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Study on Step-Stress Accelerated Life Testing for The Burr-XII Distribution Using Cumulative Exposure Model Under Progressive Type-II Censoring with Real Data Example

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Abstract: A simple step-stress accelerated life test (ALT) under progressive censoring of Type-II is considered in this paper. To minimize lifespan and decrease test cost, progressive type-II censoring and accelerated life testing are given. When the lifetime of test units matches the distribution of Burr-XII, cumulative exposure model is assumed. Also, model parameters maximum likelihood estimates (MLEs) are obtained. Furthermore, to demonstrate the proposed methods, actual dataset is analyzed. Lastly, estimators' estimated confidence intervals (CIs) are extracted.
Keywords: Accelerated life testing with step-stress, progressive type-II censoring, Bayes estimation, Burr-XII distribution, cumulative exposure model, simulation analysis

1. Introduction

The investigator is also interested in extreme factors or changing stress factors such as temperature, voltage and load on experimental units' lifetimes in reliability and life testing studies. Stage stress testing (SST), a special class of accelerated life testing (ALT), enables the experimenter to increase the stress levels during the experiment at set times to gain information on life distribution parameters faster than under normal operating conditions. Nelson [1] researched the phase stress model and data analysis of accelerated life research. Miller et al. [2] proposed optimal simple stage stress plans for accelerated life testing. The optimum simple step-stress accelerated life tests with censoring were proposed by Bai et al. [3]. Rend et al. developed a Bayes model for accelerated life testing in step-stress [4]. With type-II censored exponential results, Xiong [5] studied inference on a simple step-stress model. In simple step-stress models, Watkins [6] studied inferences. For a simple step-stress model with type-II censoring, Balakrishnan et al. [7] studied point and interval estimation. Using log-logistic distribution with known scale parameters, Al-Masri [8] studied optimum times for the step-stress cumulative exposure model. Three related aspects of the maximum probability estimate of parameters were considered in Jalali [9] for the two Burr XII distribution parameters. The distribution function of Burr (c,k) [Burr Type XII] is

$$F(x) = 1 - (1 + x^c)^{-k}, \quad x > 0, \tag{1}$$

where parameter $c > 0$ and $k > 0$ are the shape parameters of the distribution. Its density function is

$$f(x) = ckx^{c-1}(1 + x^c)^{-(k+1)}, \quad x > 0, c > 0 \; and \; k > 0 \tag{2}$$

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The distribution is unimodal with the mode at $x = \left(\frac{c-1}{ck+1}\right)^{0.5}$ and median at $x = \left(2^{\frac{1}{k}} - 1\right)^{0.5}$. Lewis [10] studied Burr distribution as a general parametric family in applications for the theory of survivorship and reliability. He found the r^{th} moment about zero as follows

$$\mu'_r = k\beta\left(1+\frac{r}{c}, k-\frac{r}{c}\right), \quad k > \frac{r}{c}, \quad r = 0,1,2,\dots \ , \tag{3}$$

and the mean and the variance are, respectively,

$$\mu = k\beta\left(1+\frac{1}{c}, k-\frac{1}{c}\right) \text{ and} \tag{4}$$

$$V(X) = k\beta\left(1+\frac{2}{c}, k-\frac{2}{c}\right) - \left[k\beta\left(1+\frac{1}{c}, k-\frac{1}{c}\right)\right]^2, \tag{5}$$

where $\beta(.,.)$ is the standard beta function. Shape parameters c and k and the cumulative distribution functions associated with some special cases of Burr (c, k) distribution (Burr Type XII) are displayed in table 1. See Lewis [10].

Table 1: The Burr Type-XII distribution and its special cases.

c	k	**Distribution**	$F(x)$
c	k	Burr (c, k)	$1-(1+x^c)^{-k}$
4.874	6.158	Approximate normal	$1-(1+x^{4.874})^{-6.158}$
c	∞	Weibull	$1-exp\ (-x^c)$
1	∞	Exponential	$1-exp\ (-x)$
∞	k	Generalized logistic	$1-(1+exp\ (x))^{-k}$
∞	1	Logistic	$1-(1+exp\ (x))^{-1}$
∞	∞	Gompertz	$1-(1+exp\ (x))$
1	1	Pareto	$1-(1+x)^{-1}$

As a lifetime model, the Burr (c, k) distribution can be used, at least when there is a large incidence of early failures dominating the distribution of lifetime ($c > 1$). Accelerated life testing and repair time are examples of such situations. The two common distributions of survival or failure time, the Weibull and the exponential are both special or limiting cases of Burr (c, k) [Burr Type]

2. Assumptions to Obtain the Simple Step-Stress Model

2.1 Notation

ALT	Accelerated Life Testing
SST	Step Stress Testing
S_0, S_1	Stress levels
PDF	Probability density function
CDF	Cumulative distribution function
$G(t)$	Cumulative exposure distribution (CED) function
$g(t)$	Probability exposure density (PED) function
n	Identical units under an initial stress level S_0
$t_{1:r:n} < t_{2:r:n} <\ldots< t_{n:r:n}$	The ordered failure times of the n unit under test
τ_1	A fixed time before which the stress level is changed from S_1 to S_2
$t_{r:r:n}$	The time when the r^{th} failure occurs; the experiment is terminated
N_1	Number of units that fail before time τ_1 at stress level S_1
N_2	Number of units that fail before time τ_2 at stress level S_2
c_i, k	The shape parameters of the Burr Type XII distribution
L(.)	Likelihood function
$logL(.)$	The logarithm of the likelihood function
F	Fisher information matrix
MSE	Mean square error
MLEs	Maximum likelihood estimates

2.2 Model Description

Suppose that the data comes from a step-stress model based on progressive Type-II censored with two stress levels, S_0 and S_1. The PDF and the CDF of Burr XII distribution are given by

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$$f_i(t;c_i) = c_i k t^{c_i-1}(1+t^{c_i})^{-(k+1)}, \quad t>0 \quad k,c_i>0 \quad i=1,2, \tag{6}$$

$$F_i(t;c_i) = 1-(1+t^{c_i})^{-k}, \quad t>0 \quad k,c_i>0 \quad i=1,2, \tag{7}$$

We deduce the cumulative exposure model (see, Alhadeed [12]) under Burr type-XII distribution.
The CDF function of time to failure under a particular step-stress pattern can be expressed mathematically as follows:
The cumulative population fraction of specimen failing by time t in stress level S_0 is

$$G(t) = F_1(t) = 1-(1+t^{c_1})^{-k}, \quad 0 \le t < \tau_1. \tag{8}$$

The cumulative population fraction of specimen failing by time t in stress level S_1 is

$$G(t) = F_2[(t-\tau_1)+u_1] = 1-[1+((t-\tau_1)+u_1)^{c_2}]^{-k}, \quad \tau_1 \le t < \tau_2, \tag{9}$$

where u_1,the equivalent starting time, is the solution of

$$F_2(u_1) = F_1(\tau_1), \implies u_1^{c_2} = \tau_1^{c_1}. \tag{10}$$

Then we get

$$u_1 = \tau_1^{\frac{c_1}{c_2}}. \tag{11}$$

Then we can rewrite $G(t)$ in stress level S_1 as follows:

$$G(t) = 1-\left[1+\left((t-\tau_1)+\tau_1^{\frac{c_1}{c_2}}\right)^{c_2}\right]^{-k}, \quad \tau_1 \le t < \tau_2. \tag{12}$$

The simple step stress model is a particular case from the cumulative exposure model, so we can say that the cumulative exposure distribution (CED) function for Burr type-XII distribution at two stress level is given as:

$$G(t) = \begin{cases} G_1(t) = 1-(1+t^{c_1})^{-k}, & 0 \le t < \tau_1, \\ G_2(t) = 1-\left[1+\left(t-\tau_1+\tau_1^{\frac{c_1}{c_2}}\right)^{c_2}\right]^{-k}, & \tau_1 \le t < \infty. \end{cases} \tag{13}$$

The corresponding probability exposure density (PED) function becomes

$$g(t) = \begin{cases} g_1(t) = kc_1 t^{c_1-1}(1+t^{c_1})^{-(k+1)}, & 0 \le t < \tau_1, \\ g_2(t) = kc_2\left[t-\tau_1+\tau_1^{\frac{c_1}{c_2}}\right]^{c_2-1} \times \left[1+\left(t-\tau_1+\tau_1^{\frac{c_1}{c_2}}\right)^{c_2}\right]^{-(k+1)} & \tau_1 \le t < \infty. \end{cases} \tag{14}$$

Suppose that the time data for failure comes from a cumulative exposure model. We consider a simple step-stress model based on progressive type-II censorship with only two levels of stress, S_0 and S_1.
We have a simple step-stress model under progressive censoring of type-II for equivalent units at an initial stress level and are defined in advance. At the time of the first failure, R_1 of $n-1$ surviving units are randomly removed from the experiment, at the time of the second failure, R_2 of $n-2-R_1$ surviving units are randomly removed from the experiment, and so on, the stress level is changed to S_1 at a pre-fixed time τ_1. The life-testing experiment is terminated when the r^{th} failure time, $t_{r:r:n}$, occurs at a time in which all remaining $R_r = n-r- \cdots - R_{r-1}$ surviving units are removed. Figure (1) depicts such a simple step-stress model under progressive type-II censoring.

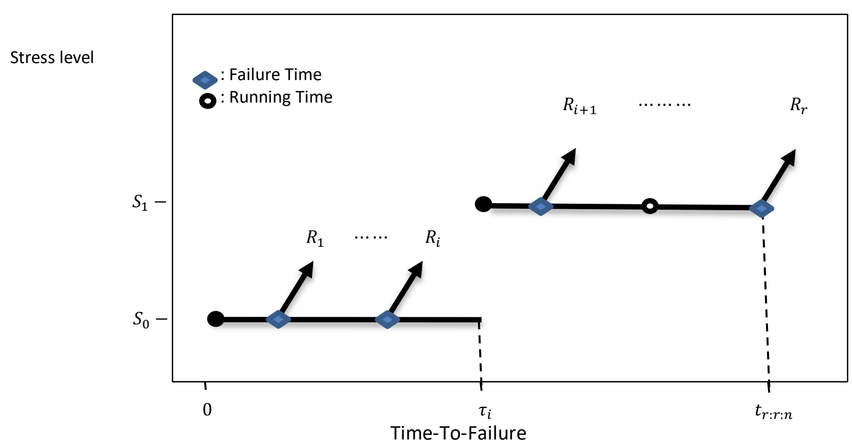

Fig. 1: Simple step-stress model under progressive type-II censoring

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Under the assumption of the cumulative exposure model, the corresponding cumulative exposure distribution, CED $G(t)$, and PED $g(t)$ are given in Eqs. (13) and (14), respectively. We will observe the following progressively type-II censored data:

$$t = \{t_{1:r:n} < \cdots \leq t_{N_1:r:n} < \cdots < t_{r:r:n}\}. \tag{15}$$

With the corresponding progressive censoring scheme

$$R = (R_1, \cdots, R_r), \tag{16}$$

where $\sum_{j=1}^{r} R_j = n - r$.

3. Maximum Likelihood Estimation

In this section, we consider the likelihood function based on the observed progressively type-II censored data given in Eq. (15) and then obtain the MLEs of the unknown parameters, c_1, c_2 and k. Let $t_{1:r:n} < \cdots < t_{r:r:n}$ denote the observed progressively type-II censored sample. Then, the likelihood function of this censored sample seen by Balakrishnan and Aggarwal [13] can be written as

$$L(c_1, c_2, k|\boldsymbol{t}) = C_p \prod_{i=1}^{r} g(t_{i:r:n})\{1 - G(t_{i:r:n})\}^{R_i}, \quad t_{1:r:n} < t_{2:r:n} < \cdots < t_{r:r:n}\,, \tag{17}$$

where $= N_1 + N_2$, t is the observed failure time data, and

$$C_p = n(n-1-R_1)(n-2-R_1-R_2)\cdots(n-r+1-\textstyle\sum_{i=1}^{r-1} R_i) = \prod_{j=1}^{r} R_j^*\,, \tag{18}$$

where $\sum_{i=j}^{r-1}(R_i + 1) = R_j^*$. From the CED in Eq. (13) and the corresponding PED in Eq. (14), we obtain the likelihood function of c_1, c_2 and k based on the progressively type-II censored sample in Eq. (17) as follows:

If $N_1 = r$ and $N_2 = 0$ in Eq. (17), the likelihood function of c_1, c_2 and k in Eq. (17) is

$$L(c_1, c_2, k|\boldsymbol{t}) = C_p\{\{\prod_{i=1}^{r} g_1(t_{i:r:n})\}[1 - G_1(t_{i:r:n})]^{R_i}\}$$
$$= C_p\{\prod_{i=1}^{r} k c_1 t_{i:r:n}^{c_1-1}(1 + t_{i:r:n}^{c_1})^{-(k+1)}\}\{(1 + t_{i:r:n}^{c_1})^{-kR_i}\}. \tag{19}$$

1. If $N_1 = 0$ and $N_2 = r$ in Eq. (17), the likelihood function of c_1, c_2 and k in Eq. (17) is

$$L(c_1, c_2, k|\boldsymbol{t}) = C_p\{\prod_{i=1}^{r} g_2(t_{i:r:n})[1 - G_2(t_{i:r:n})]^{R_i}\},$$

$$L(c_1, c_2, k|\boldsymbol{t}) = C_p\left\{\left[\prod_{i=1}^{r} k c_2 \left[t_{i:r:n} - \tau_1 + \tau_1^{\frac{c_1}{c_2}}\right]^{c_2-1} \times \left[1 + \left(t_{i:r:n} - \tau_1 + \tau_1^{\frac{c_1}{c_2}}\right)^{c_2}\right]^{-(k+1)}\right]\left[1 + \left(t_{i:r:n} - \tau_1 + \tau_1^{\frac{c_1}{c_2}}\right)^{c_2}\right]^{-kR_i}\right\}. \tag{20}$$

2. In all other cases, the likelihood function of c_1, c_2 and k in Eq. (17) is

$$L(c_1, c_2, k|\boldsymbol{t}) = C_p\{[\prod_{i=1}^{N_1} g_1(t_{i:r:n})][1 - G_1(t_{i:r:n})]^{R_i}\} \times \{[\prod_{i=N_1+1}^{r} g_2(t_{i:r:n})][1 - G_2(t_{i:r:n})]^{R_i}\},$$

$$L(c_1, c_2, k|\boldsymbol{t}) = C_p\{[\prod_{i=1}^{N_1} k c_1 t_{i:r:n}^{c_1-1}(1 + t_{i:r:n}^{c_1})^{-(k+1)}] \times (1 + t_{i:r:n}^{c_1})^{-kR_i}$$
$$\times\left[\prod_{i=N_1+1}^{r} k c_2 \left[t_{i:r:n} - \tau_1 + \tau_1^{\frac{c_1}{c_2}}\right]^{c_2-1} \times \left[1 + \left(t_{i:r:n} - \tau_1 + \tau_1^{\frac{c_1}{c_2}}\right)^{c_2}\right]^{-(k+1)}\right]$$
$$\times\left[1 + \left(t_{i:r:n} - \tau_1 + \tau_1^{\frac{c_1}{c_2}}\right)^{c_2}\right]^{-kR_i}\Bigg\}. \tag{21}$$

From the likelihood function in Eq. (19) - Eq. (21), we observe the following:

1. If $N_1 = r$ and $N_2 = 0$ in Eq. (15), the likelihood function in Eq. (19) reveals that the MLEs of c_2 does not exits.
2. If $N_1 = 0$ and $N_2 = r$ in Eq. (15), Eq. (20) 's likelihood function reveals that the MLEs of c_1, c_2 and k do exist.
3. If at least one failure before $\tau_1$and at least one failure after $\tau_1$occur in Eq. (15), the likelihood function in Eq. (21) reveals that the MLEs of c_1, c_2, and k do exist.

In the situation, where the log-likelihood function of c_1, c_2 and k is obtained from Eq. (21) as follows:

$$\mathcal{L} = \log L(c_1, c_2, k|\boldsymbol{t}) = \log[C_P] + r\log[k] + N_1\log[c_1] + N_2\log[c_2] + (c_1 - 1)\sum_{i=1}^{N_1}\log[t_{i:r:n}]$$
$$- \sum_{i=1}^{N_1}(k(1 + R_i) + 1)\log[1 + t_{i:r:n}^{c_1}] + (c_2 - 1)\sum_{i=N_1+1}^{r}\log\left[t_{i:r:n} - \tau_1 + \tau_1^{\frac{c_1}{c_2}}\right]$$
$$- \sum_{i=N_1+1}^{r}(k(1 + R_i) + 1)\log\left[1 + \left(t_{i:r:n} - \tau_1 + \tau_1^{\frac{c_1}{c_2}}\right)^{c_2}\right]. \tag{22}$$

Then, we obtain the estimators of c_1, c_2 and k by differentiating Eq. (22) with respect to c_1, c_2 and k, respectively, and equating to zero, in this case, we have

$$\frac{\partial\mathcal{L}}{\partial k} = \frac{r}{k} - \sum_{i=1}^{N_1}\text{Log}[1 + t_{i:r:n}^{c_1}](1 + R_i) - \sum_{i=1+N_1}^{r}\log[1 + B_i^{c_2}]\,(1 + R_i)\Big|_{\theta=\hat{\theta}} = 0\,, \tag{23}$$

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$$\frac{\partial \mathcal{L}}{\partial c_1} = \frac{N_1}{c_1} + \sum_{i=1}^{N_1} \log[t_{i:r:n}] - \sum_{i=1}^{N_1} \frac{\log[t_{i:r:n}] A_i t_{i:r:n}^{c_1}}{1+t_{i:r:n}^{c_1}} + \sum_{i=N_1+1}^{r} \frac{(c_2-1)\log[\tau_1]}{c_2 B_i} \tau_1^{\frac{c_1}{c_2}} - \sum_{i=N_1+1}^{r} \frac{\log[\tau_1] A_i B_i^{c_2-1}}{1+B_i^{c_2}} \tau_1^{\frac{c_1}{c_2}} \Bigg|_{\underline{\theta}=\hat{\underline{\theta}}} = 0, \quad (24)$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = \frac{N_2}{c_2} + \sum_{i=N_1+1}^{r} \log[B_i] - \sum_{i=N_1+1}^{r} \frac{A_i}{c_2(1+B_i^{c_2})} \left(\log[B_i]\, c_2 B_i^{c_2} - \log[\tau_1]\, c_1 \tau_1^{\frac{c_1}{c_2}} B_i^{c_2-1} \right) - \frac{c_1(c_2-1)}{c_2^2} \sum_{i=N_1+1}^{r} \frac{1}{B_i} \log[\tau_1]\, \tau_1^{\frac{c_1}{c_2}} \Bigg|_{\underline{\theta}=\hat{\underline{\theta}}} = 0, \quad (25)$$

Where $\underline{\theta} = (c_1, c_2, k)$, $\hat{\underline{\theta}} = (\hat{c}_1, \hat{c}_2, \hat{k})$, $A_i = 1 + k(1 + R_i)$ and $B_i = t_{i:r:n} - \tau_1 + \tau_1^{\frac{c_1}{c_2}}$. Since the closed-form solution to the nonlinear equations system (23-25) is tough to obtain, we use the Newton-Raphson method to solve the previous nonlinear equations simultaneously to obtain $\hat{c}_1, \hat{c}_2,$ and $\hat{k}$, see tables (5-9).

4. Asymptotic Variances and Covariance Matrix Under Progressively Type-II Censored

The asymptotic variance and covariance matrix of maximum likelihood estimates are given by the elements of the inverse of the Fisher information matrix as follows:

$$I_{ij}(\underline{\theta}) \cong E\left\{-\frac{\partial^2 \ln L}{\partial \theta_i \partial \theta_j}\right\}. \quad (26)$$

Unfortunately, the exact mathematical expressions for the previous expectation are complicated to obtain. Therefore, the Fisher information matrix is given by

$$I_{ij}(\underline{\theta}) \cong \left\{-\frac{\partial^2 \ln L}{\partial \theta_i \partial \theta_j}\right\}, \quad (27)$$

which is obtained by approximating the expectation on operation E and replacing c_1, c_2 and k with $\hat{c}_1, \hat{c}_2$ and $\hat{k}$, respectively, Cohen [14]. The asymptotic variance and covariance matrix F of the maximum likelihood estimates can be written as follows:

$$F^{-1} = \begin{pmatrix} -\frac{\partial^2 \ln L}{\partial k^2} & -\frac{\partial^2 \ln L}{\partial k \partial c_1} & -\frac{\partial^2 \ln L}{\partial k \partial c_2} \\ -\frac{\partial^2 \ln L}{\partial c_1 \partial k} & -\frac{\partial^2 \ln L}{\partial c_1^2} & -\frac{\partial^2 \ln L}{\partial c_1 \partial c_2} \\ -\frac{\partial^2 \ln L}{\partial c_2 \partial k} & -\frac{\partial^2 \ln L}{\partial c_2 \partial c_1} & -\frac{\partial^2 \ln L}{\partial c_2^2} \end{pmatrix}^{-1}. \quad (28)$$

The elements of matrix F are given as the following:

$$\frac{\partial^2 \mathcal{L}}{\partial k^2} = -\frac{r}{k^2}, \quad (29)$$

$$\frac{\partial^2 \mathcal{L}}{\partial k \partial c_1} = -\sum_{i=1}^{N_1} \left(\frac{t_{i:r:n}^{c_1}}{1+t_i^{c_1}} \right) \log[t_{i:r:n}]\,(1+R_i) - \sum_{i=N_1+1}^{r} \left(\frac{B_i^{c_2-1}}{1+B_i^{c_2}} \right) \log[\tau_1]\,(1+R_i) \tau_1^{\frac{c_1}{c_2}}, \quad (30)$$

$$\frac{\partial^2 \mathcal{L}}{\partial k \partial c_2} = -\sum_{i=N_{1+1}}^{r} \frac{(1+R_i)}{1+B_i^{c_2}} \left(\log[B_i]\, B_i^{c_2} - \frac{c_1}{c_2} \log[\tau_1]\, \tau_1^{\frac{c_1}{c_2}} B_i^{c_2-1} \right), \quad (31)$$

$$\frac{\partial^2 \mathcal{L}}{\partial c_1^2} = -\frac{N_1}{c_1^2} - \sum_{i=1}^{N_1} \left(-\frac{\log[\tau_1]^2 A_i t_{i:r:n}^{2c_1}}{\left(1+t_{i:r:n}^{c_1}\right)^2} + \frac{\log[\tau_1]^2 A_i t_{i:r:n}^{c_1}}{1+t_{i:r:n}^{c_1}} \right) + (c_2-1) \sum_{i=N_1+1}^{r} \left(\frac{1}{c_2^2 B_i} - \frac{\tau_1^{\frac{c_1}{c_2}}}{c_2^2 B_i^2} \right) \log[\tau_1]^2\, \tau_1^{\frac{c_1}{c_2}}$$
$$- \sum_{i=N_1+1}^{r} \left[\left(\frac{B_i^{c_2-1}}{c_2(1+B_i^{c_2})} \right) \log[\tau_1]^2\, A_i \tau_1^{\frac{c_1}{c_2}} - \left(\frac{B_i^{2(c_2-1)}}{\left(1+B_i^{c_2}\right)^2} \right) \log[\tau_1]^2\, A_i \tau_1^{\frac{2c_1}{c_2}} \right]$$
$$+ \sum_{i=N_1+1}^{r} \frac{\log[\tau_1] A_i \tau_1^{\frac{c_1}{c_2}}}{1+B_i^{c_2}} \left(\log[\tau_1]\, \tau_1^{\frac{c_1}{c_2}} B_i^{c_2-2} - \frac{\log[\tau_1] \tau_1^{\frac{c_1}{c_2}} B_i^{c_2-2}}{c_2} \right), \quad (32)$$

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$$\frac{\partial^2\mathcal{L}}{\partial c_2 \partial c_1} = \sum_{i=1+N_1}^{r} \left(\frac{\log[\tau_1]}{c_2 B_i}\right) \tau_1^{\frac{c_1}{c_2}} + (c_2-1) \sum_{i=1+N_1}^{r} \left(\frac{\log[\tau_1] c_1 \tau_1^{\frac{c_1}{c_2}}}{c_2^3 B_i^2} - \frac{\log[\tau_1] c_1}{c_2^3 B_i} - \frac{1}{c_2^2 B_i}\right) \log[\tau_1]\, \tau_1^{\frac{c_1}{c_2}}$$
$$- \sum_{i=1+N_1}^{r} \left[\log[\tau_1]\, A_i \tau_1^{\frac{c_1}{c_2}} B_i^{c_2-1} \left(\frac{\mathrm{Log}[\tau_1] c_1 \tau_1^{\frac{c_1}{c_2}} B_i^{c_2-1}}{c_2\left(1+B_i^{c_2}\right)^2} - \frac{\log[B_i] B_i^{c_2}}{\left(1+B_i^{c_2}\right)^2}\right) - \frac{B_i^{c_2-1}}{c_2^2\left(1+B_i^{c_2}\right)} \log[\tau_1]^2\, c_1 A_i \tau_1^{\frac{c_1}{c_2}}\right]$$
$$+ \sum_{i=1+N_1}^{r} \frac{\log[\tau_1] A_i \tau_1^{\frac{c_1}{c_2}}}{1+B_i^{c_2}} \left(\frac{\log[\tau_1] c_1 \tau_1^{\frac{c_1}{c_2}} B_i^{c_2-2}}{c_2^2} - \frac{\log[\tau_1] c_1 \tau_1^{\frac{c_1}{c_2}} B_i^{c_2-2}}{c_2} + \log[B_i] B_i^{c_2-1}\right), \tag{33}$$

$$\frac{\partial^2\mathcal{L}}{\partial c_2^2} = -\frac{N_2}{c_2^2} - 2 \sum_{i=N_1+1}^{r} \frac{\log[\tau_1] c_1 \tau_1^{\frac{c_1}{c_2}}}{c_2^2 B_i} + (c_2-1) \sum_{i=N_1+1}^{r} \left(-\frac{\log[\tau_1]^2 c_1^2 \tau_1^{\frac{2c_1}{c_2}}}{c_2^4 (B_i)^2} + \frac{\log[\tau_1]^2 c_1^2 \tau_1^{\frac{c_1}{c_2}}}{c_2^4 (B_i)} + \frac{2\log[\tau_1] c_1 \tau_1^{\frac{c_1}{c_2}}}{c_2^3 (B_i)}\right)$$
$$- \sum_{i=N_1+1}^{r} \left\{A_i \left(-\frac{\log[\tau_1] c_1 \tau_1^{\frac{c_1}{c_2}} B_i^{c_2-1}}{c_2} + \log[B_i]\, B_i^{c_2}\right)\left(\frac{\log[\tau_1] c_1 \tau_1^{\frac{c_1}{c_2}} B_i^{c_2-1}}{c_2\left(1+B_i^{c_2}\right)^2} - \frac{\log[B_i] B_i^{c_2}}{\left(1+B_i^{c_2}\right)^2}\right)\right.$$
$$+ \sum_{i=N_1+1}^{r} \frac{A_i}{1+B_i^{c_2}} \left(\frac{\log[\tau_1]^2 c_1^2 \tau_1^{\frac{c_1}{c_2}} B_i^{c_2-1}}{c_2^3} + \log[B_i]\left(-\frac{\log[\tau_1] c_1 \tau_1^{\frac{c_1}{c_2}} B_i^{c_2-1}}{c_2} + \log[B_i]\, B_i^{c_2}\right) - \right.$$
$$\left.\left.\frac{\log[\tau_1] c_1 \tau_1^{\frac{c_1}{c_2}} \left(\frac{\log[\tau_1] c_1 \tau_1^{\frac{c_1}{c_2}} B_i^{c_2-2}}{c_2^2} - \frac{\log[\tau_1] c_1 \tau_1^{\frac{c_1}{c_2}} B_i^{c_2-2}}{c_2} + \log[B_i] B_i^{c_2-1}\right)}{c_2}\right)\right\}, \tag{34}$$

where, $A_i = 1 + k(1+R_i)$, $B_i = t_{i:r:n} - \tau_1 + \tau_1^{\frac{c_1}{c_2}}$. Consequently, maximum likelihood estimators $\hat{c}_1$, $\hat{c}_2$ and $\hat{k}$ for c_1, c_2 and k, respectively, have an asymptotic variance-covariance matrix defined by inverting Fisher information matrix F and substituting $\underline{\hat{\theta}} = (\hat{c}_1,\ \hat{c}_2, \hat{k})$ for $\underline{\theta} = (c_1, c_2, k)$, see Tables (5-9).

5. Confidence Interval for Burr-XII Distribution Under Progressive Type-II Censoring Data

In this part of the paper, we concluded the parameters' upper and lower bound using a 95% confidence interval. The approximate confidence results are tabulated in Tables (5-9).

6. The Numerical Algorithm Used in the Paper

This section clarifies the algorithm used to generate a progressive type-II censored sample, and estimators $\hat{c}_1$, $\hat{c}_2, \hat{k}$, besides, the mean square error (MSE).

Step 1. Given τ_1 and the original progressive type-II censored sample with censoring scheme $R = (R_1, \cdots, R_r)$, we obtain $\hat{c}_1$, $\hat{c}_2$ and $\hat{k}$ from Eq. (17).

Step 2. Based on $n, r, R, \tau_1, \hat{c}_1, \hat{c}_2$, and $\hat{k}$, we generate a random sample of size n from Uniform (0, 1) distribution, and obtain progressive type-II censored uniform sample $(U_{1:r:n}, \cdots, U_{r:r:n})$.

Step 3. Find N_1 such that

$$U_{N_1:r:n} \le 1 - \left(1 - \tau_1^{c_1}\right)^{-k} < U_{N_1+1:r:n}\,. \tag{35}$$

Then, for $1 \le i < N_1$, we set

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$$t_{i:r:n} = \left[(1 - U_{i:r:n})^{-\frac{1}{k}} - 1\right]^{\frac{1}{c_1}}. \tag{36}$$

And for $N_1 + 1 \le i < r$, we set

$$t_{i:r:n} = \left[(1 - U_{i:r:n})^{-\frac{1}{k}} - 1\right]^{\frac{1}{c_2}} + \tau_1 - \tau_1^{\frac{c_1}{c_2}}. \tag{37}$$

Step 4. Based on r, N_1, τ_1 and the progressive Type-II censored observations,

$$\{t_{1:r:n}, \dots, t_{N_1:r:n}, t_{N_1+1:r:n}, \dots, t_{r:r:n}\}. \tag{38}$$

Step 5. Repeat steps 2-4 M times and arrange $\hat{c}_1, \hat{c}_2$ and $\hat{k}$ in an ascending order to obtain

$$\{\hat{c}_1^{[1]}, \dots, \hat{c}_1^{[H]}\}, \quad \{\hat{c}_2^{[1]}, \dots, \hat{c}_2^{[H]}\} \quad and \quad \{\hat{k}^{[1]}, \dots, \hat{k}^{[H]}\}. \tag{39}$$

Then, we get the estimators as follows:

$$\hat{c}_1^{\,*} = \frac{1}{H}\sum_{i=1}^{H} \hat{c}_1^{[i]}, \quad \hat{c}_2^{\,*} = \frac{1}{H}\sum_{i=1}^{H} \hat{c}_2^{[i]} \quad and \quad \hat{k}^* = \frac{1}{H}\sum_{i=1}^{H} \hat{k}^{[i]}. \tag{40}$$

We were substituting the values of parameters $\hat{c}_1^{\,*}$, $\hat{c}_2^{\,*}$, and $\hat{k}^*$ to get (MSE). Furthermore, the asymptotic variance, covariance matrix and two-sided confidence intervals of the estimators are obtained.

7. Application on Real Data Example

In this section, we introduce a real data example. The real data in Table 2 was collected from chapter 5 of Zhu [15]. For more information about this data, see Zhu [15]. The failure times of the light bulbs are displayed in Table 1, and the unit removed from the test before failure is noted by +. In this case, the number of units under stress is $\boldsymbol{n = 64}$, and 11 of these units were removed from the test before the failure.

Table 2: The failure times in hours of 64 light bulbs

No.	Failure time	No.	Failure time	No.	Failure time	No.	Failure time
1	12.07	17	91.22	33	14.00	49	94.38
2	19.50	18	102.10	34	17.95	50	97.71
3	22.10	19	105.10	35	24.00	51	101.53
4	23.11	20	109.20	36	26.46	52	105.11
5	24.00	21	114.40	37	26.58	53	112.11
6	25.10	22	117.90	38	28.06	54	119.58
7	26.90	23	121.90	39	34.00	55	120.20
8	36.64	24	122.50	40	36.13	56	126.95
9	44.10	25	123.60	41	40.85	57	129.25
10	46.30	26	126.50	42	41.11	58	136.31
11	54.00	27	130.10	43	42.63	59	140+
12	58.09	28	140+	44	52.51	60	140+
13	64.17	29	140+	45	62.68	61	140+
14	72.25	30	140+	46	73.13	62	140+
15	86.90	31	140+	47	83.63	63	140+
16	90.09	32	140+	48	91.56	64	140+

To determine whether the data makes a good fit for the Burr distribution, we made a modified Kolmogorov Smirnov goodness-of-fit test for the progressive type-II censored data. This method was done by Pakyari and Balakrishnan [16]. The results of p-values for each stress level $S_i, i = 1, 2$, are tabulated in Table 3. the results shows us that the distributions provide an excellent fit to the given data because all p-values exceed 0.05. The MLEs of c_1, c_2, k are introduced in Table 4.

Table 3: p-values and the value of the statistic for each level

Stress (voltage)	2.25 V	2.44 V
D	1.1006927863141185	0.013692047887372413
p-value	0.30	0.965

Our distribution makes a good fit for the data used by Zhu [15]. The MLEs are introduced in Table 3.

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Table 4: Values of the parameters for the real dataset

Parameter	MLEs	Lower bound	Upper bound
c_1	6.628880641589399	2.4194947611505766	18.161667165397084
c_2	18.88352001192684	$5.114098462680956 \times 10^{-55}$	$6.972633214689974 \times 10^{56}$
k	0.027029	$7.320913602421803 \times 10^{-58}$	$9.979352642043303 \times 10^{53}$

The relation between scale parameter and acceleration model has the following form $\ln(\sigma_i) = a + b\ln(S_i),\ b > 0, i = 0, 1, 2$. By estimating the acceleration model's parameters, we can find that $a = -8.42764859826418, b = 12.913306559462917, k = 0.024$; from the previous values, we can calculate the scale parameter under normal conditions, $\sigma_0 = 1.68737$.
By using the estimated values of k and σ_0, the reliability function under use conditions is given by:
$R(t) = (1+t^{1.687373})^{-0.024}$ (41)

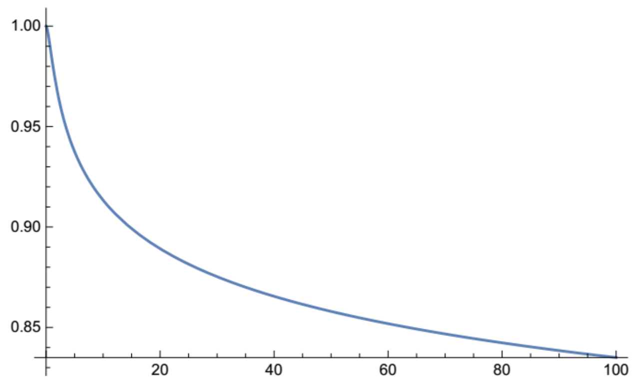

Fig 2: Reliability function under average condition

8. Simulation Study

Simulation studies are conducted in this section to examine the performances of the MLEs, 95% approximate CI length. The algorithm is as follows:

1. Assign values for n, m, τ.
2. Generate a simple random sample of size m from Uniform (0, 1) distribution, $(U_1, U_2, \dots, U_m)$.
3. Determine the values of the censored scheme, $R_i, i = 1, 2, \dots, m$, such that $\sum_{i=1}^{m} R_i = n - m$.
4. Set $E_i = U_i^{\frac{1}{\left(i+\sum_{d=m-i+1}^{m} R_d\right)}}, i = 1, 2, \dots, m$.
5. Obtain the progressive type-II censored sample, $U_{1:m:n}^*, U_{2:m:n}^*, \dots, U_{m:m:n}^*$, where $U_{i:m:n}^* = 1 - \prod_{d=m-i+1}^{m} E_d$, $i = 1, 2, \dots, m$.
6. Find n_1, such that $U_{n_1:m:n}^* < F_1(\tau) \le U_{n_1+1:m:n}^*$.
7. From Steps (2)-(6), the order observations, $t_{1:m:n}, t_{2:m:n}, \dots, t_{n_1:m:n}, t_{n_1+1:m:n}, \dots, t_{m:m:n}$, are calculated as follows:
$$t_{i:m:n} = \begin{cases} \left[(1-U_{i:r:n})^{-\frac{1}{k}} - 1\right]^{\frac{1}{c_1}}, & i = 1, 2, \dots, n_1, \\ \left[(1-U_{i:r:n})^{-\frac{1}{k}} - 1\right]^{\frac{1}{c_2}} + \tau_1 - \tau_1^{\frac{c_1}{c_2}}, & i = n_1 + 1, \dots, m. \end{cases}$$
8. Solve the nonlinear system in Eqs. (23-25) and then evaluate 95% CI for the three parameters, k, c_1 and c_2.

Table 5: $n = 30, m = 15, R_1 = R_2 = R_3 = 5, c_1 = 0.25856, c_2 = 0.27358, k = 0.5, \tau_1 = 4.2$.

Parameter	MLEs mean value	MSE	Lower bound	Upper bound	Coverage probability	CI length
c_1	0.246	0.0048	0.094	0.3984	0.93	0.3047
c_2	0.2963	0.0142	0.0606	0.532	0.92	0.3378
k	0.29883	0.048	0.0842	0.5146	0.66	0.4715

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Table 6: $n = 30, m = 30, R_1 = R_2 = \cdots = R_m = 0, c_1 = 0.25856, c_2 = 0.27358, k = 0.5$ and $\tau_1 = 4.2$.

Parameter	MLEs mean value	MSE	Lower bound	Upper bound	Coverage probability	CI length
c_1	0.27445	0.0046	0.13	0.4193	0.99	0.2898
c_2	0.28014	0.0063	0.113	0.4473	0.95	0.3063
k	0.51689	0.0272	0.2208	0.8129	0.96	0.5921

Table 7: $n = 50, m = 30, R_1 = R_2 = 10, R_3 \ldots = R_m = 0, c_1 = 0.25856, c_2 = 0.27358, k = 0.5$ and $\tau_1 = 4.2$.

Parameter	MLEs mean value	MSE	Lower bound	Upper bound	Coverage probability	CI length
c_1	0.26125	0.00311	0.1376	0.38491	0.97	0.24731
c_2	0.34075	0.01093	0.14307	0.53842	0.99	0.24183
k	0.32164	0.03741	0.12397	0.51931	0.61	0.39535

Table 8: $n = 50, m = 50, R_1 = R_2 = R_3 \ldots = R_m = 0, c_1 = 0.25856, c_2 = 0.27358, k = 0.5$ and $\tau_1 = 4.2$.

Parameter	MLEs mean value	MSE	Lower bound	Upper bound	Coverage probability	CI length
c_1	0.26829	0.00333	0.15754	0.37905	0.95000	0.22151
c_2	0.28030	0.00437	0.15266	0.40793	0.93000	0.22638
k	0.49723	0.01388	0.36960	0.62487	0.79000	0.25527

Table 9: $n = 70, m = 56, R_1 = R_2 = R_3 \ldots = R_7 = 2, c_1 = 0.25856, c_2 = 0.27358, k = 0.5$ and $\tau_1 = 4.2$.

Parameter	MLEs mean value	MSE	Lower bound	Upper bound	Coverage probability	CI length
c_1	0.26628	0.00265	0.16929	0.36327	0.93000	0.19398
c_2	0.30472	0.00520	0.17563	0.43382	0.97000	0.18765
k	0.40029	0.01629	0.23476	0.56582	0.79000	0.33106

9. Conclusion

From the results in Tables (4-8), we have observed the following:

1. The MSEs of MLEs of the considered parameters decrease as the sample size increases, except for a few cases. This may be due to fluctuations in data.
2. The length of approximate CIs decreases as the sample size increases, except for a few cases. This may be due to fluctuations in data.
3. For the real data sets, the Burr-XII distribution gives a good fit for the real data.
4. Before acceleration, the real data was fitted using the usual Kolmogorov-Smirnov method, which provides a good fit for the data.
5. The real data after acceleration was fitted using the modified Kolmogorov-Smirnov method, which provides a good fit for the data.
6. We estimated the distribution parameters under normal conditions. The reliability function was graphed under the standard condition, as in figure 2.
7. At time equals zero, the reliability function gives a 100% efficiency, and as time increases, the reliability function becomes a decreasing function.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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