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Some fixed point theorems in fuzzy 2–metric spaces under ψ–contractive mappings

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Abstract: In this article, we estiblish some fixed point theorem by using the concept of ψ–contractive mapping on fuzzy 2–metric space. Also, we prove a unique common fixed point of the mappings into itself from complete (F2M)-space by using the definition of weakly commuting and ψ–contractive mappings.

Keywords: (F2M)–space, ψ–contractive mapping, weakly commuting.
Mathematics Subjects Classification: 03E72 , 54E35

1 Introduction

The definition of the fuzzy sets is one of the vital concepts in analysis and topology. It is given by Zadeh in 1965 [1]. This concept has been being attracted the focus of many researchers to develop its theory and applications such as the fixed-point theorems that is emerge in mathematics with vigorous hope and vital trust. Based on the concept of the fuzzy sets, Kramosil and Michalek were introduced the fuzzy metric in 1975 [2]. The principle contraction in fuzzy metric spaces was proved by Grabiec in 1988 [3]. Additionally, the cooperation between George and Veeramani was produced the modified concept of the fuzzy metric spaces with the help of t-norms in 1994 [4]. While, the contraction mappings type in 2-metric spaces was given by Iseki et al. [5]. Moreover, the common fixed-point theorem was proves by Sharma for three mappings in fuzzy 2-metric spaces [6]. These theorems have been being studied by many other researchers [7], [8], [9], [10], [11]. Under strict contractive condition, some common fixed-point theorems in fuzzy 2-metric spaces were proved by Amita Joshi [3] while the common fixed point theorem in fuzzy 2- metric spaces was improved by Chauhan et al., on six self-mappings using the concept of sub compatibility of type A and commutativity changing the concept of sub–compatibility [3]. While, the notion of the reciprocal continuity of mappings in metric spaces in defined by Pant [12] while, Jungck and Rhoades concepted a pair of self-mappings to be weakly compatible if they commute at their coincidence points [13].

In this paper, some fixed-point theorems are investigated by using the concept of ψ–contractive mapping on fuzzy 2–metric space. Also, a unique common fixed point of the mappings into itself from complete (F2M)-space is proved by using the definition of weakly commuting and ψ–contractive mappings. Now we begin with some known definitions and preliminary concepts.

2 preliminaries

Definition 21 *[?] The ordered pair (X,d) is said to be 2–metric space if X is an arbitrary set, the function d on X^3 into R^+ if the function d achieve the following conditions :*

(i) For distinct points $x,y \in X\,(x \neq y)$, there exists a point $z \in X$ such that $d(x,y,z) \neq 0$,

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(ii) $d(x,y,z) = 0$ *if at least two of* x*,* y *and* z *are equal,*
(iii) $d(x,y,z) = d(x,z,y) = d(y,z,x)$ *for all* $x,y,z \in X$*,*
(iv) $d(x,y,z) \leq d(x,y,u) + d(x,u,z) + d(u,y,z)$ *for all* $x,y,z,u \in X$*.*

Definition 22 *[3] Let* $* : [0,1]^3 \longrightarrow [0,1]$ *be binary operation and is said to be continuous t–norm if* $([0,1],*)$ *is an abelian topological monoid with until 1 such that* $a*b*c \leq d*e*f$ *whenever* $a \leq d$*,* $b \leq e$*,* $c \leq f$ *for all* $a,b,c,d,e,f \in [0,1]$*.*

Definition 23 *[3] The triple-tuple* $(X,F,*)$ *is said to be fuzzy 2–metric space if* $X \neq \phi$*,* $*$ *is a continuous t–norm and* F *is fuzzy set in* $X^3 \times [0,\infty)$ *satisfy the following conditions:*
(F2M-1) $F(x,y,z,0) = 0$*;*
(F2M-2) $F(x,y,z,t) = 1$*,* $t > 0$ *and when at least two point of* x,y,z *are equal ;*
(F2M-3) $F(x,y,z,t) = F(x,z,y,t) = F(y,z,x,t)$ *;*
(F2M-4) $F(x,y,z,t_1+t_2+t_3) \geq F(x,y,u,t_1) * F(x,u,z,t_2) * F(u,y,z,t_3)$ *for all* $x,y,z,u \in X$ *and* $t_1,t_2,t_3 > 0$*;*
(F2M-5) $F(x,y,z,.) : [0,\infty) \longrightarrow [0,1]$ *is left continuous .*

Definition 24 *[3] Let* $(X,F,*)$ *be fuzzy 2–metric space, then the sequence* $\{x_n\}$ *in* X *is said to be convergent to a point* $x \in X$ *if* $\lim_{t\longrightarrow\infty} F(x_n,x,z,t) = 1$ *for all* $z \in X$ *and* $t > 0$ *.*

Definition 25 *[3] Let* $(X,F,*)$ *be fuzzy 2–metric space, then the sequence* $\{x_n\}$ *in* X *is said to Cauchy sequence iff* $\lim_{t\longrightarrow\infty} F(x_{n+p},x_n,z,t) = 1$ *for all* $z \in X$*,* $p > 0$ *and* $t > 0$

Definition 26 *[3] A fuzzy 2–metric space is said to be Complete if every Cauchy sequence is convergent.*

Example 2.1 *Let* $X = [0,\infty)$*,* $a*b = \min(a,b)$ *for all* $a,b \in [0,1]$ *, Let* $F(x,y,z,t) = \frac{t}{t+d(x,y,z)}$ *for all* $x,y,z \in X$ *and* $t > 0$*. Then* $(X,F,*)$ *is fuzzy 2–metric space .*

Definition 27 *[3] The class of all* ψ *function will be denoted by* Ψ *if* $\psi : [0,1] \longrightarrow [0,1]$ *such that* ψ *is continuous nondecreasing function and* $\psi(t) > t$ *for all* $t \in (0,1)$*.*

Definition 28 *Let* $(X,F,*)$ *be fuzzy 2–metric space and* $\psi \in \Psi$*. A mapping* $T : X \longrightarrow X$ *is said to be* ψ*–contractive mapping on fuzzy 2–metric space if verify the condition:*

$$F(x,y,z,t) > 0 \text{ then } F(T(x),T(y),z,t) \geq \psi(F(x,y,z,t)).$$

Definition 29 *[14] Let A and B be a self mappings of fuzzy 2–metric space* $(X,F,*)$ *These mappings are called weakly commuting if*

$$F(ABx,BAx,z,t) \geq F(Ax,Bx,z,t) \text{ for all } x \in X \text{ and } t > 0.$$

Lemma 2.1 If $\psi \in \Psi$. Then $\psi(1) = 1$.

Lemma 2.2. $\psi \in \Psi$. Then $\lim_{n\longrightarrow\infty^+} \psi^n(t) = 1$ for all $t \in (0,1)$.

Proposition 2.1 Let $(X,F,*)$ be a fuzzy 2–metric space achieving the condition $F(x,y,z,t) > 0$ $\forall t > 0$ and $T : X \longrightarrow X$ be ψ–contractive mapping on fuzzy 2–metric space . Then T has at most one fixed point.

Proof. If x,y are fixed point T such that $T(x) = x$, $T(y) = y$,then

$$F(T(x),T(y),z,t) \geq \psi(F(x,y,z,t)) \quad \forall t > 0$$

Suppose that $x \neq y$. Then $F(x,y,z,s) < 1$ for some $s > 0$, that is, double inequality: $0 < F(x,y,z,s) < 1$ holds, then

$$F(x,y,z,s) \geq \psi(F(x,y,z,s)) > F(x,y,z,s)$$

this is contradiction, therefore, $x = y$.

3 Main results

Definition 31*Let $(X,F,*)$ be fuzzy 2–metric space if X be an arbitrary set ,$*$ is continuous t-norm and F fuzzy set on $X^3\times[0,\infty)$ .Let $A,B:X\longrightarrow X$, (A,B) is a pair of ψ–contractive mappings on fuzzy 2–metric space if there exists $\psi\in\Psi$ such that for all $x,y,z\in X$ and $t\in(0,1)$ with $F(x,y,z,t)>0$ the following condition holds:*

$$F(A(x),B(y),z,t)\geq\psi(\min\{F(x,y,z,t),F(A(x),x,z,t),F(y,B(y),z,t)\}).$$

Lemma 3.1 Let $(X,F,*)$ be fuzzy 2–metric space. Let $A,B:X\longrightarrow X$ and assume that $F(x_o,A(x_o),z,t)>0$. Then $\lim_{n\longrightarrow\infty^+}F(x_{n+1},x_n,z,t)=1$ where $\{x_n\}$ is the(A,B)–sequence of initial point x_o.

Proof. Let $x_o\in X$ be an arbitrary point and we define the sequence $\{x_n\}$ by $x_1=A(x_o)$, $x_2=B(x_1)$, ..., $x_{2n+1}=A(x_{2n})$, $x_{2n+2}=B(x_{2n+1})$, ...
If $F(x_{n+1},x_n,z,t)=1\quad\forall n\in N$, then $F(x_{m+1},x_m,z,t)=1\quad\forall m>n$. Assume that $F(x_{n+1},x_n,z,t)<1\quad\forall n\in N$, from $F(x_o,A(x_o),z,t)=F(x_o,x_1,z,t)>0$, it follows that $F(x_2,x_1,z,t)=F(B(x_1),A(x_o),z,t)$

$$\geq\psi(\min\{F(x_1,x_o,z,t),F(A(x_o),x_o,z,t),F(x_1,B(x_1),z,t)\})$$

$$\geq\psi(F(x_o,x_1,z,t))$$

$$F(x_3,x_2,z,t)=F(A(X_2),B(x_1),z,t)$$

$$\geq\psi(\min\{F(x_2,x_1,z,t),F(A(x_2),x_2,z,t),F(x_1,B(x_1),z,t)\})$$

$$\geq\psi(F(x_2,x_1,z,t))$$

$$\geq\psi^2(F(x_o,x_1,z,t))$$

for each n, we get, $F(x_{n+1},x_n,z,t)\geq\psi^n(F(x_o,x_1,z,t))$. By lemma 2.2 as $n\longrightarrow\infty$ we obtain $\lim_{n\longrightarrow\infty}F(x_{n+1},x_n,z,t)=1$.

Theorem 1. *Let A and B be self mappings of a complete fuzzy 2–metric space $(X,F,*)$ such that $a*b=\min(a,b)$ for all $a,b\in X$. Assume $F(x_o,A(x_o),z,t)>0$ for $x_o\in x$ and $t>0$ satisfying the conditions:*

(1)A,B are two continuous and ψ–contractive mappings on fuzzy 2–metric space.
(2)$F(A(x),B(y),z,t)\geq(\min\{F(x,y,z,t),F(A(x),x,z,t),F(y,B(y),z,t)\})\ \forall x,y,z\in X$ and $t>0$.
(3)$\{x_n\}$ is (A,B)–sequence of initial point x_o.

Then A and B have a unique common fixed point in X.

Proof. *From (3) $\{x_n\}$ is (A,B)–sequence of initial point x_o.*
Let $x_o\in X$ and we define sequence $\{x_n\}$ by $x_1=A(x_o),x_2=B(x_1),\ldots,x_{2n+1}=A(x_{2n}),x_{2n+2}=B(x_{2n+1}),\ldots$ from lemma 3.1. We obtain $\{x_n\}$ is cauchy sequence.Since X is complete, then there exist point $x\in X$ such that $x_n\longrightarrow x$ for all $t>0$,

$$F(x_{2n+1},A(x),z,t)=F(A(x_{2n}),A(x),z,t)$$

$$\geq\psi(F(x_{2n},x,z,t))$$

Since A is continuous, letting $n\longrightarrow\infty$, it follow that

$$F(x,A(x),z,t)\geq\psi(1)=1.$$

Then $A(x)=x$. Similiarity, we get $B(x)=x$ and x is a common fixed point of A and B. Now we proved the uniqueness of the common fixed points of A and B. Assume that $x,y\in X$ are two common fixed points $t>0$ such that $0<F(x,y,z,t)<1$ and $F(x,y,z,t)=F(A(x),B(y),z,t)$

$$\geq\psi(\min\{F(x,y,z,t),F(A(x),x,z,t),F(y,B(y),z,t)\})$$

$$\geq\psi(F(x,y,z,t))>F(x,y,z,t)$$

this is contradiction. Therefore $x=y$.
This completes the proof.

Theorem 2. *Let $(X,F,*)$ be a complete fuzzy 2–metric space such that $a*b=\min(a,b)$ for all $a,b\in X$ and let A,B and T be self mappings of X satisfying the following conditions:*

(1)A is ψ–contractive mapping on fuzzy 2–metric space and A,T are two continuous mappings,
(2)$A(x)\subset B(x)\cap T(x)$ and (A,B), (A,T) are weakly commuting,
(3)$F(Ax,Ay,z,t)\geq\psi(\min\{F(Bx,Ty,z,t),F(Bx,Ax,z,t),F(Bx,Ay,z,t),$

$$F(Ty,Ay,z,t)\}) \qquad \forall x,y,z\in X \text{ and } t>0,$$

Then A,B and T have a unique common fixed point in X.

Proof. *Let $x_o\in X$ be an arbitrary point. Since $A(x)\subset B(x)$, then there exists a point $x_1\in X$ such that $Ax_o=Bx_1$. Also, since $A(X)\subset T(x)$, there exists another point $x_2\in X$ such that $Ax_1=Tx_2$.*
In general, we get a sequence $\{y_n\}$ repeated as $y_n=Bx_{n+1}=Ax_n$ and $y_{n+1}=Tx_{n+2}=Ax_{n+1}$ $n\in N$. Let $F_n=F(y_{n+1},y_n,z,t)=F(Ax_{n+1},Ax_n,z,t)$ and $F(y_o,y_1,z,t)>0$. Then, $F_{n+1}=F(y_{n+2},y_{n+1},z,t)=F(Ax_{n+2},Ax_{n+1},z,t)$
From (3), we get,

$$\begin{aligned}F_{n+1}=F(Ax_{n+2},Ax_{n+1},z,t)&\geq\psi(\min\{F(Bx_{n+2},Tx_{n+1},z,t),F(Bx_{n+2},Ax_{n+2},z,t)\\ &\quad F(Bx_{n+2},Ax_{n+1},z,t),F(Tx_{n+1},Ax_{n+1},z,t)\}),\\ &=\psi(\min\{F(Ax_{n+1},Ax_n,z,t),F(Ax_{n+1},Ax_{n+2},z,t)\\ &\quad F(Ax_{n+1},Ax_{n+1},z,t),F(Ax_n,Ax_{n+1},z,t)\}),\\ &=\psi(\min\{F_n,F_{n+1},1,F_n\})\end{aligned}$$

If $F_n>F_{n+1}$, then by definition of ψ function, we have $F_{n+1}\geq\psi(F_{n+1})>F_{n+1}$
this is contradiction, therefore

$$F_{n+1}\geq\psi(F_n)$$

Thus, we get,

$$F(y_{n+2},y_{n+1},z,t)\geq\psi(F(y_{n+1},y_n,z,t))\ \forall n\in N,\ t>0.$$

Hence repeating this inequality n times, we get

$$F(y_n,y_{n+1},z,t)\geq\psi^n(F(y_o,y_1,z,t))$$

Letting $n\longrightarrow\infty$ and from lemma 2.1. We get $\lim_{n\longrightarrow\infty}F(y_n,y_{n+1},z,t)=1$.
$\{y_n\}$ is a cauchy aequence. Since X is a complete fuzzy 2–metric space, then there exists a point $p\in X$ such that $y_n\longrightarrow p$. Hence $(Ax_n)\longrightarrow p$. Since A is ψ–contractive mapping on F2-M space.

$$F(y_n,Ap,z,t)=F(A(x_n),Ap,z,t)\geq\psi(F(x_n,p,z,t))$$

By taking limit as $n\longrightarrow\infty$, we get $\lim_{n\longrightarrow\infty}F(p,Ap,z,t)=1$.Hence $Ap=p$. Since $(Ax_n)\longrightarrow p\in X$.Hence the subsequence (Bx_n) and (Tx_n) of (Ax_n) have the same limit. Since B is continuous, then we have $BAx_n\longrightarrow BA$, $BBx_n\longrightarrow Bp$. Also (A,B) is weakly commuting, we have $ABx_n\longrightarrow Bp$. Let $x=Bx_n$, $y=x_n$ from the inequality (3), we get,

$$\begin{aligned}F(ABx_n,Ax_n,z,t)\geq\psi(\min\{&F(BBx_n,ABx_n,z,t),F(BBx_n,Tx_n,z,t)\\ &F(BBx_n,Ax_n,z,t),F(Tx_n,Ax_n,z,t)\}).\end{aligned}$$

Taking limit $n\longrightarrow\infty$

$$\begin{aligned}F(Bp,p,z,t)\geq\psi(\min\{&F(Bp,Bp,z,t),F(Bp,Bp,z,t)\\ &F(Bp,p,z,t),F(p,p,z,t)\})\end{aligned}$$

Therefore $Bp=p$. Since T is continuous, then we have,$TTx_n\longrightarrow Tp$, $TAx_n\longrightarrow Tp$. Also (A,T) is weakly commuting, $ATx_n\longrightarrow Tp$ from the inequality (3), we get,

$$\begin{aligned}F(Ax_n,ATx_n,z,t)\geq\psi(\min\{&F(Bx_n,TTx_n,z,t),F(Bx_n,Ax_n,z,t)\\ &F(Bx_n,ATx_n,z,t),F(TTx_n,ATx_n,z,t)\})\end{aligned}$$

Taking limit $n \longrightarrow \infty$, we get,

$$F(p,Tp,z,t) \geq \psi(\min\{F(p,Tp,z,t),F(p,p,z,t),F(p,Tp,z,t)F(Tp,Tp,z,t)\})$$
$$= \psi(\min\{F(p,Tp,z,t),1,F(p,Tp,z,t),1\})$$
$$= \psi(F(p,Tp,z,t)) > F(p,Tp,z,t)$$

Therefore $Tp = p$. Hence p is common fixed point of A, B, T. Now we proved the uniqueness of common fixed point of A, B, T. Let v be another common fixed point of A, B, T, then $Av = Bv = Tv = v$. Put $x = p$, $y = v$ from (3), we get,

$$F(p,v,z,t) \geq \psi(\min\{F(Bp,Tv,z,t),F(Bp,Ap,z,t),F(Bp,Av,z,t),F(Tv,Ap,z,t)\})$$
$$= \psi(F(p,v,z,t)) > F(p,v,z,t)$$

Then $v = p$, therefore p is a unique common fixed point A,B and T.
This completes the proof.

Corollary 3.1 Let $(X,F,*)$ be a complete fuzzy 2–metric space such that $a*b = \min(a,b)$ for all $a,b \in X$. Let A and B be self mappings of X verify the conditions:

(1)A *is* ψ–contractive mappings on fuzzy 2–metric space and B is a continuous mapping;
(2)$A(x) \subset B(x)$ and (A,B) is weakly commuting;
(3)$F(Ax,Ay,z,t) \geq \psi(\min\{F(Bx,By,z,t),F(Bx,Ax,z,t),F(Bx,Ay,z,t),$

$$F(By,Ay,z,t)\}) \qquad \forall x,y,z \in X \text{ and } t > 0$$

Then A and B have a unique common fixed point in X.

References

[1] Zadeh, Lotfi A. "Fuzzy sets, information and control." vol 8 (1965): 338-353.
[2] Kramosil, Ivan, and Jiří Michálek. "Fuzzy metrics and statistical metric spaces." Kybernetika 11, no. 5 (1975): 336-344.
[3] Chauhan, Surjeet Singh, and Kiran Utreja. "A Common Fixed Point Theorem in Fuzzy 2-Metric Space." Int. J. Contemp. Math. Sciences 8, no. 2 (2013): 85-91.
[4] George, A., and P. Veeramani. "On some results in fuzzy metric spaces." Fuzzy sets and systems 64, no. 3 (1994): 395-399.
[5] Iséki, K., P. L. Sharma, and B. K. Sharma. "Contraction type mapping on 2-metric space." Math. Japon 21, no. 1 (1976): 67-70.
[6] Sharma, Sushil. "On fuzzy metric space." Southeast Asian Bulletin of Mathematics 26, no. 1 (2003): 133-145.
[7] Ansari, Zaheer K., Rajesh Shrivastava, Gunjan Ansari, and Arun Garg. "Some Fixed Point Theorems in" Fuzzy 2-Metric and Fuzzy 3-Metric Spaces." Int. J. Contemp. Math. Sciences 6, no. 46 (2011): 2291-2301.
[8] Han, Jinkyu. "A common fixed point theorem on fuzzy 2-metric spaces." Journal of the Chung Cheong Mathematical Society 23, no. 4 (2010).
[9] Kumar, Sanjay. "Common fixed point theorem in fuzzy 2-metric spaces." Uni. Din. Bacau. Studii Si Cercetiri Sciintifice, Serial: Mathematical Nr 18 (2008): 111-116.
[10] Tokse, Rajesh, Sanjay Choudhari, and Kamal Wadhwa. "Common Fixed Point Theorem in Fuzzy 2-Metric Space Using Implicit Relation." International Journal of Theoretical and Applied Sciences 6, no. 1 (2014): 117.
[11] Abu-Donia, H. M., Atia, H. A., & Khater, O. M. (2020). Fixed point theorem by using Ψ–contraction and (φ,ϕ)–contraction in probabilistic 2–metric spaces. Alexandria Engineering Journal.
[12] Pant, R. P. "Common fixed points of four mappings." Bull. Cal. Math. Soc. 90 (1998): 281-286.
[13] Jungck, Gerald, and B. E. Rhoades. "Fixed points for set valued functions without continuity." Indian Journal of pure and applied mathematics 29 (1998): 227-238.
[14] Turkoglu, Duran, and Muzeyyen Sangurlu. "Fixed point theorems for fuzzy ψ-contractive mappings in fuzzy metric spaces." Journal of Intelligent & Fuzzy Systems 26, no. 1 (2014): 137-142.