$$
E\left\{f^{(T)}_{abc}(\mu,\nu)\right\}=\frac{1}{L}\int_0^L E\left\{I^{l(N-M)}_{abc}(\mu,\nu)\right\}du \tag{27}
$$

then by (20), the proof of (25) is completed. From (19), we get

$$
Cov\left\{f^{(T)}_{a_1b_1c_1}(\mu_1,\lambda_2),f^{(T)}_{a_2b_2c_2}(\lambda_3,\lambda_4)\right\}=\frac{1}{L^2}\int_0^L\int_0^L Cov\left\{I^{l(N-M)}_{a_1b_1c_1}(\lambda_1,\lambda_2),I^{l(N-M)}_{a_2b_2c_2}(\lambda_3,\lambda_4)\right\}du_1du_2. \tag{28}
$$

Then by (21), the proof of (26) is completed.

5 Conclusion

In this paper, we discussed the bispectral analysis of a strictly stationary r-vector valued continuous time series with randomly missing observations in joint segments of observations. The modified biperiodogram and smoothed biperiodogram were defined to estimate the bispectral density function. Finally, we explored the statistical properties of both the modified biperiodogram and the smoothed biperiodogram.

Conflict of Interest

The authors declare that they have no conflict of interest.

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