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Solution of new nonlinear second order singular perturbed Lane-Emden equation by the numerical spectral collocation method

M. A. Abdelkawy[1,2,*] *and Zulqurnain Sabir*[3]

[1]Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt
[2]Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University, Riyadh, Saudi Arabia
[3]Department of Mathematics and Statistics, Hazara University Mansehra, Pakistan

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Abstract: The aim of the present study is to design a new nonlinear second order singularly perturbed Lane-Emden equation and report numerical solutions by using a well-known spectral collocation technique. The idea of the present study has been taken from the standard Lane-Emden equation. For the model validation, three different examples based on the singular perturbed Lane-Emden equation along with three cases have been presented and the solutions are numerically investigated by using a spectral collocation technique. Comparison of the present outcomes with the exact solutions shows the exactness, correctness and stability of the designed model as well as the present scheme. Moreover, absolute error and convergence are derived in the form of plots as well as tables.

Keywords: Nonlinear singular perturbed Lane-Emden; spectral collocation technique; shape factor.

1 Introduction

Singular Lane-Emden equation was introduced first time by famous astrophysicist Jonathan Homer Lane [1] and Robert Emden [2] working on the thermal execution of a spherical cloud of gas and thermodynamics classical laws [3]. The models based on singularity describe a various phenomena in the studies of physical science [4], catalytic diffusion reactions [5], density profile of gaseous star [6], isothermal gas spheres [7], stellar structure [8], catalytic diffusion reactions [9], electromagnetic theory [10], classical and quantum mechanics [11], mathematical physics [12], oscillating magnetic fields [13], morphogenesis[16], isotropic continuous media [14] and dusty fluid models [15].

It is always very challenging, difficult and hard to find the numerical and analytic solutions of the singular models because of the singularity at the origin. There are few existing methods to handle such nonlinear singular Lane-Emden type of models. To mention some of the methods for solving the singular equations, Bender et al [16] suggested a perturbative scheme. Shawagfeh [17] proposed Adomian decomposition technique (ADT). Wazwaz [18] also implemented ADT to avoid the singularity difficulty. Liao [19] used an analytic scheme to handle the singularity, Parand and Razzaghi [20] discussed a numerical approach for the solution of singular models. Nouh [21] established power series technique by using the Pade approximation method as well as Euler-Abel transformation.

The present study is relevant to model the nonlinear singular perturbed (SP) Lane-Emden equation. The modeled form of the SP Lane-Emden equation is written as:

$$\begin{aligned} & x^{-\eta}\frac{d}{dx}\left(\varepsilon x^{\eta}\frac{d}{dx}\right)\mathscr{Y}+\mathscr{G}(\mathscr{Y})=\mathscr{H}(x), \\ & y(0)=A, \quad y^{'}(0)=0, \end{aligned} \tag{1}$$

* Corresponding author e-mail: melkawy@yahoo.com

the updated form of equation (1) is

$$\begin{aligned} &\varepsilon \mathscr{Y}''(x) + \frac{\varepsilon\eta}{x}\mathscr{Y}'(x) + \mathscr{F}(x)\mathscr{G}(\mathscr{Y}) = \mathscr{H}(x), \\ &y(0) = A, \quad y'(0) = 0, \end{aligned} \tag{2}$$

where $0 \leq \varepsilon \leq 1$, $\eta \geqslant 1$ is the shape factor and $\mathscr{F}(x)$, $\mathscr{G}(\mathscr{Y})$, $\mathscr{H}(x)$ are given functions. The nonlinear SP Lane-Emden is basically obtained from the standard Lane-Emden equation. The designed model (2) is verified by solving the four different examples based on nonlinear SP second order Lane-Emden equations using the numerical spectral collocation technique. The major features of the present study are investigated as:

- –The mathematical model of the nonlinear SP Lane-Emden equation is successfully designed and verified by solving three examples using the spectral collocation method.
- –For the correctness and exactness of the designed model, the obtained numerical outcomes from the numerical spectral collocation technique are compared with the exact results.
- –Manipulation of the designed technique by applying the designed SP Lane-Emden model provided genius outcomes with greater accuracy and larger reliability.
- –The reliability, exactness and correctness of the SP Lane-Emden model is verified through the reliable and consistent absolute error values of the present and exact outcomes.
- –The nonlinear singular second order perturbed Lane-Emden model is always difficult to tackle numerically because of perturbed form, singularity, non-linearity. However, the spectral collocation technique is one of the best selection and great choice to challenge such types of complex models.

Recently, there are more interest of appointing the spectral methods to treat with various kinds of differential and integral equations [22,23,24,25], due to their applicability to finite and infinite domains [26,27,28,29]. The convergence speed, exponential convergence rates and high accuracy level are the major advantages of spectral method. The spectral method has been classified to four classes, collocation [31], tau [32], galerkin [33] and Petrov galerkin [34] method. The collocation ones is a particular kind of spectral methods, that is widely applicable for almost types of differential equations.

Our paper is coordinated as next. Firstly, facts about shifted Jacobi polynomials are mentioned. The mentioned approach is executed for the nonlinear second order singular perturbed Lane-Emden in section 3. Three test examples are solved in Section 4. Finally, conclusions are outlined.

2 Shifted Jacobi polynomials

We consider the Jacobi polynomials $\mathscr{J}_k^{(\rho,\sigma)}(x)$, which satisfy the following properties:

$$\begin{aligned} &\mathscr{J}_{k+1}^{(\rho,\sigma)}(x) = (a_k^{(\rho,\sigma)}x - b_k^{(\rho,\sigma)})\mathscr{J}_k^{(\rho,\sigma)}(x) - c_k^{(\rho,\sigma)}\mathscr{J}_{k-1}^{(\rho,\sigma)}(x), k \geq 1, \\ &\mathscr{J}_0^{(\rho,\sigma)}(x) = 1, \mathscr{J}_1^{(\rho,\sigma)}(x) = \frac{1}{2}(\rho+\sigma+2)x + \frac{1}{2}(\rho-\sigma), \\ &\mathscr{J}_k^{(\rho,\sigma)}(-x) = (-1)^k\mathscr{J}_k^{(\rho,\sigma)}(x), \mathscr{J}_k^{(\rho,\sigma)}(-1) = \frac{(-1)^k\Gamma(k+\sigma+1)}{k!\Gamma(\sigma+1)}, \end{aligned} \tag{3}$$

where $\rho,\ \sigma > -1$, $x \in [-1,1]$ and

$$\begin{aligned} a_k^{(\rho,\sigma)} &= \frac{(2k+\rho+\sigma+1)(2k+\rho+\sigma+2)}{2(k+1)(k+\rho+\sigma+1)}, \\ b_k^{(\rho,\sigma)} &= \frac{(\sigma^2-\rho^2)(2k+\rho+\sigma+1)}{2(k+1)(k+\rho+\sigma+1)(2k+\rho+\sigma)}, \\ c_k^{(\rho,\sigma)} &= \frac{(k+\rho)(k+\sigma)(2k+\rho+\sigma+2)}{(k+1)(k+\rho+\sigma+1)(2k+\rho+\sigma)}. \end{aligned}$$

the rth derivative of $\mathscr{J}_j^{(\rho,\sigma)}(x)$, is computed as

$$D^r\mathscr{J}_j^{(\rho,\sigma)}(x) = \frac{\Gamma(j+\rho+\sigma+q+1)}{2^r\Gamma(j+\rho+\sigma+1)}\mathscr{J}_{j-r}^{(\rho+r,\sigma+r)}(x), \tag{4}$$

where r is an integer. For the shifted Jacobi polynomial $\mathscr{J}_{\mathscr{L},k}^{(\rho,\sigma)}(x) = \mathscr{J}_{k}^{(\rho,\sigma)}(\frac{2x}{\mathscr{L}}-1)$, $\mathscr{L}>0$, the explicit analytic form is written as

$$\begin{aligned} P_{\mathscr{L},k}^{(\rho,\sigma)}(x) &= \sum_{j=0}^{k}(-1)^{k-j}\frac{\Gamma(k+\sigma+1)\Gamma(j+k+\rho+\sigma+1)}{\Gamma(j+\sigma+1)\Gamma(k+\rho+\sigma+1)(k-j)!j!\mathscr{L}^{j}}x^{j} \\ &= \sum_{j=0}^{k}\frac{\Gamma(k+\rho+1)\Gamma(k+j+\rho+\sigma+1)}{j!(k-j)!\Gamma(j+\rho+1)\Gamma(k+\rho+\sigma+1)\mathscr{L}^{j}}(x-\mathscr{L})^{j}. \end{aligned} \tag{5}$$

Thereby, we conclude the next

$$\begin{aligned} P_{\mathscr{L},k}^{(\rho,\sigma)}(0) &= (-1)^{k}\frac{\Gamma(k+\sigma+1)}{\Gamma(\sigma+1)\,k!}, \\ \mathscr{J}_{\mathscr{L},k}^{(\rho,\sigma)}(\mathscr{L}) &= \frac{\Gamma(k+\rho+1)}{\Gamma(\rho+1)\,k!}, \end{aligned} \tag{6}$$

$$D^{r}\mathscr{J}_{\mathscr{L},k}^{(\rho,\sigma)}(0) = \frac{(-1)^{k-r}\Gamma(k+\sigma+1)(k+\rho+\sigma+1)_{r}}{L^{r}\Gamma(k-r+1)\Gamma(r+\sigma+1)}, \tag{7}$$

$$D^{r}\mathscr{J}_{\mathscr{L},k}^{(\rho,\sigma)}(\mathscr{L}) = \frac{\Gamma(k+\rho+1)(k+\rho+\sigma+1)_{r}}{L^{r}\Gamma(k-r+1)\Gamma(r+\rho+1)}, \tag{8}$$

$$D^{r}\mathscr{J}_{\mathscr{L},k}^{(\rho,\sigma)}(x) = \frac{G(r+k+\rho+\sigma+1)}{\mathscr{L}^{r}G(k+\rho+\sigma+1)}\mathscr{J}_{\mathscr{L},k-r}^{(\rho+r,\sigma+r)}(x). \tag{9}$$

3 Methodology of Shifted Jacobi collocation method

The collocation technique is the simplest of the weighted residuals method. For the first time, Lanczos [35] mentioned that a convenient selection of the trial function and the collocation nodes are pivotal to the solution accuracy. This work was refreshed by the authors in [36,37,38].These works included applications of Chebyshev polynomial approaches to initial value problems. Here, we introduce a numerical method based on shifted Jacobi collocation method to solve new nonlinear second order singular perturbed Lane-Emden:

$$\varepsilon\mathscr{Y}''(x) + \frac{\varepsilon\eta}{x}\mathscr{Y}'(x) + \mathscr{F}(x)\mathscr{G}(\mathscr{Y}) = \mathscr{H}(x),\ \eta>0, < x < \mathscr{L}, \tag{10}$$

related to the initial conditions

$$\mathscr{Y}(0)=\zeta_{1}, \quad \mathscr{Y}'(0)=\zeta_{2}. \tag{11}$$

The solution of Eq. (10) is approximated as.

$$\mathscr{Y}_{\mathscr{K}}(x) = \sum_{j=0}^{\mathscr{K}}\varsigma_{j}\mathscr{J}_{\mathscr{L},j}^{(\rho,\sigma)}(x) = \Delta_{\mathscr{L},\mathscr{K}}^{(\rho,\sigma)}(x). \tag{12}$$

We approximate the independent variable using shifted Jacobi collocation method at $x_{\mathscr{L},\mathscr{K},j}^{(\rho,\sigma)}$ nodes. Thus, the required derivatives of first and second orders of the approximate solutions are then estimated as

$$\begin{aligned} \frac{d^{r}\mathscr{Y}(x)}{dx^{r}} = \mathscr{Y}_{\mathscr{K}}^{(r)}(x) &= \sum_{j=0}^{\mathscr{K}}\varsigma_{j}\frac{d^{r}}{dx^{r}}\left(\mathscr{J}_{\mathscr{L},j}^{(\rho,\sigma)}(x)\right) \\ &= \sum_{j=0}^{\mathscr{K}}\varsigma_{j}\frac{G(r+j+\rho+\sigma+1)}{\mathscr{L}^{r}G(j+\rho+\sigma+1)}\mathscr{J}_{\mathscr{L},j-r}^{(\rho+r,\sigma+r)}(x) \\ &= \wp_{\mathscr{L},\mathscr{K}}^{(\rho,\sigma,r)}(x), \end{aligned} \tag{13}$$

then, we can estimated the residual of (10) as

$$\varepsilon\wp_{\mathscr{L},\mathscr{K}}^{(\rho,\sigma,2)}(x) + \frac{\varepsilon\eta}{x}\wp_{\mathscr{L},\mathscr{K}}^{(\rho,\sigma,1)}(x) + \mathscr{F}(x)\mathscr{G}(\Delta_{\mathscr{L},\mathscr{K}}^{(\rho,\sigma)}(x)) = \mathscr{H}(x). \tag{14}$$

For the current method, the residual (14) is permit to be zero at $x^{(\rho,\sigma)}_{\mathscr{L},\mathscr{K},j}$ (Jacobi-Gauss-Lobatto nodes)

$$\begin{aligned}&\varepsilon\wp^{(\rho,\sigma,2)}_{\mathscr{L},\mathscr{K}}(x^{(\rho,\sigma)}_{\mathscr{L},\mathscr{K},i})+\frac{\varepsilon\eta}{x^{(\rho,\sigma)}_{\mathscr{L},\mathscr{K},i}}\wp^{(\rho,\sigma,1)}_{\mathscr{L},\mathscr{K}}(x^{(\rho,\sigma)}_{\mathscr{L},\mathscr{K},i})\\&+\mathscr{F}(x^{(\rho,\sigma)}_{\mathscr{L},\mathscr{K},i})\mathscr{G}(\Delta^{(\rho,\sigma)}_{\mathscr{L},\mathscr{K}}(x^{(\rho,\sigma)}_{\mathscr{L},\mathscr{K},i}))=\mathscr{H}(x^{(\rho,\sigma)}_{\mathscr{L},\mathscr{K},i}),\end{aligned} \tag{15}$$

where $i=1,2,\ldots,\mathscr{K}-1$. So, we have $\mathscr{K}-1$ algebraic equations for $\mathscr{K}+1$ unknowns, the remaining equations can obtained from the conditions (11) as

$$\Delta^{(\rho,\sigma)}_{\mathscr{L},\mathscr{K}}(0)=\zeta_1,\quad \wp^{(\rho,\sigma,1)}_{\mathscr{L},\mathscr{K}}(0)=\zeta_2, \tag{16}$$

A system of nonlinear algebraic equations are acquired from Eqs. (15) and (16). This system may be solved for unknown coefficients ς_j, $j=0,\ldots,\mathscr{K}$.

4 Numerical results and comparisons

Using the algorithm presented in the previous section, we give in this section some numerical results. For the model validation, three different examples based on the singular perturbed Lane-Emden equation along with three cases have been presented and the solutions are numerically investigated by using a spectral collocation technique. Comparison of the present outcomes with the exact solutions shows the exactness, correctness and stability of the designed model as well as the present scheme. Moreover, absolute error and convergence are derived in the form of plots as well as tables.

4.1 Problem I

We start the following nonlinear second order singular perturbed Lane-Emden

$$\begin{aligned}&\varepsilon\mathscr{Y}''(x)+\frac{\varepsilon}{x}\mathscr{Y}'(x)+\mathscr{Y}=x^5-x^4+25x^3\varepsilon-16x^2\varepsilon,\ 0\le x\le 1,\\&\mathscr{Y}(0)=0,\quad \mathscr{Y}'(0)=0,\end{aligned} \tag{17}$$

the exact solution is given by $\mathscr{Y}(x)=x^5-x^4$.
In Table (1), we listed the numerical solutions ($\mathscr{Y}_{\mathscr{K}}$) of Problem I in case of taking $\mathscr{K}=5$, and different values of parameters (ρ,σ). We note from the results in Table (1), that we have obtained more accurate results. Also, we see in Fig. 1 the prefect matching of the approximate and exact solutions. In Fig. 2, the curve of the absolute errors (E) of Problem I is displayed.

4.2 Problem II

Here, we test the following nonlinear second order singular perturbed Lane-Emden

$$\begin{aligned}&\varepsilon\mathscr{Y}''(x)+\frac{\varepsilon}{x}\mathscr{Y}'(x)+e^{\mathscr{Y}}=e^{x^3+1}+9x\varepsilon,\ 0\le x\le 1,\\&\mathscr{Y}(0)=1,\quad \mathscr{Y}'(0)=0,\end{aligned} \tag{18}$$

the exact solution is given by $\mathscr{Y}(x)=1+x^3$.

In Table (2), we listed the numerical solutions ($\mathscr{Y}_{\mathscr{K}}$) of Problem II in case of taking $\mathscr{K}=3$, and different values of parameters (ρ,σ). We note from the results in Table (2), that we have obtained more accurate results.

Table 1: Numerical solutions of Problem I.

(ρ,σ)	$\varepsilon=0.01.$
(0,0)	$-1.04083\times10^{-17}-5.55112\times10^{-17}x+6.10623\times10^{-16}x^2-2.22045\times10^{-15}x^3-x^4+x^5$
$(\frac{1}{2},\frac{1}{2})$	$-1.38778\times10^{-17}+2.77556\times10^{-17}x-8.04912\times10^{-16}x^2+5.32907\times10^{-15}x^3-x^4+x^5$
$(\frac{1}{2},-\frac{1}{2})$	$-1.11022\times10^{-16}x^2+1.38778\times10^{-15}x^3-x^4+x^5$
$(-\frac{1}{2},-\frac{1}{2})$	$8.67362\times10^{-18}-2.77556\times10^{-17}x-7.91034\times10^{-16}x^2+2.22045\times10^{-15}x^3-x^4+x^5$
	$\varepsilon=0.03.$
(0,0)	$-3.60822\times10^{-16}x^2-x^4+x^5$
$(\frac{1}{2},\frac{1}{2})$	$2.77556\times10^{-17}x+2.39392\times10^{-16}x^2-8.88178\times10^{-16}x^3-x^4+x^5$
$(\frac{1}{2},-\frac{1}{2})$	$6.93889\times10^{-18}+2.77556\times10^{-16}x^3-x^4+x^5$
$(-\frac{1}{2},-\frac{1}{2})$	$2.77556\times10^{-17}+8.32667\times10^{-17}x-9.15934\times10^{-16}x^2+3.55271\times10^{-15}x^3-x^4+x^5$
	$\varepsilon=0.05.$
(0,0)	$-3.1225\times10^{-17}+6.93889\times10^{-17}x^2+4.44089\times10^{-16}x^3-x^4+x^5$
$(\frac{1}{2},\frac{1}{2})$	$1.38778\times10^{-17}-2.77556\times10^{-17}x+2.01228\times10^{-16}x^2-8.88178\times10^{-16}x^3-x^4+x^5$
$(\frac{1}{2},-\frac{1}{2})$	$2.77556\times10^{-17}+2.77556\times10^{-17}x-1.11022\times10^{-16}x^2+3.33067\times10^{-16}x^3-x^4+x^5$
$(-\frac{1}{2},-\frac{1}{2})$	$-3.81639\times10^{-17}+5.55112\times10^{-17}x+1.11022\times10^{-16}x^2-x^4+x^5$

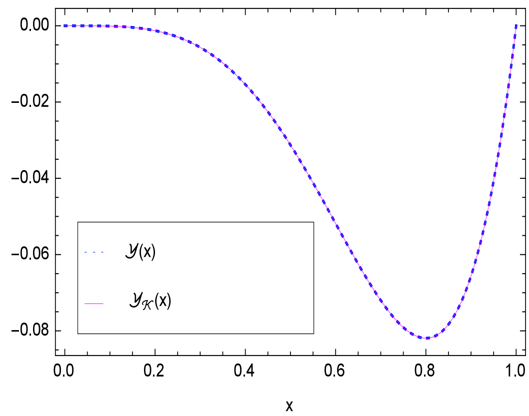

Fig. 1: Curves of the exact and numerical solutions ($\mathscr{Y}$ and $\mathscr{Y}_{\mathscr{K}}$)of Problem I where $\varepsilon=0.05$, $\rho=\sigma=0$, and $\mathscr{K}=5$.

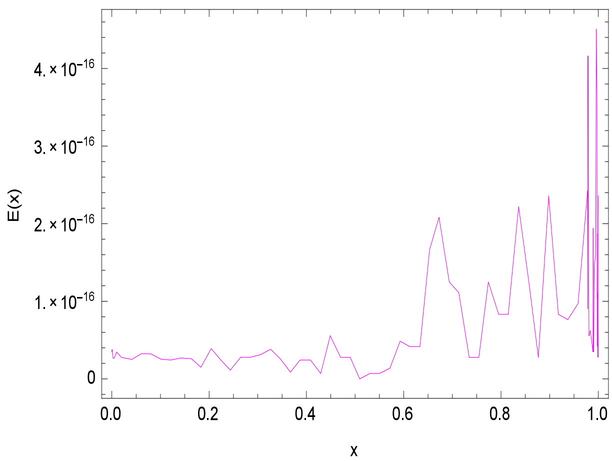

Fig. 2: Curve of the absolute errors (E) of Problem I where $\rho=\sigma=0$, $\varepsilon=0.05$, and $\mathscr{K}=5$.

Table 2: Numerical solutions of Problem II.

(ρ,σ)	$\varepsilon=0.01.$	$\varepsilon=0.03.$
(0,0)	$+1.11022\times10^{-16}x-1.55431\times10^{-15}x^2+x^3$	$-1.11022\times10^{-16}x+1.77636\times10^{-15}x^2+x^3$
$(\frac{1}{2},0)$	$-2.22045\times10^{-16}x^2+x^3$	$+3.10862\times10^{-15}x^2+x^3$
$(\frac{1}{2},\frac{1}{2})$	$-4.44089\times10^{-16}x^2+x^3$	$+2.22045\times10^{-16}x-2.66454\times10^{-15}x^2+x^3$
$(-\frac{1}{2},-\frac{1}{2})$	$+1.77636\times10^{-15}x^2+x^3$	$+4.44089\times10^{-16}x-2.22045\times10^{-15}x^2+x^3$
(ρ,σ)	$\varepsilon=0.05.$	$\varepsilon=0.07.$
(0,0)	$-1.11022\times10^{-16}x+3.55271\times10^{-15}x^2+x^3$	$-2.22045\times10^{-16}x-2.66454\times10^{-15}x^2+x^3$
$(\frac{1}{2},0)$	$-2.22045\times10^{-16}x^2+x^3$	$+2.22045\times10^{-16}x+1.33227\times10^{-15}x^2+x^3$
$(\frac{1}{2},\frac{1}{2})$	$-3.33067\times10^{-16}x-4.66294\times10^{-15}x^2+x^3$	$+1.11022\times10^{-16}x-8.88178\times10^{-16}x^2+x^3$
$(-\frac{1}{2},-\frac{1}{2})$	$-1.11022\times10^{-16}x+6.66134\times10^{-16}x^2+x^3$	$+2.22045\times10^{-16}x-1.11022\times10^{-15}x^2+x^3$

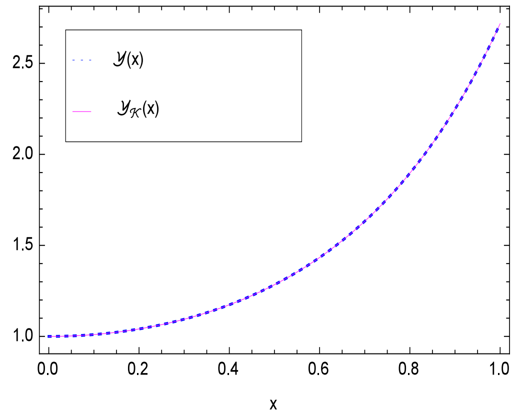

Fig. 3: Curves of the exact and numerical solutions ($\mathscr{Y}$ and $\mathscr{Y}_{\mathscr{K}}$)of Problem III where $\varepsilon=0.01$, $\rho=\sigma=0$, and $\mathscr{K}=18$.

4.3 Problem III

Here, we test the following nonlinear second order singular perturbed Lane-Emden

$$\begin{aligned}
&\varepsilon\mathscr{Y}''(x)+\frac{\varepsilon}{x}\mathscr{Y}'(x)+\mathscr{Y}=4e^{x^2}x^2\varepsilon+4e^{x^2}\varepsilon+e^{x^2},0\leq x\leq 1,\\
&\mathscr{Y}(0)=1,\quad \mathscr{Y}'(0)=0,
\end{aligned}\tag{19}$$

the exact solution is given by $\mathscr{Y}(x)=e^{x^2}$. Table 3 appears the accurate results for the $\mathscr{M}_{E_{\mathscr{Y}}}$ of our method. Also, we see in Fig. 3 the prefect matching of the approximate and exact solutions. In Fig. 4, the curve of the absolute errors (E) of Problem III is displayed. Moreover, we sketched in Figs. 5 the logarithmic graphs of $\mathscr{M}_{E_{\mathscr{Y}}}$ obtained by the present method with different values of ε and $\rho=\sigma=\frac{1}{2}$ and ($\mathscr{K}=2,4,6,\cdots,18$).

Taking $\rho=\sigma=-\frac{1}{2}$, $\varepsilon=0.04$, we obtain the numerical solution of Problem III as:

$$\begin{aligned}
\mathscr{Y}_{18}(x)=&1+2.53232\times10^{-16}x+x^2-9.8576\times10^{-11}x^3+0.5x^4\\
&-6.79226\times10^{-8}x^5+0.166667x^6-6.326\times10^{-6}x^7\\
&+0.0417019x^8-0.000143255x^9+0.00876706x^{10}-\\
&0.00099037x^{11}+0.00310202x^{12}-0.00223737x^{13}\\
&+0.00237687x^{14}-0.00154102x^{15}+0.000779421x^{16}\\
&-0.000232208x^{17}+0.0000376921x^{18}.
\end{aligned}$$

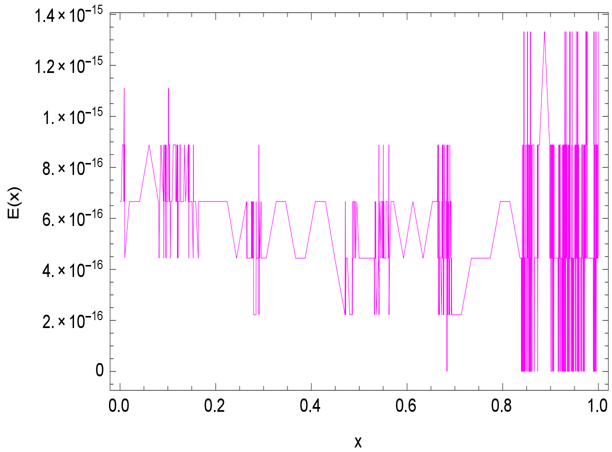

Fig. 4: Curve of the absolute errors (E) of Problem I where $\varepsilon = 0.01$, $\rho = \sigma = 0$, and $\mathscr{K} = 18$.

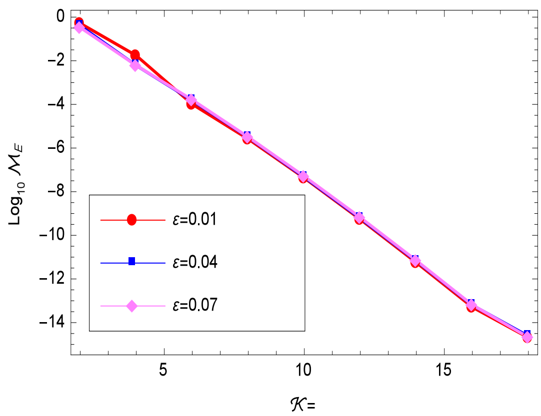

Fig. 5: $\mathscr{M}_{E_{\mathscr{Q}}}$ convergence of Problem III.

Table 3: $\mathscr{M}_{\mathscr{E}_{\mathscr{Y}}}$ and $\mathscr{M}_{\mathscr{E}_{\mathscr{Q}}}$ of Problem III.

	$\varepsilon = 0.01$			
$\mathscr{K}$	$\rho = \sigma = 0$	$\rho = \frac{1}{2}, \sigma = 0$	$\rho = \sigma = \frac{1}{2}$	$\rho = \sigma = -\frac{1}{2}$
2	0.5175	0.553157	0.5175	0.5175
6	1.49218×10^{-4}	1.67181×10^{-4}	1.01688×10^{-4}	2.38637×10^{-4}
10	2.15552×10^{-8}	2.57041×10^{-8}	4.2423×10^{-8}	3.29858×10^{-8}
14	2.54621×10^{-12}	2.7609×10^{-12}	5.61429×10^{-12}	3.39999×10^{-12}
18	1.12882×10^{-15}	1.55431×10^{-15}	2.04666×10^{-15}	1.98936×10^{-15}
	$\varepsilon = 0.04$			
2	0.399183	0.454502	0.399183	0.399183
6	1.15194×10^{-4}	1.16808×10^{-4}	1.51759×10^{-4}	1.38596×10^{-4}
10	2.4376×10^{-8}	2.46543×10^{-8}	4.64442×10^{-8}	2.96568×10^{-8}
14	2.63312×10^{-12}	2.68985×10^{-12}	6.65124×10^{-12}	3.31475×10^{-12}
18	1.24565×10^{-15}	1.60982×10^{-15}	2.38909×10^{-15}	2.4361×10^{-15}

5 Conclusion

The task to model representing the singular nonlinear perturbed Lane-Emden equation along with numerical expressions was difficult to tackle. However the numerical outcomes of the model are described by using the spectral collection technique. Three different examples along with three cases of each example has been numerically solved and compared with the exact solutions that depicts the exactness and correctness of the designed model. The graphs of absolute error and accuracy with good measures for all examples have been plotted. For solving such kind of complicated, linear/nonlinear, singular/non-singular, perturbed Lane-Emden type of problems, the spectral numerical collocation technique can be the best choice to handle. So one can understand that the proposed numerical spectral technique is not only suitable but also effective. The present surveys represent that the numerical spectral collocation technique is an effective, efficient and appropriate technique for solving the nonlinear singular second order perturbed Lane-Emden equations. In future, the proposed technique will be used to solve nonlinear system of perturbed Lane-Emden equations, perturbed third order Lane-Emden equation and system of perturbed Lane-Emden equations.

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