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Controllability of Hilfer Fractional Non-Autonomous Evolution Equations with Nonlocal Initial Conditions

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Abstract: This paper addresses the controllability of a new class of Hilfer fractional non-autonomous evolution equations with nonlocal initial conditions in Banach spaces. The investigation of the aforesaid system mainly depends on both the Krasnoselskii fixed point theorem and the properties of evolution operators. An example is proposed to clarify the theoretical results.

Keywords: Controllability, Hilfer fractional derivative, Krasnoselskii's fixed point theorem.

1 Introduction

In the past two decades, the study of the qualitative properties and applications of fractional differential equations has captured the interest of many authors due to their applications in different branches of applied science and describing physical phenomena more realistically (see [1,2,3,4,5]), especially with the emergence of several new definitions of fractional differentiation (see [6]-[11],[12]). Among these new definitions is the so-called generalized Riemann-Liouville fractional derivative $D^{\alpha,\beta}$ of order α with $0<\alpha<1$ and type β with $0\leq\beta\leq 1$. It has been initially presented by Hilfer [13,14,15], accordingly many authors called it Hilfer fractional derivative. Such derivative includes the Riemann-Liouville and Caputo fractional derivative by introducing only one additional real parameter $\beta\in[0,1]$ (cf. Definition 3). Some properties and applications of Hilfer fractional derivative are presented in [15]. Also, because of the singularities found in the traditional fractional operators which are reckoned to make some difficulties in the modeling process, new types of non-singular fractional operators were proposed in [16,17].

The notion of controllability plays a decisive role in the advance of modern mathematical control theory. Exact controllability allows steering the system to arbitrary final state, whereas approximate controllability implies that the system can steer an arbitrary small neighborhood of final state. There are considerable relationships between observability, controllability, optimal control, and stabilizability in both finite dimensional and infinite dimensional control systems. For more contributions relevant to the controllability of evolution fractional differential equations and inclusions, we refer the interested reader to the recent papers [18,19,20,21,22,23,24,25,26] and the references therein.

Many authors attempted to study approximate controllability of Hilfer fractional evolution equations by supposing that the operator A is an infinitesimal generator of an analytic semigroup. We will briefly list some of these papers.

Recently, Liu et al. [27] investigated the finite approximate controllability of the Hilfer fractional evolution system in Hilbert space

$$\begin{cases} D_{0^+}^{\nu,\mu}x(t)=Ax(t)+f(t,x(t))+Bu(t), & t\in(0,b],\\ I_{0^+}^{(1-\nu)(1-\mu)}x(0)=x_0, & 0<\nu<1,\ 0\leq\mu\leq 1, \end{cases}$$

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Ahmed et al. [28] studied the existence and established a set of sufficient conditions for controllability for the nonlinear delay Hilfer fractional differential equation with impulsive condition

$$\begin{cases} D_{0^+}^{\mu,\nu}x(t) = Ax(t) + Bu(t) + f(t, x(\gamma_1(t)), \int_0^t h(t,s)g(s, x(\gamma_2(s)))ds), \quad t \in (0,b], t \neq t_k \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, 2, \cdots, m \\ I_{0^+}^{(1-\nu)(1-\mu)}x(0) = x_0, \quad 0 < \mu < 1, \; 0 \leq \nu \leq 1, \end{cases}$$

Min Yang and Qi-Ru Wang [29] discussed the approximate controllability of the Hilfer fractional differential inclusions with nonlocal conditions in a Banach space X

$$\begin{cases} D_{0^+}^{\beta;\gamma}x(t) \in Ax(t) + F(t, x(t)) + Bu(t), \quad t \in (0,b], \\ I_{0^+}^{(1-\beta)(1-\gamma)}x(0) + h(x) = x_0, \quad \frac{1}{2} < \gamma < 1, \; 0 \leq \beta \leq 1, \end{cases}$$

Jun Du et al. [30] investigated the approximate controllability of the impulsive fractional differential inclusions involving Hilfer fractional derivative

$$\begin{cases} D_{0^+}^{q,p}x(t) \in Ax(t) + F(t, x(t)) + Bu(t), \quad t \in (0,b], \; t \neq t_k \\ \Delta I_{0^+}^{1-\nu}x(t)|_{t=t_k} = G_k(t_k, x(t_k^-)), \; k = 1, 2, \cdots, m, \\ I_{0^+}^{1-\nu}x(t)|_{t=0} = x_0, \quad \frac{1}{2} < q \leq 1, \; 0 \leq p \leq 1, \; \nu = p + q - pq. \end{cases}$$

In contrast, the controllability of Hilfer fractional differential equations is still in its early stages, especially with regard to non-autonomous fractional differential evolution equations involving a family $\{A(t), t \in J\}$ of closed linear operators which generates an evolution operators $U(t,s), \; 0 \leq s \leq t \leq b$.

In the present work, we discuss the controllability of a new class of non-autonomous evolution equations with Hilfer fractional derivative of the form:

$$\begin{cases} D_{0^+}^{\alpha,\beta}x(t) = A(t)x(t) + Bu(t) + f(t, x(t)), \quad t \in J := [0,b], \; b > 0, \\ I_{0^+}^{1-\gamma}x(0) + g(x) = x_0, \quad \alpha \leq \gamma = \alpha + \beta(1-\alpha) < 1, \end{cases} \tag{1}$$

where $D_{0^+}^{\alpha,\beta}$ denotes the Hilfer fractional derivative of order α, $(0 < \alpha < 1)$ and type β, $(0 \leq \beta \leq 1)$, $I_{0^+}^{1-\gamma}$ denotes the Riemann-Liouville fractional integral of order $1-\gamma$, the state $x(\cdot)$ takes values in the Banach space $(X, \|\cdot\|)$ and $u_0 \in X$. Let $\{A(t), t \in J\}$ be a closed linear operator defined on a dense domain $D(A)$ in X into X such that $D(A)$ is independent of t and $A(t)$ generates an evolution operator $U(t,s), \; 0 \leq s \leq t \leq b$ in the Banach space X, the control function $u(\cdot)$ is given in $L^2(J, U')$, a Banach space of admissible control functions with U' as a Banach space, and B is a bounded linear operator from U to X. Finally, the function $f : J \times X \to X$ satisfies the Carathèodory condition, whereas $g : C_{1-\gamma}(J, X) \to X$ is continuous function.

2 Preliminaries

In this section, we recall some notations, definitions, and lemmas which play a pivotal role throughout the paper.

We set $J = [0,b]$, where $b > 0$ is a constant. Let $C(J,X)$ be the Banach space of all $X-$valued continuous functions from J into X equipped with the norm $\|x\|_C = \sup_{t \in J} \|x(t)\|$, $\forall x \in C(J,X)$.
Let $L^1(J,X)$ be the Banach space of all X-value Bochner integrable functions defined on J with the norm $\|x\|_1 = \int_0^b \|x(t)\| dt$.

Define a weighted space $C_{1-\gamma}(J,X)$ of all $X-$valued continuous functions w by

$$C_{1-\gamma}(J,X) := \{w : J \to X : \; t^{1-\gamma}w(t) \in C(J,X)\}, \quad 0 \leq \gamma < 1.$$

It is clear that $C_{1-\gamma}(J,X)$ is a Banach space with the norm

$$\|w\|_{C_{1-\gamma}} = \|t^{1-\gamma}w(t)\|_C = \sup_{t \in J} \|t^{1-\gamma}w(t)\|.$$

First, we survey some essential definitions of the Riemann-Liouville fractional integral and derivative which will be explained to the Hilfer fractional derivative, see [2,31].

Definition 1. *The left-sided Riemann-Liouville fractional integral of order* $\alpha > 0$ *of a function* $y \in L^1([0,+\infty),\mathbb{R})$ *is defined as*

$$I_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds, \ \ t > 0, \tag{2}$$

where $\Gamma(\cdot)$ *is the Gamma function.*

Definition 2. *The left-sided Riemann-Liouville fractional derivative of order* α *of a function* $y : [0,+\infty) \to \mathbb{R}$ *is defined as*

$$D_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} y(s)\, ds, \ \ t > 0, \tag{3}$$

where $n-1 < \alpha < n, \ n \in \mathbb{N}$.

Remark. If y is an abstract function with values in X, then the integrals in Definitions 1 and 2 are taken in the sense of Bochner. That is a measurable function y maps from $[0,+\infty)$ to X is Bochner integrable, if $\|y\|$ is Lebesgue integrable.

Definition 3. *The right-sided Hilfer fractional derivative operator of order* α $(0 < \alpha < 1)$ *and type* β $(0 \le \beta \le 1)$ *is defined as*

$$D_{0^+}^{\alpha,\beta} y(t) = \left(I_{0^+}^{\beta(1-\alpha)} D \left(I_{0^+}^{(1-\beta)(1-\alpha)} y\right)\right)(t), \tag{4}$$

where $D := \frac{d}{dt}$.

This generalization (4) will be reduced to the Riemann-Liouville fractional derivative as $\beta = 0$. In addition, it will be reduced to the Caputo fractional derivative as $\beta = 1$.
In the light of the Laplace transform of the Hilfer derivative ([32], formula (1.6))

$$\mathscr{L}[D_{0^+}^{\alpha,\beta} y](s) = s^{\alpha} \mathscr{L}[y](s) - s^{\beta(\alpha-1)} (I_{0^+}^{(1-\alpha)(1-\beta)} y)(0^+), \tag{5}$$

it is obvious that the initial conditions must be taken as $(I_{0^+}^{(1-\alpha)(1-\beta)} y)(0^+)$, where $1-\gamma = (1-\alpha)(1-\beta)$.
For more properties and applications of the generalized Riemann-Liouville fractional derivative, we refer the reader to [13].

Next, we recall the following renowned definition.

Definition 4.*(see [33, 34, 35]) The family* $\{A(t), t \in J\}$ *generates a unique linear evolution operator* $U(t,s)$, $0 \le s \le t \le b$ *such that the following statements are satisfied:*

(a) $U(t,s) \in L(X)$, *the space of all linear transformations on* X, *whenever* $0 \le s \le t \le b$ *and for each* $x \in X$, *the mapping* $(t,s) \to U(t,s)x$ *is continuous;*
(b) $U(t,s)U(s,\tau) = U(t,\tau)$ *for* $0 \le \tau \le s \le t \le b$*;*
(c) $U(t,t) = I$ *(identity);*
(d) $U(t,s)$ *is a compact operator whenever* $t - s > 0$*;*
(e) $\frac{\partial U}{\partial t}(t,s)x = -A(t)U(t,s)$ *for* $s < t$*;*
(f) There exists a constant $M_1 \ge 1$ *such that* $\|U(t,s)\| \le M_1, \ 0 \le s \le t \le b$*;*
(g) If $0 < h < 1$, $t - s > h$, *and* $0 < \nu < 1$, *then*

$$\|U(t+h,s) - U(t,s)\| \le \frac{M_2 h^{\nu}}{|t-s|^{\nu}}, \ \ M_2 > 0;$$

(h) If $f(t)$ *is continuous on* J, *then the mapping* $t \to \int_0^t U(t,s) f(s) ds$ *is Hölder continuous with an exponent* $0 < \nu < 1$.

For more facts about evolution family of bounded linear operators, controllability, control function and mild solutions; we recommend reading the books by Friedman [34], Pazy [35], Bensoussan [36], Tanabe [37] and Yosida [38].

Now, we define of the mild solution of the nonlocal Hilfer fractional system (1).

Definition 5. *A function* $x \in C_{1-\gamma}(J,X)$ *is said to be a mild solution of problem (1) if* $I_{0^+}^{1-\gamma} x(0) + g(x) = x_0$, *then* x *satisfies the integral equation*

$$x(t) = U(t,0)(x_0 - g(x)) \frac{t^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) Bu(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) f(s,x(s))\, ds. \tag{6}$$

Definition 6. *The nonlocal Hilfer fractional system (1) is said to be controllable on J if, for every $x_1 \in X$, there exists a control $u \in L^2(J,U')$ such that the mild solution $x(t)$ of (1) satisfies $x(b)+g(x)=x_1$.*

For ending this section, we present the following Krasnoselskii's fixed point theorem.

Theorem 1. *(see [39]) Let Ω be a closed, convex, and non-empty subset of a Banach space $(E,\|\cdot\|)$. Let $\mathscr{A},\mathscr{B}:\Omega\to E$ be two operators such that the following assumptions are satisfied:*

(i) $\mathscr{A}x+\mathscr{B}y\in\Omega$ whenever $x,y\in\Omega$;
(ii) $\mathscr{A}$ is a contraction operator;
(iii) $\mathscr{B}$ is compact and continuous.

Then, $\mathscr{A}+\mathscr{B}$ has a fixed point $z\in\Omega$.

3 Controllability Results

For investigating the controllability of the nonlocal Hilfer fractional system (1), the following assumptions will be imposed:

$(H1)$ The function $f:J\times X\to X$ satisfied the Caratheodory condition, i.e. $f(t,\cdot)$ is continuous for a.e. $t\in J$ and $f(\cdot,x)$ is measurable for all $x\in X$.
$(H2)$ There exist constants $L_1>0$ and $L_2>0$ such that

$$\|f(t,x)-f(t,y)\|\le L_1\|x-y\|,\ \text{for each } t\in J,\ x,y\in X,$$

and $L_2=\sup_{t\in J}\|f(t,0)\|$.
$(H3)$ $g:C_{1-\gamma}(J,X)\to X$ is a continuous function and there exists a constant $L_3>0$ such that

$$\|g(\varphi)-g(\psi)\|\le L_3\|\varphi-\psi\|_{C_{1-\gamma}},\ \text{for all } \varphi,\psi\in C_{1-\gamma}(J,X).$$

$(H4)$ The linear operator $\mathscr{W}:L^2(J,U')\to X$, defined by

$$\mathscr{W}u=\frac{1}{\Gamma(\alpha)}\int_0^b(b-s)^{\alpha-1}U(b,s)Bu(s)\,ds$$

possesses motivated inverse $\mathscr{W}^{-1}$ that occupies values in $L^2(J,U')/\ker\mathscr{W}$, where the kernel space of $\mathscr{W}$ is realized by $\ker\mathscr{W}=\{x\in L^2(J,U'):\mathscr{W}x=0\}$ and subsists two non-negative constants N_1,N_2 such that $\|B\|\le N_1$ and $\|\mathscr{W}^{-1}\|\le N_2$.

Let us define the set $Z_r=\{x\in C_{1-\gamma}(J,X):\|x\|_{C_{1-\gamma}}\le r\}$, where

$$r\ge\frac{\omega}{1-\ell},$$

where

$$\ell:=\frac{M_1L_3}{\Gamma(\gamma)}+\frac{M_1N_1b^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\Upsilon_2+\frac{M_1L_1b^{\alpha}\mathbf{B}(\alpha,\gamma)}{\Gamma(\alpha)}<1,$$
$$\omega:=\frac{M_1}{\Gamma(\gamma)}(\|x_0\|+\|g(0)\|)+\frac{M_1N_1b^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\Upsilon_1+\frac{M_1L_2b^{1-\gamma+\alpha}}{\Gamma(\alpha+1)},$$

and $\mathbf{B}(\cdot,\cdot)$ denotes the beta function defined by

$$\mathbf{B}(\alpha,\gamma)=\int_0^1(1-y)^{\alpha-1}y^{\gamma-1}\,dy,\quad \alpha>0,\ \gamma>0.$$

Clearly, Z_r is closed, convex, and non-empty subset of $C_{1-\gamma}(J,X)$.

Using the assumption $(H4)$, for an arbitrary function $x(\cdot)\in C_{1-\gamma}(J,X)$, we define the control

$$u_x(t)=\mathscr{W}^{-1}\left[x_1-g(x)-U(b,0)(x_0-g(x))\frac{b^{\gamma-1}}{\Gamma(\gamma)}-\frac{1}{\Gamma(\alpha)}\int_0^b(b-s)^{\alpha-1}U(b,s)f(s,x(s))\,ds\right](t).\tag{7}$$

First, we shall prove that $\|u_x(t)\| \leq \Upsilon_1 + \Upsilon_2 r$, where

$$\Upsilon_1 := N_2\left(\|x_1\| + \|g(0)\| + \frac{M_1 b^{\gamma-1}}{\Gamma(\gamma)}(\|x_0\| + \|g(0)\|) + \frac{M_1 L_2 b^{\alpha}}{\Gamma(\alpha+1)}\right),$$

$$\Upsilon_2 := N_2\left(L_3 + \frac{M_1 L_3 b^{\gamma-1}}{\Gamma(\gamma)} + \frac{M_1 L_1 b^{\alpha+\gamma-1}\mathbf{B}(\alpha,\gamma)}{\Gamma(\alpha)}\right).$$

For each $t \in J$ and $x \in Z_r$, we have

$$\begin{aligned}
\|u_x(t)\| &\leq \|\mathscr{W}^{-1}\| \Big(\|x_1\| + \|g(x) - g(0)\| + \|g(0)\| + \|U(b,0)\| \big(\|x_0\| + \|g(x) - g(0)\| + \|g(0)\|\big) \frac{b^{\gamma-1}}{\Gamma(\gamma)} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} \|U(b,s)\| \big(\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\|\big) ds \Big) \\
&\leq N_2 \left(\|x_1\| + L_3\|x\|_{C_{1-\gamma}} + \|g(0)\| + \frac{M_1 b^{\gamma-1}}{\Gamma(\gamma)} (\|x_0\| + L_3\|x\|_{C_{1-\gamma}} + \|g(0)\|) \right. \\
&\quad \left. + \frac{M_1}{\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} (L_1\|x(s)\| + L_2) ds \right) \\
&\leq N_2 \left(\|x_1\| + L_3 r + \|g(0)\| + \frac{M_1 b^{\gamma-1}}{\Gamma(\gamma)} (\|x_0\| + L_3 r + \|g(0)\|) \right. \\
&\quad \left. + \frac{M_1 L_2 b^{\alpha}}{\Gamma(\alpha+1)} + \frac{M_1 L_1}{\Gamma(\alpha)} \|x\|_{C_{1-\gamma}} \int_0^b (b-s)^{\alpha-1} s^{\gamma-1} ds \right) \\
&\leq N_2 \left(\|x_1\| + L_3 r + \|g(0)\| + \frac{M_1 b^{\gamma-1}}{\Gamma(\gamma)} (\|x_0\| + L_3 r + \|g(0)\|) \right. \\
&\quad \left. + \frac{M_1 L_2 b^{\alpha}}{\Gamma(\alpha+1)} + \frac{M_1 L_1 b^{\alpha+\gamma-1}\mathbf{B}(\alpha,\gamma)}{\Gamma(\alpha)} r \right) \\
&= \Upsilon_1 + \Upsilon_2 r.
\end{aligned}$$

Hence, the desired result follows.
It is convenient, in the sequel, to use the constant $0 \leq \Delta < 1$, where

$$\Delta := \frac{M_1 L_2}{\Gamma(\gamma)} + \frac{M_1 N_1 N_2 L_2 b^{1-\gamma+\alpha}}{\Gamma(\alpha+1)} + \frac{M_1^2 N_1 N_2 L_2 b^{\alpha}}{\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{M_1^2 N_1 N_2 L_1 b^{2\alpha}\mathbf{B}(\alpha,\gamma)}{\Gamma(\alpha+1)}.$$

Remark. It is easy to verify that $x(b) = x_1 - g(x)$. Substituting $t = b$ in (6) with the above control (7) and using the definition of $\mathscr{W}$ in assumption $(H4)$, we get

$$\begin{aligned}
x(b) &= U(b,0)(x_0 - g(x)) \frac{b^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} U(b,s) B u(s)\, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} U(b,s) f(s,x(s))\, ds \\
&= U(b,0)(x_0 - g(x)) \frac{b^{\gamma-1}}{\Gamma(\gamma)} + \mathscr{W}\mathscr{W}^{-1} \left[x_1 - g(x) - U(b,0)(x_0 - g(x)) \frac{b^{\gamma-1}}{\Gamma(\gamma)} \right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} U(b,s) f(s,x(s))\, ds \right] + \frac{1}{\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} U(b,s) f(s,x(s))\, ds \\
&= x_1 - g(x).
\end{aligned}$$

Theorem 2. *If the assumptions $(H1)-(H5)$ hold. Then, the nonlocal Hilfer fractional system (1) is controllable on J.*

Proof. Using the control (7), we define the operator $H : C_{1-\gamma}(J,X) \to C_{1-\gamma}(J,X)$ by

$$(Hx)(t) = U(t,0)(x_0 - g(x)) \frac{t^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) B u_x(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) f(s,x(s))\, ds. \quad (8)$$

By virtue of Definition 5, it is easy to show that the existence of mild solution of the nonlocal Hilfer fractional system (1) analogous to the fixed point of the operator H.

We decompose the operator H into two operators H_1 and H_2 $(H = H_1 + H_2)$ on Z_r, where

$$(H_1x)(t) = U(t,0)(x_0 - g(x))\frac{t^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}U(t,s)Bu_x(s)\,ds,$$

and

$$(H_2x)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}U(t,s)f(s,x(s))\,ds.$$

We shall prove that $H_1 + H_2$ has a fixed point in Z_r. The proof will involve several steps.

Step 1. We show that $(H_1 + H_2)Z_r \subset Z_r$.

For every $x, y \in Z_r$ and $t \in J$, one has

$$\begin{aligned}
\|t^{1-\gamma}\big((H_1x)(t) + (H_2y)(t)\big)\| &\le \left\|\frac{1}{\Gamma(\gamma)}U(t,0)(x_0 - g(x)) + \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}U(t,s)Bu_x(s)\,ds\right\| \\
&+ \left\|\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}U(t,s)f(s,y(s))\,ds\right\| \\
&\le \frac{\|U(t,0)\|}{\Gamma(\gamma)}(\|x_0\| + \|g(x) - g(0)\| + \|g(0)\|) \\
&+ \frac{b^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|U(t,s)\|\|B\|\|u_x(s)\|\,ds \\
&+ \frac{b^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|U(t,s)\|\big(\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\|\big)\,ds \\
&\le \frac{M_1}{\Gamma(\gamma)}(\|x_0\| + L_3 r + \|g(0)\|) + \frac{M_1N_1b^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}(\Upsilon_1 + \Upsilon_2 r) \\
&+ \frac{M_1L_2b^{1-\gamma+\alpha}}{\Gamma(\alpha+1)} + \frac{M_1L_1b^{1-\gamma}}{\Gamma(\alpha)}\|x\|_{C_{1-\gamma}}\int_0^b (b-s)^{\alpha-1}s^{\gamma-1}ds \\
&\le \frac{M_1}{\Gamma(\gamma)}(\|x_0\| + L_3 r + \|g(0)\|) \\
&+ \frac{M_1N_1b^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}(\Upsilon_1 + \Upsilon_2 r) + \frac{M_1L_2b^{1-\gamma+\alpha}}{\Gamma(\alpha+1)} + \frac{M_1L_1b^{\alpha}\mathbf{B}(\alpha,\gamma)}{\Gamma(\alpha)}r \\
&= \left(\frac{M_1}{\Gamma(\gamma)}(\|x_0\| + \|g(0)\|) + \frac{M_1N_1b^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\Upsilon_1 + \frac{M_1L_2b^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\right) \\
&\quad + \left(\frac{M_1L_3}{\Gamma(\gamma)} + \frac{M_1N_1b^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\Upsilon_2 + \frac{M_1L_1b^{\alpha}\mathbf{B}(\alpha,\gamma)}{\Gamma(\alpha)}\right)r \\
&\le r.
\end{aligned}$$

Thus, $\|H_1x + H_2y\|_{C_{1-\gamma}} \le r$, which shows that $H_1x + H_2y \in Z_r$ for every $x, y \in Z_r$.

Step 2. The operator H_1 is contraction on Z_r.
For each $x, y \in Z_r$ and $t \in J$, one has

$$
\begin{aligned}
&\|t^{1-\gamma}\big((H_1x)(t)-(H_1y)(t)\big)\| \\
&\le \left\|\frac{1}{\Gamma(\gamma)}U(t,0)(g(x)-g(y))\right\| + \left\|\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}U(t,s)B(u_x(s)-u_y(s))\,ds\right\| \\
&\le \frac{M_1L_3}{\Gamma(\gamma)}\|x-y\|_{C_{1-\gamma}} + \frac{t^{1-\gamma}}{\Gamma(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1}U(t,s)B\mathscr{W}^{-1}\left[(g(x)-g(y))\right.\right. \\
&+\ U(b,0)(g(x)-g(y))\frac{b^{\gamma-1}}{\Gamma(\gamma)} + \left.\left.\frac{1}{\Gamma(\alpha)}\int_0^b (b-\tau)^{\alpha-1}U(b,\tau)(f(\tau,x(\tau))-f(\tau,y(\tau)))\,d\tau\right](s)\,ds\right\| \\
&\le \frac{M_1L_3}{\Gamma(\gamma)}\|x-y\|_{C_{1-\gamma}} + \frac{M_1N_1N_2t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[L_3\|x-y\|_{C_{1-\gamma}} + \frac{M_1L_3b^{\gamma-1}}{\Gamma(\gamma)}\|x-y\|_{C_{1-\gamma}}\right. \\
&+ \left.\frac{M_1L_1}{\Gamma(\alpha)}\int_0^b (b-\tau)^{\alpha-1}\|x(\tau)-y(\tau)\|\,d\tau\right]\,ds \\
&\le \frac{M_1L_3}{\Gamma(\gamma)}\|x-y\|_{C_{1-\gamma}} + \frac{M_1N_1N_2b^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[L_3\|x-y\|_{C_{1-\gamma}} + \frac{M_1L_3b^{\gamma-1}}{\Gamma(\gamma)}\|x-y\|_{C_{1-\gamma}}\right. \\
&+ \left.\frac{M_1L_1b^{\alpha+\gamma-1}\mathbf{B}(\alpha,\gamma)}{\Gamma(\alpha)}\|x-y\|_{C_{1-\gamma}}\right]\,ds \\
&\le \left(\frac{M_1L_3}{\Gamma(\gamma)} + \frac{M_1N_1N_2L_3b^{1-\gamma+\alpha}}{\Gamma(\alpha+1)} + \frac{M_1^2N_1N_2L_3b^{\alpha}}{\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{M_1^2N_1N_2L_1b^{2\alpha}\mathbf{B}(\alpha,\gamma)}{\Gamma(\alpha+1)}\right)\|x-y\|_{C_{1-\gamma}} \\
&= \Delta\|x-y\|_{C_{1-\gamma}}.
\end{aligned}
$$

Hence, $\|H_1x-H_1y\|_{C_{1-\gamma}} \le \Delta\|x-y\|_{C_{1-\gamma}},\ \ 0\le\Delta<1$, which implies that the operator H_1 is contraction on Z_r.

Step 3. The operator H_2 is compact and continuous.

First, we shall show that the operator H_2 is continuous.
Let $\{x_n\}$ be a sequence such that $x_n \to x$ as $n\to\infty$ in Z_r. Then, for each $t\in J$, one has

$$
\begin{aligned}
\|t^{1-\gamma}\big((H_2x_n)(t)-(H_2x)(t)\big)\| &\le \left\|\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}U(t,s)(f(s,x_n(s))-f(s,x(s)))\,ds\right\| \\
&\le \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|U(t,s)\|\|f(s,x_n(s))-f(s,x(s))\|ds \\
&\le \frac{M_1b^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\gamma-1}\|f(\cdot,x_n(\cdot))-f(\cdot,x(\cdot))\|_{C_{1-\gamma}}ds \\
&\le \frac{M_1b^{\alpha}\mathbf{B}(\alpha,\gamma)}{\Gamma(\alpha)}\|f(\cdot,x_n(\cdot))-f(\cdot,x(\cdot))\|_{C_{1-\gamma}},
\end{aligned}
$$

which implies that

$$
\|H_2x_n-H_2x\|_{C_{1-\gamma}} \le \frac{M_1b^{\alpha}\mathbf{B}(\alpha,\gamma)}{\Gamma(\alpha)}\|f(\cdot,x_n(\cdot))-f(\cdot,x(\cdot))\|_{C_{1-\gamma}}.
$$

The R.H.S. of the above inequality tends to zero as $n\to\infty$, which proves the continuity of H_2.

Next, we show that $\{H_2x : x \in Z_r\}$ is uniformly bounded. For any $t \in J$, one has

$$
\begin{aligned}
\|t^{1-\gamma}(H_2x)(t)\| &\leq \left\| \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) f(s,x(s))\, ds \right\| \\
&\leq \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\| \|f(s,x(s))\| ds \\
&\leq \frac{M_1 b^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big(\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\|\big)\, ds \\
&\leq \frac{M_1 b^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (L_1\|x(s)\| + L_2) ds \\
&\leq \frac{M_1 L_2 b^{1-\gamma+\alpha}}{\Gamma(\alpha+1)} + \frac{M_1 L_1 b^{\alpha} \mathbf{B}(\alpha,\gamma)}{\Gamma(\alpha)} \|x\|_{C_{1-\gamma}},
\end{aligned}
$$

which implies that

$$
\|H_2x\|_{C_{1-\gamma}} \leq \frac{M_1 L_2 b^{1-\gamma+\alpha}}{\Gamma(\alpha+1)} + \frac{M_1 L_1 b^{\alpha} \mathbf{B}(\alpha,\gamma)}{\Gamma(\alpha)} r.
$$

Therefore, H_2 is uniformly bounded on Z_r.
We show that $\{H_2x : x \in Z_r\}$ is an equicontinuous set.
For $t_1, t_2 \in J$, $t_1 < t_2$ and any $x \in Z_r$, using the assumptions $(H1)$-$(H3)$ jointly with Definition 4, one has

$$
\begin{aligned}
&\|t_2^{1-\gamma}(H_2x)(t_2) - t_1^{1-\gamma}(H_2x)(t_1)\| \\
&\leq \left\| \frac{t_2^{1-\gamma}}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} U(t_2,s) f(s,u(s))\, ds - \frac{t_1^{1-\gamma}}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} U(t_1,s) f(s,u(s))\, ds \right\| \\
&\leq \left\| \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} t_2^{1-\gamma} (t_2-s)^{\alpha-1} U(t_2,s) f(s,u(s))\, ds \right\| \\
&+ \left\| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left(t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1} \right) U(t_2,s) f(s,u(s))\, ds \right\| \\
&\quad + \left\| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} t_1^{1-\gamma}(t_1-s)^{\alpha-1} \left(U(t_2,s) - U(t_1,s)\right) f(s,u(s))\, ds \right\| \\
&\leq \frac{M_1\|f\|_{C_{1-\gamma}}}{\Gamma(\alpha+1)} \left[t_2^{1-\gamma}(t_2-t_1)^{\alpha} + \left((t_2^{1-\gamma+\alpha} - t_1^{1-\gamma+\alpha}) - t_2^{1-\gamma}(t_2-t_1)^{\alpha} \right) \right] \\
&+ \frac{M_2\|f\|_{C_{1-\gamma}}}{(\alpha-\nu)\Gamma(\alpha)} t_1^{1-\gamma+\alpha-\nu} (t_2-t_1)^{\nu} \\
&\leq \frac{M_1\|f\|_{C_{1-\gamma}}}{\Gamma(\alpha+1)} (t_2^{1-\gamma+\alpha} - t_1^{1-\gamma+\alpha}) + \frac{M_2 b^{1-\gamma+\alpha-\nu}\|f\|_{C_{1-\gamma}}}{(\alpha-\nu)\Gamma(\alpha)} (t_2-t_1)^{\nu}.
\end{aligned}
$$

It follows that

$$
\|t_2^{1-\gamma}(H_2x)(t_2) - t_1^{1-\gamma}(H_2x)(t_1)\| \leq \frac{M_1\|f\|_{C_{1-\gamma}}}{\Gamma(\alpha+1)} (t_2^{1-\gamma+\alpha} - t_1^{1-\gamma+\alpha}) + \frac{M_2 b^{1-\gamma+\alpha-\nu}\|f\|_{C_{1-\gamma}}}{(\alpha-\nu)\Gamma(\alpha)} (t_2-t_1)^{\nu}.
$$

Consequently, for $\alpha \neq \nu$, the R.H.S. of the above inequality tends to zero, as $t_2 \to t_1$. This shows that H_2 is equicontinuous. Therefore, by steps 1-3 and the Arzelà-Ascoli theorem, we infer that H_2 is compact and continuous. Hence, the Krasnoskelskii's fixed point theorem (Theorem 1) guarantees that the Hilfer fractional system (1) is controllable on J.

4 Example

We consider the following Hilfer fractional differential evolution equation with nonlocal condition:

$$\begin{cases} D^{\frac{1}{2},\frac{1}{2}}\omega(t,\eta) = \omega_{\eta\eta}(t,\eta) + \upsilon(t,\eta)\omega(t,\eta) + \psi(t,\eta) + \dfrac{e^{-t}}{1+e^{t}}\left(\dfrac{\omega(t,\eta)}{1+\omega(t,\eta)}\right), \\ \omega(t,0) = \omega(t,\pi) = 0,\ t\in[0,1], \\ I_{0^+}^{\frac{1}{4}}\omega(0,\eta) + \sum_{i=1}^{m} c_i \sin(\omega(\tau_i,\eta)) = \omega_0(\eta),\ \eta\in[0,\pi], \end{cases} \tag{9}$$

where $\alpha=\beta=\frac{1}{2}$, then $\gamma=\frac{3}{4}$, $\upsilon,\psi:[0,1]\times[0,\pi]\to[0,\pi]$ are continuous functions, $0<\tau_1<\cdots<\tau_m<1$ and c_i are given real numbers for $i=1,\cdots,m$. Let us take $X=U'=L^2([0,1],\mathbb{R}^+)$ with the usual norm $\|\cdot\|_{L^2}$, and define $A(t):X\to X$ by

$$A(t)\omega = \omega'' + \upsilon(t,\eta)\omega,$$

with domain

$$D(A) = \left\{\omega(\cdot)\in X : \omega,\omega'' \text{are absolutely continuous},\ \omega''\in X,\ \omega(0)=\omega(\pi)=0\right\}.$$

It is known that $A(t)$ generates an evolution operator $U(t,s)$, see [34], and is given by

$$U(t,s) = Q(t-s)e^{\int_s^t \upsilon(\tau,\cdot)\,d\tau},$$

where $Q(t)$ is the compact analytic semigroup generated by the operator A with $A\omega=\omega''$.
Here,

$$Q(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t}(\omega,\omega_n)\omega_n, \quad \omega\in X.$$

where $\omega_n(\eta)=\sqrt{\frac{2}{\pi}}\sin(n\eta)$, $n=1,2,3,\ldots$ is the orthogonal set of eigenvectors of A, (for more details see [35]).

Next, we write $x(t)(\eta)=\omega(t,\eta)$ and the bounded linear operator $Bu(t)(\eta)=\psi(t,\eta)$. We assume that the operator $\mathscr{W}$ defined by

$$\mathscr{W}u(\cdot) = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{q-1}U(1,s)\psi(s,\cdot)\,ds,$$

has a bounded invertible $\mathscr{W}^{-1}\in L^2(J,U')/\ker\mathscr{W}$ satisfies $(H4)$.
Furthermore, the function $f:[0,1]\times X\to X$ is given by by

$$f(t,\omega(t,\eta)) = \frac{e^{-t}}{1+e^{t}}\left(\frac{\omega(t,\eta)}{1+\omega(t,\eta)}\right),$$

where $\|f(t,\omega)\|\le\frac{e^{-t}}{1+e^{t}}$ and $\|f(t,\omega_1)-f(t,\omega_2)\|\le L_1\|\omega_1-\omega_2\|$ with $L_1=L_2=\frac{1}{2}$. Hence, the assumptions $(H1)$ and $(H2)$ hold.

Finally, define the function $g:C_{1-\gamma}([0,1],X)\to X$ by

$$g(\omega)(\eta) = \sum_{i=1}^{m} c_i\sin(\omega(\tau_i,\eta)),\ 0<\tau_1<\cdots<\tau_m<1,\ \eta\in[0,\pi].$$

From the fact $|\sin a-\sin b|\le|a-b|$, for all $a,b\in\mathbb{R}$, it follows that the condition $(H3)$ is satisfied with $L_3=\max\{|c_i| : i=1,\cdots,m\}$.
Therefore, all the assumptions of Theorem 2 are satisfied. Hence, the Hilfer fractional system (9) is controllable on $[0,1]$.

Conflict of Interest

The authors declare that they have no conflict of interest.

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