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Existence and Stability of Nonlinear Implicit Caputo-Exponential Fractional Differential Equations

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Abstract: In this paper, the existence and stability of solutions for the following fractional problem with Caputo–Exponential fractional derivative are discussed

$$ {}^{e}_{c}D^{\zeta}_{0^+}\psi(\alpha) = \vartheta(\alpha, \psi(\alpha), {}^{e}_{c}D^{\zeta}_{0^+}\psi(\alpha)), \text{ for each, } \alpha \in \Theta := [0,b],\ b > 0,\ 0 < \zeta \leq 1, $$

$$ \psi(0) = \psi_0. $$

We use Schauder's fixed point theorem, the nonlinear alternative of Leray–Schauder type and Banach contraction principle to demonstrate our results. Two examples are provided to demonstrate the relevance of our results.

Keywords: Caputo's-exponential fractional derivative, implicit fractional differential equations, initial value problem, Gronwall's lemma, fractional integral, fixed point, Ulam–Hyers–Rassias stability.

1 Introduction

Recently, there has been a lot of interest in the existence of solutions to initial and boundary value problems for fractional differential equations; see for instance the books [1,2,3,4,5,6,7] and the articles [8,9,10,11,12,13,14] and references therein.

In [10], Benchohra and Lazreg studied the following initial value problem for implicit

$$ {}^{C}D^{\zeta}\psi(\alpha) = \vartheta(\alpha, \psi(\alpha), {}^{C}D^{\zeta}\psi(\alpha)),\ \alpha \in \Theta,\ 0 < \zeta \leq 1, $$

$$ \psi(0) = \psi_0, $$

where ${}^{C}D^{\zeta}$ is the Caputo fractional derivative, $\psi_0 \in \mathbb{R}$, $\Theta = [0,b]$, $b > 0$ and $\vartheta : \Theta \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function space. In [9], they studied the following initial value problem

$$ {}^{H}D^{\zeta}\psi(\alpha) = \vartheta(\alpha, \psi(\alpha), {}^{H}D^{\zeta}\psi(\alpha)), \text{ for each } \alpha \in \Theta,\ 0 < \zeta \leq 1, $$

$$ \psi(1) = \psi_1, $$

where ${}^{H}D^{\zeta}$ is the Hadamard fractional derivative, $\vartheta : \Theta \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function space, $\psi_1 \in \mathbb{R}$ and $\Theta = [1,b]$, $b > 1$,

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On the other hand, several articles treat the Ulam stability with different fractional derivatives: Hadamard derivative, Caputo derivative, Hilfer derivative and Hilfer–Hadamard derivative, etc. (see [15,16,17,18,19,9,10]).

In ([20] p. 99, Section 2.5) Kilbas *et al.* presented the definitions and some properties of the fractional integrals and fractional derivatives of a function ϑ with respect to another function γ. Let $\Theta := [a,b]$, $(-\infty \le a < b \le \infty)$ be a finite interval of the real line $\mathbb{R}$ and $\zeta > 0$. Also let $\gamma(\alpha)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\gamma'(\alpha)$ on (a,b).

The left-sided fractional integral of a function ϑ with respect to another function γ on $[a,b]$ is defined by

$$(I^{\zeta}_{a+}\vartheta)(\alpha) := \frac{1}{\Gamma(\zeta)}\int_a^{\alpha}(\gamma(\alpha)-\gamma(s))^{\zeta-1}\gamma'(s)\vartheta(s)ds \text{ for } \alpha > a.$$

If $a = 0$ and $b = \infty$, then

$$(I^{\zeta}_{0+}\vartheta)(\alpha) := \frac{1}{\Gamma(\zeta)}\int_0^{\alpha}(\gamma(\alpha)-\gamma(s))^{\zeta-1}\gamma'(s)\vartheta(s)ds \text{ for } \alpha > 0.$$

If $a = -\infty$ and $b = \infty$, then

$$(I^{\zeta}_{+}\vartheta)(\alpha) := \frac{1}{\Gamma(\zeta)}\int_{-\infty}^{\alpha}(\gamma(\alpha)-\gamma(s))^{\zeta-1}\gamma'(s)\vartheta(s)ds \text{ for } \alpha \in \mathbb{R}.$$

1. Taking $\gamma(\alpha) = \alpha$ we obtain the *Riemann–Liouville* fractional integral.
2. Taking $\gamma(\alpha) = ln(\alpha)$ we obtain the *Hadamard* fractional integral.
3. Taking $\gamma(\alpha) = e^{\alpha}$ we obtain the *exponential* fractional integral (see Definition 1).

In [21], the authors studied the following exponential initial value problem

$${}^{e}_{c}D^{\zeta}_{0+}\psi(\alpha) = \vartheta(\alpha,\psi(\alpha)),\ \alpha \in \Theta := [0,b],\ b > 0,\ 1 < \zeta \le 2,$$

$$\psi(0) = \lambda_1,\ \ {}^{e}D\psi(0) = \lambda_2,$$

where $\vartheta : \Theta \times \mathbb{R} \to \mathbb{R}$ is a given function and λ_1, λ_2 are given constants and $1 < \zeta \le 2$, and the following exponential boundary value problem

$${}^{e}_{c}D^{\zeta}_{0+}\psi(\alpha) = \vartheta(\alpha,\psi(\alpha)),\ \alpha \in \Theta := [0,b],\ b > 0,\ 0 < \zeta \le 1,\ \ and\ \ 1 < \zeta \le 2,$$

$$\upsilon_1\psi(0) + \upsilon_2\psi(b) = \upsilon_3,$$

where $\upsilon_1, \upsilon_2, \upsilon_3$ are given constants with $\upsilon_1 + \upsilon_2 \neq 0$.

Motivated by the works mentioned above, the present paper aims to establish existence and uniqueness results to the following fractional implicit differential equation (IVP):

$${}^{e}_{c}D^{\zeta}_{0+}\psi(\alpha) = \vartheta(\alpha,\psi(\alpha),\ {}^{e}_{c}D^{\zeta}_{0+}\psi(\alpha)),\ \ \alpha \in \Theta := [0,b],\ b > 0,\ 0 < \zeta \le 1, \tag{1}$$

$$\psi(0) = \psi_0, \tag{2}$$

where ${}^{e}_{c}D^{\zeta}_{0+}$ is the left-sided Caputo–Exponential type fractional derivative, $\vartheta : \Theta \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $\psi_0 \in \mathbb{R}$. In addition, the aim of this paper is to establish some types of Ulam stability for the fractional implicit differential equation (1).

Three results for problem (1)-(2) are presented in this paper. To prove the first, we use the Banach contraction principle; for the second, we use Schauder's fixed point theorem; and for the third, we use the nonlinear alternative of Leray–Schauder type. Examples are provided to demonstrate the applicability of our findings.

2 Preliminaries

By $\mathscr{V} := C(\Theta, \mathbb{R})$, we denote the Banach space of all continuous functions ξ from Θ into $\mathbb{R}$ with the norm

$$\|\xi\|_\infty := \sup_{\alpha\in\Theta} |\xi(\alpha)|.$$

The notation $L^1([0,b],\mathbb{R})$ denotes the Banach space of measurable functions $\xi : [0,b] \to \mathbb{R}$ which are Lebesgue integrable and normed by

$$\|\xi\|_{L^1} = \int_0^b |\xi(s)|ds, \text{ for all } \xi \in L^1(\Theta,\mathbb{R}).$$

Definition 1.([21,22]) *The exponential left-sided fractional integral of a function $\vartheta : \mathbb{R} \to \mathbb{R}$ of order $\zeta \geq 0$ is given by*

$$({}^eI^\zeta_{a+}\vartheta)(\alpha) := \frac{1}{\Gamma(\zeta)} \int_a^\alpha (e^\alpha - e^s)^{\zeta-1}\vartheta(s)e^s ds \text{ for } \alpha > a \text{ and } \zeta > 0, \tag{3}$$

and

$$({}^eI^0_{(\cdot)}\vartheta)(\alpha) := \vartheta(\alpha). \tag{4}$$

If $a = 0$ and $b = \infty$, then

$$({}^eI^\zeta_{0+}\vartheta)(\alpha) := \frac{1}{\Gamma(\zeta)} \int_0^\alpha (e^\alpha - e^s)^{\zeta-1}\vartheta(s)e^s ds \text{ for } \alpha > 0 \text{ and } \zeta > 0. \tag{5}$$

If $a = -\infty$, then

$$({}^eI^\zeta_{+}\vartheta)(\alpha) := \frac{1}{\Gamma(\zeta)} \int_{-\infty}^\alpha (e^\alpha - e^s)^{\zeta-1}\vartheta(s)e^s ds \text{ for } \alpha \in \mathbb{R} \text{ and } \zeta > 0. \tag{6}$$

Lemma 1.([21,22] *Semigroup property.) Let $\zeta > 0$ and $\upsilon > 0$. Then, for all $\alpha \in [a,b]$,*

$${}^eI^\zeta_a({}^eI^\upsilon_a\vartheta)(\alpha) = {}^eI^\upsilon_a({}^eI^\zeta_a\vartheta)(\alpha) = {}^eI^{\zeta+\upsilon}_a\vartheta(\alpha).$$

Lemma 2.([21,22]) *Let $\zeta > 0$ and $\upsilon > 0$. The fractional integral formula or the power exponential function is given by*

$${}^eI^\zeta_+ e^{\upsilon x} = \frac{\Gamma(\upsilon+1)}{\Gamma(\zeta+\upsilon+1)} e^{(\zeta+\upsilon)x}.$$

For $n \in \mathbb{N} := \{1,2,3,\cdots\}$, let

$$AC^n_e(\Theta) := \left\{ v \in C^{n-1}(\Theta,\mathbb{R}) : {}^eD^{(n-1)}v(\alpha) \in AC(\Theta),\ {}^eD = e^{-\alpha}\frac{d}{dt} \right\}.$$

In particular, $AC^1_e(\Theta) := AC_e(\Theta)$.

Definition 2.([21,22]) *The exponential left-sided fractional derivatives of Riemann–Liouville type of order $\zeta \geq 0$ for a function $\vartheta : \mathbb{R} \to \mathbb{R}$ is given by*

$$({}^eD^\zeta_{a+}\vartheta)(\alpha) := \frac{1}{\Gamma(n-\zeta)} \left(e^{-\alpha}\frac{d}{dt}\right)^n \int_a^\alpha (e^\alpha - e^s)^{n-\zeta-1}\vartheta(s)\frac{ds}{e^{-s}}, \text{ for } \alpha > a \text{ and } \zeta > 0, \tag{7}$$

and

$$({}^eD^0_{(\cdot)}\vartheta)(\alpha) := \vartheta(\alpha). \tag{8}$$

If $a = 0$ and $b = \infty$, then

$$({}^eD^\zeta_{0+}\vartheta)(\alpha) := \frac{1}{\Gamma(n-\zeta)} \left(e^{-\alpha}\frac{d}{dt}\right)^n \int_0^\alpha (e^\alpha - e^s)^{n-\zeta-1}\vartheta(s)\frac{ds}{e^{-s}}, \text{ for } \alpha > 0 \text{ and } \zeta > 0. \tag{9}$$

If $a = -\infty$, then

$$({}^eI^\zeta_{+}\vartheta)(\alpha) := \frac{1}{\Gamma(\zeta)} \int_{-\infty}^\alpha (e^\alpha - e^s)^{\zeta-1}\vartheta(s)e^s ds, \text{ for } \alpha \in \mathbb{R} \text{ and } \zeta > 0, \tag{10}$$

where $n = [\zeta] + 1$.

Lemma 3.([21, 22]). *The exponential fractional derivative formula of power exponential function is given by*

$$ {}^{e}D_{+}^{\zeta}e^{\upsilon x}=\frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon-\zeta+1)}e^{(\upsilon-\zeta)x}. $$

*Remark.*For $\upsilon=0$ we get

$$ {}^{e}D_{+}^{\zeta}1=\frac{e^{-\zeta x}}{\Gamma(1-\zeta)}\neq 0. $$

Definition 3.[21, 22] *The exponential left-sided fractional derivatives of Caputo type of order $\zeta\geq 0$ for a function $\vartheta:\mathbb{R}\to\mathbb{R}$ is given by*

$$ ({}_{c}^{e}D_{a^{+}}^{\zeta}\vartheta)(\alpha):=\frac{1}{\Gamma(n-\zeta)}\int_{a}^{\alpha}(e^{\alpha}-e^{s})^{n-\zeta-1}\left(e^{-s}\frac{d}{ds}\right)^{n}\vartheta(s)\frac{ds}{e^{-s}},\quad \alpha>a,\ \ \zeta>0, \tag{11} $$

and

$$ ({}_{c}^{e}D_{(\cdot)}^{0}\vartheta)(\alpha):=\vartheta(\alpha), \tag{12} $$

where $n=[\zeta]+1$.

Lemma 4.[21, 22] *If $\zeta,\upsilon>0$, then*

1. ${}^{e}I_{a}^{\zeta}(e^{\alpha}-e^{a})^{\upsilon}=\frac{\Gamma(\upsilon+1)}{\Gamma(\zeta+\upsilon+1}(e^{\alpha}-e^{a})^{\zeta+\upsilon}$.
2. ${}^{e}D_{a}^{\zeta}(e^{\alpha}-e^{a})^{\upsilon}=\frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon-\zeta+1}(e^{\alpha}-e^{a})^{\upsilon-\zeta}$.

Lemma 5.[21, 22] *Let $\zeta\geq 0$ and $n=[\zeta]+1$. Then*

$$ {}_{c}^{e}D_{a^{+}}^{\zeta}\vartheta(\alpha)={}^{e}D_{a^{+}}^{\zeta}\left[\vartheta(s)-\sum_{k=0}^{n-1}\frac{{}^{e}D^{k}\vartheta(a)}{k!}(e^{s}-e^{a})^{k}\right](\alpha), $$

where ${}^{e}D=e^{-\alpha}\frac{d}{dt}$.

Theorem 1.[21, 22] *If $0<\upsilon<\zeta$ and $1\leq p<\infty$, then for $\vartheta\in L^{p}(a,b)$ we have*

$$ {}^{e}D_{a}^{\upsilon}({}^{e}I_{a}^{\zeta}\vartheta)(\alpha)={}^{e}I_{a}^{\zeta-\upsilon}\vartheta(\alpha)\quad\text{and}\quad {}_{c}^{e}D_{a}^{\upsilon}({}^{e}I_{a}^{\zeta}\vartheta)(\alpha)={}^{e}I_{a}^{\zeta-\upsilon}\vartheta(\alpha). $$

In addition,

$$ {}^{e}D_{a}^{\zeta}({}^{e}I_{a}^{\zeta}\vartheta)(\alpha)=\vartheta(\alpha)\quad\text{and}\quad {}_{c}^{e}D_{a}^{\zeta}({}^{e}I_{a}^{\zeta}\vartheta)(\alpha)=\vartheta(\alpha). $$

Theorem 2.[21, 22] *Let $\zeta\geq 0$ and $n=[\zeta]+1$. Then,*

1. ${}^{e}I_{a}^{\zeta}({}^{e}D_{a}^{\zeta}\vartheta)(\alpha)=\vartheta(\alpha)-\sum_{k=1}^{n}\frac{(e^{s}-e^{a})^{\zeta-k}}{\Gamma(\zeta-k+1)}\,{}^{e}D^{n-k}({}^{e}I^{n-\zeta}\vartheta)(a).$
2. ${}^{e}I_{a}^{\zeta}({}_{c}^{e}D_{a}^{\zeta}\vartheta)(\alpha)=\vartheta(\alpha)-\sum_{k=0}^{n-1}\frac{(e^{s}-e^{a})^{k}}{k!}\,{}^{e}D^{k}\vartheta(a).$

Lemma 6.([23])*(Generalized Gronwall's inequality with respect to another function.) Let ψ,ν be two integrable functions and ξ continuous, with domain $[a,b]$. Let $\gamma\in C^{1}[a,b]$ an increasing function such that $\gamma'(\alpha)\neq 0$, for all $\alpha\in[a,b]$. Assume that the functions ψ, ν are nonnegative, and ξ is nonnegative and nondecreasing. If*

$$ \psi(\alpha)\leq\nu(\alpha)+\xi(\alpha)\int_{a}^{\alpha}\gamma'(s)(\gamma(\alpha)-\gamma(s))^{\zeta-1}\psi(s)ds, $$

then

$$ \psi(\alpha)\leq\nu(\alpha)+\int_{a}^{\alpha}\sum_{k=1}^{\infty}\frac{[\xi(\alpha)\Gamma(\zeta)]^{k}}{\Gamma(\zeta k)}\gamma'(s)[\gamma(\alpha)-\gamma(s)]^{\zeta k-1}\nu(s)ds,\ \text{for every}\ \alpha\in[a,b]. $$

Definition 4.([20]) *The Mittag–Leffler function is given by*

$$E_{\zeta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\zeta k+1)},\ \zeta \in \mathbb{C},\ \Re(\zeta) > 0.$$

Thus

$$E_{\zeta}(z) = E_{\zeta,1}(z),\ E_1(z) = E_{1,1}(z) = e^z.$$

We shall now define the several types of Ulam stability that were used in this study. (see [9,10]).

Definition 5.([9,10]) *The equation* (1) *is* ***Ulam–Hyers stable*** *(U-H) if there exists a real number* $c_{\vartheta} > 0$ *such that for each* $\bar{\bar{\varepsilon}} > 0$ *and for each solution* $\nu \in C^1(\Theta,\mathbb{R})$ *of the inequality*

$$|{}^{e}_{c}D^{\zeta}_{0^+}\nu(\alpha) - \vartheta(\alpha,\nu(\alpha),\ {}^{e}_{c}D^{\zeta}_{0^+}\nu(\alpha))| \le \bar{\varepsilon},\ \alpha \in \Theta, \tag{13}$$

there exists a solution $\psi \in C^1(\Theta,\mathbb{R})$ *of equation* (1) *with*

$$|\nu(\alpha) - \psi(\alpha)| \le c_{\vartheta}\bar{\varepsilon},\ \alpha \in \Theta.$$

Definition 6.([9,10]) *The equation* (1) *is* ***generalized Ulam–Hyers stable*** *(G.U-H) if there exists* $\tilde{\gamma}_{\vartheta} \in C(\mathbb{R}_+,\mathbb{R}_+)$, $\tilde{\gamma}_{\vartheta}(0) = 0$, *such that for each solution* $\nu \in C^1(\Theta,\mathbb{R})$ *of the inequality* (13) *there exists a solution* $\psi \in C^1(\Theta,\mathbb{R})$ *of the equation* (1) *with*

$$|\nu(\alpha) - \psi(\alpha)| \le \tilde{\gamma}_{\vartheta}(\bar{\varepsilon}),\ \alpha \in \Theta.$$

Definition 7.([9,10]) *The equation* (1) *is* ***Ulam–Hyers–Rassias stable*** *(U-H-R) with respect to* $\chi \in C(\Theta,\mathbb{R}_+)$ *if there exists a real number* $c_{\vartheta} > 0$ *such that for each* $\bar{\varepsilon} > 0$ *and for each solution* $\nu \in C^1(\Theta,\mathbb{R})$ *of the inequality*

$$|{}^{e}_{c}D^{\zeta}_{0^+}\nu(\alpha) - \vartheta(\alpha,\nu(\alpha),\ {}^{e}_{c}D^{\zeta}_{0^+}\nu(\alpha))| \le \bar{\varepsilon}\chi(\alpha),\ \alpha \in \Theta, \tag{14}$$

there exists a solution $\psi \in C^1(\Theta,\mathbb{R})$ *of equation (1) with*

$$|\nu(\alpha) - \psi(\alpha)| \le c_{\vartheta}\bar{\varepsilon}\chi(\alpha),\ \alpha \in \Theta.$$

Definition 8.([9,10]) *The equation* (1) *is* ***generalized Ulam–Hyers–Rassias stable*** *(G.U-H-R) with respect to* $\chi \in C(\Theta,\mathbb{R}_+)$ *if there exists a real number* $c_{\vartheta,\chi} > 0$ *such that for each solution* $\nu \in C^1(\Theta,\mathbb{R})$ *of the inequality*

$$|{}^{e}_{c}D^{\zeta}_{0^+}\nu(\alpha) - \vartheta(\alpha,\nu(\alpha),\ {}^{e}_{c}D^{\zeta}_{0^+}\nu(\alpha))| \le \chi(\alpha),\ \alpha \in \Theta, \tag{15}$$

there exists a solution $\psi \in C^1(\Theta,\mathbb{R})$ *of equation* (1) *with*

$$|\nu(\alpha) - \psi(\alpha)| \le c_{\vartheta,\chi}\chi(\alpha),\ \alpha \in \Theta.$$

*Remark.*A function $\nu \in C^1(\Theta,\mathbb{R})$ is a solution of of the inequality (13) if and only if there exists a function $\delta \in \mathscr{V}$ (which depends on ψ) such that

(i)$|\delta(\alpha)| \le \bar{\varepsilon}, \forall \alpha \in \Theta$.
(ii)${}^{e}_{c}D^{\zeta}_{0^+}\nu(\alpha) = \vartheta(\alpha,\nu(\alpha),\ {}^{e}_{c}D^{\zeta}_{0^+}\nu(\alpha)) + \delta(\alpha),\ \alpha \in \Theta$.

*Remark.*Clearly,

(i)Definition 5 $\Longrightarrow$ Definition 6.
(ii)Definition 7 $\Longrightarrow$ Definition 8.

3 Existence of solutions

Let us establish what we consider by a solution to problem (1)–(2).

Definition 9. *A function $\psi \in C^1(\Theta,\mathbb{R})$ is said to be a solution of the problem (1)–(2) if ψ satisfies equation (1) on Θ and conditions (2).*

In the sequel, we will need the following lemmas:

Lemma 7. *Let $\vartheta(\alpha,\psi,\nu):\Theta\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Then, problem (1)–(2) is equivalent to the problem:*

$$\psi(\alpha)=\psi_0+{}^{e}I^{\zeta}_{0^+}\delta(\alpha), \tag{16}$$

where $\delta\in\mathscr{V}$ satisfies the equation:

$$\delta(\alpha)=\vartheta(\alpha,\psi_0+{}^{e}I^{\zeta}_{0^+}\delta(\alpha),\delta(\alpha)).$$

Proof. If ${}^{e}_{c}D^{\zeta}_{0^+}\psi(\alpha)=\delta(\alpha)$ then ${}^{e}I^{\zeta}_{0^+}\,{}^{e}_{c}D^{\zeta}_{0^+}\psi(\alpha)={}^{e}I^{\zeta}_{0^+}\delta(\alpha)$. We obtain $\psi(\alpha)=\psi_0+{}^{e}I^{\zeta}_{0^+}\delta(\alpha)$.

Based on Banach's fixed point, we can now assert and demonstrate our existence result for the problem (1)–(2).

Theorem 3. *Assume*

($\mathscr{A}_1$) *The function $\vartheta:\Theta\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous.*
($\mathscr{A}_2$) *There exist constants $\rho_1>0$ and $0<\rho_2<1$ such that*

$$|\vartheta(\alpha,\psi,\nu)-\vartheta(\alpha,\bar{\psi},\bar{\nu})|\le\rho_1|\psi-\bar{\psi}|+\rho_2|\nu-\bar{\nu}|$$

for any $\psi,\nu,\bar{\psi},\bar{\nu}\in\mathbb{R}$ and $\alpha\in\Theta$.

Put $\rho_3=\dfrac{\rho_1}{1-\rho_2}$. If

$$\frac{\rho_3(e^b-1)^{\zeta}}{\Gamma(\zeta+1)}<1, \tag{17}$$

then there exists a unique solution for IVP (1) − (2) on Θ.

Proof. Consider the operator $\Upsilon:\mathscr{V}\to\mathscr{V}$ defined by:

$$\Upsilon(\psi)(\alpha)=\psi_0+{}^{e}I^{\zeta}_{0^+}\delta(\alpha), \tag{18}$$

where $\delta\in\mathscr{V}$ satisfies the following

$$\delta(\alpha)=\vartheta(\alpha,\psi(\alpha),\delta(\alpha)).$$

The fixed points of operator Υ are clearly solutions of problem (1)–(2). Let $\psi,\nu\in\mathscr{V}$. Then for $\alpha\in\Theta$,

$$(\Upsilon\psi)(\alpha)-(\Upsilon\nu)(\alpha)=\frac{1}{\Gamma(\zeta)}\int_0^{\alpha}(e^{\alpha}-e^{s})^{\zeta-1}(\delta(s)-\sigma(s))e^{s}ds,$$

where $\delta,\sigma\in\mathscr{V}$ is such that

$$\delta(\alpha)=\vartheta(\alpha,\psi(\alpha),\delta(\alpha)),$$
$$\sigma(\alpha)=\vartheta(\alpha,\nu(\alpha),\sigma(\alpha)).$$

Then, for $\alpha\in\Theta$

$$|(\Upsilon\psi)(\alpha)-(\Upsilon\nu)(\alpha)|\le\frac{1}{\Gamma(\zeta)}\int_0^{\alpha}(e^{\alpha}-e^{s})^{\zeta-1}|\delta(s)-\sigma(s)|e^{s}ds. \tag{19}$$

By ($\mathscr{A}_2$) we deduce

$$\begin{aligned}|\delta(\alpha)-\sigma(\alpha)|&=|\vartheta(\alpha,\psi(\alpha),\delta(\alpha))-\vartheta(\alpha,\nu(\alpha),\sigma(\alpha))|\\&\le\rho_1|\psi(\alpha)-\nu(\alpha)|+\rho_2|\delta(\alpha)-\sigma(\alpha)|.\end{aligned}$$

Thus

$$|\delta(\alpha)-\sigma(\alpha)|\le\rho_3|\psi(\alpha)-\nu(\alpha)|.$$

And by (19)

$$|(\Upsilon\psi)(\alpha)-(\Upsilon\nu)(\alpha)| \leq \frac{\rho_3}{\Gamma(\zeta)}\int_0^\alpha (e^\alpha - e^s)^{\zeta-1}|\psi(s)-\nu(s)|e^s ds$$
$$\leq \frac{\rho_3(e^b-1)^\zeta}{\Gamma(\zeta+1)}\|\psi-\nu\|_\infty.$$

Then

$$\|\Upsilon\psi-\Upsilon\nu\|_\infty \leq \frac{\rho_3(e^b-1)^\zeta}{\Gamma(\zeta+1)}\|\psi-\nu\|_\infty.$$

By (17), the operator Υ is a contraction. Consequently, by using Banach's contraction principle ([24]), we deduce that Υ has a unique fixed point.

Schauder's fixed point theorem provides the basis for our next existence result.

Theorem 4. *Assume $(\mathscr{A}_1)$,$(\mathscr{A}_2)$ and the following hypothesis holds.*

$(\mathscr{A}_3)$ *There exist $\theta_1,\theta_2,\theta_3 \in C(\Theta,\mathbb{R}_+)$ with $\theta_3{}^* = \sup_{\alpha\in\Theta}\theta_3(\alpha) < 1$ such that*

$$|\vartheta(\alpha,\psi,\nu)| \leq \theta_1(\alpha)+\theta_2(\alpha)|\psi|+\theta_3(\alpha)|\nu| \text{ for } \alpha\in\Theta \text{ and } \psi,\nu\in\mathbb{R}.$$

If

$$\frac{\theta_2{}^*(e^b-1)^\zeta}{(1-\theta_3{}^*)\Gamma(\zeta+1)} < 1, \tag{20}$$

where $\theta_1{}^* = \sup_{\alpha\in\Theta}\theta_1(\alpha)$, and $\theta_2{}^* = \sup_{\alpha\in\Theta}\theta_2(\alpha)$. Then, problem (1)-(2) has at least one solution.

Proof. Consider the operator Υ given in (18). theorem.

Claim 1: *Υ is continuous.*
Let $\{\psi_n\}$ be a sequence such that $\psi_n \to \psi$ in $\mathscr{V}$. Then for each $\alpha\in\Theta$

$$|\Upsilon(\psi_n)(\alpha)-\Upsilon(\psi)(\alpha)| \leq \frac{1}{\Gamma(\zeta)}\int_0^\alpha (e^\alpha-e^s)^{\zeta-1}|\delta_n(s)-\delta(s)|e^s ds, \tag{21}$$

where $\delta_n,\delta\in\mathscr{V}$ such that

$$\delta_n(\alpha) = \vartheta(\alpha,\psi_n(\alpha),\delta_n(\alpha)),$$

and

$$\delta(\alpha) = \vartheta(\alpha,\psi(\alpha),\delta(\alpha)).$$

By $(\mathscr{A}_2)$ we have

$$|\delta_n(\alpha)-\delta(\alpha)| = |\vartheta(\alpha,\psi_n(\alpha),\delta_n(\alpha))-\vartheta(\alpha,\psi(\alpha),\delta(\alpha))|$$
$$\leq \rho_1|\psi_n(\alpha)-\psi(\alpha)|+\rho_2|\delta_n(\alpha)-\delta(\alpha)|.$$

Then

$$|\delta_n(\alpha)-\delta(\alpha)| \leq \rho_3|\psi_n(\alpha)-\psi(\alpha)|.$$

Since $\psi_n\to\psi$, then we get $\delta_n(\alpha)\to\delta(\alpha)$ as $n\to\infty$ for each $\alpha\in\Theta$, and let $\tilde\beta>0$ be such that, for each $\alpha\in\Theta$, we have $|\delta_n(\alpha)|\leq\tilde\beta$ and $|\delta(\alpha)|\leq\tilde\beta$, then we have

$$(e^\alpha-e^s)^{\zeta-1}e^s|\delta_n(s)-\delta(s)| \leq (e^\alpha-e^s)^{\zeta-1}e^s[|\delta_n(s)|+|\delta(s)|]$$
$$\leq 2\tilde\beta(e^\alpha-e^s)^{\zeta-1}e^s.$$

For each $\alpha\in\Theta$, the function $s\to 2\tilde\beta(e^\alpha-e^s)^{\zeta-1}e^s$ is integrable on $[0,\alpha]$, and the Lebesgue dominated convergence theorem and (21) imply that

$$|\Upsilon(\psi_n)(\alpha)-\Upsilon(\psi)(\alpha)| \to 0 \text{ as } n\to\infty,$$

so

$$\|\Upsilon(\psi_n) - \Upsilon(\psi)\|_\infty \to 0 \text{ as } n \to \infty.$$

Consequently, Υ is continuous.

Let

$$\Omega \geq \frac{|\psi_0| + M\theta_1{}^*}{1 - Mq^*},$$

where $M := \frac{(e^b-1)^\zeta}{(1-\theta_3{}^*)\Gamma(\zeta+1)}$ and define the set

$$D_\Omega = \{\psi \in \mathscr{V} : \|\psi\|_\infty \leq \Omega\}.$$

It is obvious that D_Ω is a closed, convex and bounded subset of $\mathscr{V}$.

Claim 2: $\Upsilon(D_\Omega) \subset D_\Omega$.

Let $\psi \in D_\Omega$ we show that $\Upsilon\psi \in D_\Omega$. We have, for each $\alpha \in \Theta$

$$|\Upsilon\psi(\alpha)| \leq |\psi_0| + \frac{1}{\Gamma(\zeta)} \int_0^\alpha (e^\alpha - e^s)^{\zeta-1} |\delta(s)| e^s ds. \tag{22}$$

By $(\mathscr{A}_3)$ and for all $\alpha \in \Theta$, we have

$$\begin{aligned} |\delta(\alpha)| &= |\vartheta(\alpha, \psi(\alpha), \delta(\alpha))| \\ &\leq \theta_1(\alpha) + \theta_2(\alpha)|\psi(\alpha)| + \theta_3(\alpha)|\delta(\alpha)| \\ &\leq \theta_1(\alpha) + \theta_2(\alpha)\Omega + \theta_3(\alpha)|\delta(\alpha)| \\ &\leq \theta_1{}^* + \theta_2{}^*\Omega + \theta_3{}^*|\delta(\alpha)|. \end{aligned}$$

Then

$$|\delta(\alpha)| \leq \frac{\theta_1{}^* + \theta_2{}^*\Omega}{1 - \theta_3{}^*} := \widetilde{M}.$$

Thus (22) implies that

$$\begin{aligned} |\Upsilon\psi(\alpha)| &\leq |\psi_0| + \frac{(\theta_1{}^* + \theta_2{}^*\Omega)(e^b - 1)^\zeta}{(1 - \theta_3{}^*)\Gamma(\zeta + 1)} \\ &\leq |\psi_0| + (\theta_1{}^* + \theta_2{}^*\Omega)M \\ &\leq \Omega. \end{aligned}$$

Then $\Upsilon(D_\Omega) \subset D_\Omega$.

Claim 3: $\Upsilon(D_\Omega)$ *is relatively compact.*

Let $\omega_1, \omega_2 \in \Theta$, $\omega_1 < \omega_2$, and let $\psi \in D_\Omega$. Then

$$\begin{aligned} |\Upsilon(\psi)(\omega_2) - \Upsilon(\psi)(\omega_1)| &= \left| \frac{1}{\Gamma(\zeta)} \int_0^{\omega_1} \left[(e^{\omega_2} - e^s)^{\zeta-1} - (e^{\omega_1} - e^s)^{\zeta-1} \right] \delta(s) e^s ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\zeta)} \int_{\omega_1}^{\omega_2} (e^{\omega_2} - e^s)^{\zeta-1} \delta(s) e^s ds \right| \\ &\leq \frac{\widetilde{M}}{\Gamma(\zeta+1)} \left((e^{\omega_2} - 1)^\zeta - (e^{\omega_2} - 1)^\zeta \right). \end{aligned}$$

As $\omega_1 \to \omega_2$, the right-hand side of the above inequality tends to zero.

We can now deduce by Arzelá-Ascoli theorem and Claims 1 to 3 that $\Upsilon : \mathscr{V} \to \mathscr{V}$ is continuous and compact. Thus, by Schauder's fixed point theorem ([24]), we conclude that Υ has a fixed point.

The basis for our third existence result is the nonlinear alternative of Leray–Schauder type.

Theorem 5. *Assume* $(\mathscr{A}_1), (\mathscr{A}_2), (\mathscr{A}_3)$ *hold. Then, the I.V.P. (1)-(2) has at least one solution.*

Proof. We will demonstrate that Υ verifies the assumption of Leray–Schauder fixed point theorem.

Claim 1: *Obviously Υ is continuous.*

Claim 2: *Υ maps bounded sets into bounded sets in $\mathscr{V}$.*

We will prove that there exist a positive constant μ such that for each $\psi \in B_\mu = \{\psi \in \mathscr{V} : \|\psi\|_\infty \leq \mu\}$, we have $\|\Upsilon(\psi)\|_\infty \leq \mu$.

For $\psi \in B_\mu$, we have

$$|\Upsilon\psi(\alpha)| \leq |\psi_0| + \frac{1}{\Gamma(\zeta)} \int_0^\alpha (e^\alpha - e^s)^{\zeta-1} |\delta(\alpha)| e^s ds. \tag{23}$$

By $(\mathscr{A}_3)$, we obtain

$$\begin{aligned}|\delta(\alpha)| &= |\vartheta(\alpha, \psi(\alpha), \delta(\alpha))| \\ &\leq \theta_1(\alpha) + \theta_2(\alpha)|\psi(\alpha)| + \theta_3(\alpha)|\delta(\alpha)| \\ &\leq \theta_1(\alpha) + \theta_2(\alpha)\mu + \theta_3(\alpha)|\delta(\alpha)| \\ &\leq \theta_1{}^* + \theta_2{}^*\mu + \theta_3{}^*|\delta(\alpha)|.\end{aligned}$$

Then

$$|\delta(\alpha)| \leq \frac{\theta_1{}^* + \theta_2{}^*\mu}{1 - \theta_3{}^*} := M^*.$$

Thus (23) implies that

$$|\Upsilon\psi(\alpha)| \leq |\psi_0| + \frac{M^*(e^b - 1)^\zeta}{\Gamma(\zeta + 1)}.$$

Thus

$$\|\Upsilon\psi\|_\infty \leq |\psi_0| + \frac{M^*(e^b - 1)^\zeta}{\Gamma(\zeta + 1)} := \mu.$$

Claim 3: *It is clear that Υ maps bounded sets into equicontinuous sets of $\mathscr{V}$.*

We deduce that $\Upsilon : \mathscr{V} \longrightarrow \mathscr{V}$ is continuous and completely continuous.

Claim 4: *A priori bounds.*

Now, we show that there exists an open set $\mathscr{X} \subseteq \mathscr{V}$ with $\psi \neq \tau\Upsilon(\psi)$, for $\tau \in (0,1)$ and $\psi \in \partial\mathscr{X}$. Let $\psi \in \mathscr{V}$ and $\psi = \tau\Upsilon(\psi)$ for some $0 < \tau < 1$. Thus, for $\alpha \in \Theta$, we get

$$\psi(\alpha) = \tau\psi_0 + \frac{\tau}{\Gamma(\zeta)} \int_0^\alpha (e^\alpha - e^s)^{\zeta-1} \delta(s) e^s ds.$$

This implies by $(\mathscr{A}_2)$ that, for each $\alpha \in \Theta$,

$$|\psi(\alpha)| \leq |\psi_0| + \frac{1}{\Gamma(\zeta)} \int_0^\alpha (e^\alpha - e^s)^{\zeta-1} |\delta(s)| e^s ds. \tag{24}$$

And, by $(\mathscr{A}_3)$, for each $\alpha \in \Theta$,

$$\begin{aligned}|\delta(\alpha)| &= |\vartheta(\alpha, \psi(\alpha), \delta(\alpha))| \\ &\leq \theta_1(\alpha) + \theta_2(\alpha)|\psi(\alpha)| + \theta_3(\alpha)|\delta(\alpha)| \\ &\leq \theta_1{}^* + \theta_2{}^*|\psi(\alpha)| + \theta_3{}^*|\delta(\alpha)|.\end{aligned}$$

Thus

$$|\delta(\alpha)| \leq \frac{1}{1 - \theta_3{}^*}(\theta_1{}^* + \theta_2{}^*|\psi(\alpha)|).$$

Hence

$$|\psi(\alpha)| \le |\psi_0| + \frac{\theta_1{}^*(e^b-1)^\zeta}{(1-\theta_3{}^*)\Gamma(\zeta+1)} + \frac{\theta_2{}^*}{(1-\theta_3{}^*)\Gamma(\zeta)} \int_0^\alpha (e^\alpha - e^s)^{\zeta-1} |\psi(s)| e^s ds.$$

Then Lemma 6 implies that, for each $\alpha \in \Theta$,

$$\begin{aligned}
|\psi(\alpha)| &\le \left[|\psi_0| + \frac{\theta_1{}^*(e^b-1)^\zeta}{(1-\theta_3{}^*)\Gamma(\zeta+1)}\right]\left[1 + \int_0^\alpha \sum_{k=1}^{\infty} \frac{\left(\frac{\theta_2{}^*}{1-\theta_3{}^*}\right)^k}{\Gamma(k\zeta)} (e^\alpha - e^s)^{k\zeta-1} e^s ds\right] \\
&\le \left[|\psi_0| + \frac{\theta_1{}^*(e^b-1)^\zeta}{(1-\theta_3{}^*)\Gamma(\zeta+1)}\right]\left[1 + \sum_{k=1}^{\infty} \frac{\left(\frac{\theta_2{}^*}{1-\theta_3{}^*}\right)^k}{\Gamma(k\zeta)} \frac{(e^\alpha-1)^{k\zeta}}{k\zeta}\right] \\
&\le \left[|\psi_0| + \frac{\theta_1{}^*(e^b-1)^\zeta}{(1-\theta_3{}^*)\Gamma(\zeta+1)}\right]\left[1 + \sum_{k=1}^{\infty} \frac{\left(\frac{\theta_2{}^*}{1-\theta_3{}^*}\right)^k}{\Gamma(k\zeta+1)} (e^\alpha-1)^{k\zeta}\right] \\
&\le \left[|\psi_0| + \frac{\theta_1{}^*(e^b-1)^\zeta}{(1-\theta_3{}^*)\Gamma(\zeta+1)}\right] E_\zeta\left(\frac{\theta_2{}^*}{1-\theta_3{}^*}(e^b-1)\right).
\end{aligned}$$

Thus

$$\|\psi\|_\infty \le \left[|\psi_0| + \frac{\theta_1{}^*(e^b-1)^\zeta}{(1-\theta_3{}^*)\Gamma(\zeta+1)}\right] E_\zeta\left(\frac{\theta_2{}^*}{1-\theta_3{}^*}(e^b-1)\right) := \overline{M}. \tag{25}$$

Let

$$\mathscr{X} = \{\psi \in \mathscr{V} : \|\psi\|_\infty < \overline{M} + 1\}.$$

Because of our pick of $\mathscr{X}$, there is no $\psi \in \partial\mathscr{X}$ such that $\psi = \tau\Upsilon(\psi)$, for $\tau \in (0,1)$. By Leray–Schauder's theorem ([24]), we conclude that Υ has a fixed point.

4 Ulam–Hyers stability

Theorem 6. *Assume that $(\mathscr{A}_1)$, $(\mathscr{A}_2)$ and (17) are met. Then, equation (1) is (U-H) stable.*

Proof. Let $\nu \in \mathscr{V}$ be a solution of (13), then

$$|{}^e_cD^\zeta_{0^+}\nu(\alpha) - \vartheta(\alpha, \nu(\alpha), {}^e_cD^\zeta_{0^+}\nu(\alpha))| \le \bar{\varepsilon}, \ \alpha \in \Theta. \tag{26}$$

By $\psi \in \mathscr{V}$ we denote the unique solution of the problem

$$\begin{gathered}
{}^e_cD^\zeta_{0^+}\psi(\alpha) = \vartheta(\alpha, \psi(\alpha), {}^e_cD^\zeta_{0^+}\psi(\alpha)), \text{ for each, } \alpha \in \Theta, \ 0 < \zeta \le 1, \\
\psi(0) = \nu(0).
\end{gathered}$$

Using Lemma 7, we have

$$\psi(\alpha) = \nu(0) + \frac{1}{\Gamma(\zeta)} \int_0^\alpha (e^\alpha - e^s)^{\zeta-1} \delta_\psi(s) e^s ds,$$

where $\delta_\psi \in \mathscr{V}$ satisfies

$$\delta_\psi(\alpha) = \vartheta(\alpha, \psi(0) + {}^eI^\zeta_{0^+}\delta_\psi(\alpha), \delta_\psi(\alpha)).$$

However, by integration (26) we get

$$\begin{aligned}
\left|\nu(\alpha) - \nu(0) - \frac{1}{\Gamma(\zeta)} \int_0^\alpha (e^\alpha - e^s)^{\zeta-1} \delta_\nu(s) e^s ds\right| &\le \frac{\bar{\varepsilon}(e^\alpha-1)^\zeta}{\Gamma(\zeta+1)} \\
&\le \frac{\bar{\varepsilon}(e^b-1)^\zeta}{\Gamma(\zeta+1)},
\end{aligned} \tag{27}$$

where $\delta_\nu \in \mathscr{V}$ satisfies

$$\delta_\nu(\alpha) = \vartheta(\alpha, \nu(0) + {}^e I_{0+}^{\zeta} \delta_\nu(\alpha), \delta_\nu(\alpha)).$$

For each $\alpha \in \Theta$, we have

$$\begin{aligned}
|\nu(\alpha) - \psi(\alpha)| &= \left| \nu(\alpha) - \nu(0) - \frac{1}{\Gamma(\zeta)} \int_0^\alpha (e^\alpha - e^s)^{\zeta-1} \delta_\psi(s) e^s ds \right| \\
&= \left| \nu(\alpha) - \nu(0) - \frac{1}{\Gamma(\zeta)} \int_0^\alpha (e^\alpha - e^s)^{\zeta-1} \delta_\nu(s) e^s ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\zeta)} \int_0^\alpha (e^\alpha - e^s)^{\zeta-1} \left(\delta_\nu(s) - \delta_\psi(s)\right) e^s ds \right| \\
&\leq \left| \nu(\alpha) - \nu(0) - \frac{1}{\Gamma(\zeta)} \int_0^\alpha (e^\alpha - e^s)^{\zeta-1} \delta_\nu(s) e^s ds \right| \\
&\quad + \frac{1}{\Gamma(\zeta)} \int_0^\alpha (e^\alpha - e^s)^{\zeta-1} |\delta_\nu(s) - \delta_\psi(s)| e^s ds, \qquad (28)
\end{aligned}$$

where

$$\delta_\psi(\alpha) = \vartheta(\alpha, \psi(\alpha), \delta_\psi(\alpha)),$$

and

$$\delta_\nu(\alpha) = \vartheta(\alpha, \nu(\alpha), \delta_\nu(\alpha)).$$

By $(\mathscr{A}_2)$, we obtain

$$\begin{aligned}
|\delta_\nu(\alpha) - \delta_\psi(\alpha)| &= |\vartheta(\alpha, \nu(\alpha), \delta_\nu(\alpha)) - \vartheta(\alpha, \psi(\alpha), \delta_\psi(\alpha))| \\
&\leq \rho_1 |\nu(\alpha) - \psi(\alpha)| + \rho_2 |\delta_\nu(\alpha) - \delta_\psi(\alpha)|.
\end{aligned}$$

Then

$$|\delta_\nu(\alpha) - \delta_\psi(\alpha)| \leq \rho_3 |\nu(\alpha) - \psi(\alpha)|. \qquad (29)$$

Thus, by (27), (28), and (29) we get

$$|\nu(\alpha) - \psi(\alpha)| \leq \frac{\bar{\varepsilon}(e^b - 1)^\zeta}{\Gamma(\zeta + 1)} + \frac{\rho_3}{\Gamma(\zeta)} \int_0^\alpha (e^\alpha - e^s)^{\zeta-1} |\nu(s) - \psi(s)| e^s ds.$$

Then Lemma 6 implies the following

$$\begin{aligned}
|\nu(\alpha) - \psi(\alpha)| &\leq \frac{\bar{\varepsilon}(e^b - 1)^\zeta}{\Gamma(\zeta + 1)} \left[1 + \int_0^\alpha \sum_{k=1}^\infty \frac{\rho_3{}^k}{\Gamma(k\zeta)} (e^\alpha - e^s)^{k\zeta - 1} e^s ds \right] \\
&\leq \frac{\bar{\varepsilon}(e^b - 1)^\zeta}{\Gamma(\zeta + 1)} \left[1 + \sum_{k=1}^\infty \frac{\rho_3{}^k}{\Gamma(k\zeta)} \frac{(e^\alpha - 1)^{k\zeta}}{k\zeta} \right] \\
&\leq \frac{\bar{\varepsilon}(e^b - 1)^\zeta}{\Gamma(\zeta + 1)} \left[1 + \sum_{k=1}^\infty \frac{\rho_3{}^k}{\Gamma(k\zeta + 1)} (e^\alpha - 1)^{k\zeta} \right] \\
&\leq \frac{\bar{\varepsilon}(e^b - 1)^\zeta}{\Gamma(\zeta + 1)} E_\zeta \left(\rho_3 (e^b - 1) \right) := c\bar{\varepsilon}.
\end{aligned}$$

Hence, Equation (1) is (U-H) stable. Taking $\tilde{\gamma}(\bar{\varepsilon}) = c\bar{\varepsilon}$, $\tilde{\gamma}(0) = 0$ yields that Equation (1) is (G.U-H).

5 Ulam–Hyers–Rassias stability

Theorem 7. *Assume* $(\mathscr{A}_1)$, $(\mathscr{A}_2)$, (17) *and*

$(\mathscr{A}_4)$ *The function* $\chi \in C(\Theta, \mathbb{R}_+)$ *is increasing and there exists* $\lambda_\chi > 0$ *such that, for each* $\alpha \in \Theta$, *we have*

$${}^e I_{0+}^{\zeta} \chi(\alpha) \leq \lambda_\chi \chi(\alpha).$$

Then, Equation (1) *is (U-H-R) stable with respect to* χ.

Proof. Let $\nu \in \mathscr{V}$ be a solution (14), then

$$|{}_{c}^{e}D_{0+}^{\zeta}\nu(\alpha) - \vartheta(\alpha, \nu(\alpha),\ {}_{c}^{e}D_{0+}^{\zeta}\nu(\alpha))| \leq \bar{\varepsilon}\chi(\alpha),\ \alpha \in \Theta,\ \bar{\varepsilon} > 0. \tag{30}$$

By $\psi \in \mathscr{V}$, we denote the unique solution of the problem

$$\begin{aligned} {}_{c}^{e}D_{0+}^{\zeta}\psi(\alpha) &= \vartheta(\alpha, \psi(\alpha),\ {}_{c}^{e}D_{0+}^{\zeta}\psi(\alpha)),\ \text{for each},\ \alpha \in \Theta,\ 0 < \zeta \leq 1, \\ \psi(0) &= \nu(0). \end{aligned}$$

Using Lemma 7, we have

$$\psi(\alpha) = \nu(0) + \frac{1}{\Gamma(\zeta)}\int_0^{\alpha}(e^{\alpha} - e^{s})^{\zeta-1}\delta_{\psi}(s)e^{s}ds,$$

where $\delta_{\psi} \in \mathscr{V}$ satisfies

$$\delta_{\psi}(\alpha) = \vartheta(\alpha, \psi(0) + {}^{e}I_{0+}^{\zeta}\delta_{\psi}(\alpha), \delta_{\psi}(\alpha)).$$

By integration (30) and by $(\mathscr{A}_3)$, we get

$$\begin{aligned} \left|\nu(\alpha) - \nu(0) - \frac{1}{\Gamma(\zeta)}\int_0^{\alpha}(e^{\alpha} - e^{s})^{\zeta-1}\delta_{\nu}(s)e^{s}ds\right| &\leq \frac{\bar{\varepsilon}}{\Gamma(\zeta)}\int_0^{\alpha}(e^{\alpha} - e^{s})^{\zeta-1}\chi(s)e^{s}ds \\ &\leq \bar{\varepsilon}\lambda_{\chi}\chi(\alpha). \end{aligned} \tag{31}$$

where $\delta_{\nu} \in \mathscr{V}$

$$\delta_{\nu}(\alpha) = \vartheta(\alpha, \nu(0) + {}^{e}I_{0+}^{\zeta}\delta_{\nu}(\alpha), \delta_{\nu}(\alpha)).$$

For each $\alpha \in \Theta$, we have

$$\begin{aligned} |\nu(\alpha) - \psi(\alpha)| &= \left|\nu(\alpha) - \nu(0) - \frac{1}{\Gamma(\zeta)}\int_0^{\alpha}(e^{\alpha} - e^{s})^{\zeta-1}\delta_{\psi}(s)e^{s}ds\right| \\ &= \left|\nu(\alpha) - \nu(0) - \frac{1}{\Gamma(\zeta)}\int_0^{\alpha}(e^{\alpha} - e^{s})^{\zeta-1}\delta_{\nu}(s)e^{s}ds\right. \\ &+ \left.\frac{1}{\Gamma(\zeta)}\int_0^{\alpha}(e^{\alpha} - e^{s})^{\zeta-1}\left(\delta_{\nu}(s) - \delta_{\psi}(s)\right)e^{s}ds\right| \\ &\leq \left|\nu(\alpha) - \nu(0) - \frac{1}{\Gamma(\zeta)}\int_0^{\alpha}(e^{\alpha} - e^{s})^{\zeta-1}\delta_{\nu}(s)e^{s}ds\right| \\ &+ \frac{1}{\Gamma(\zeta)}\int_0^{\alpha}(e^{\alpha} - e^{s})^{\zeta-1}|\delta_{\nu}(s) - \delta_{\psi}(s)|e^{s}ds, \end{aligned} \tag{32}$$

where

$$\delta_{\psi}(\alpha) = \vartheta(\alpha, \psi(\alpha), \delta_{\psi}(\alpha)),$$

and

$$\delta_{\nu}(\alpha) = \vartheta(\alpha, \nu(\alpha), \delta_{\nu}(\alpha)).$$

By $(\mathscr{A}_2)$, we have

$$\begin{aligned} |\delta_{\nu}(\alpha) - \delta_{\psi}(\alpha)| &= |\vartheta(\alpha, \nu(\alpha), \delta_{\nu}(\alpha)) - \vartheta(\alpha, \psi(\alpha), \delta_{\psi}(\alpha))| \\ &\leq \rho_1|\nu(\alpha) - \psi(\alpha)| + \rho_2|\delta_{\nu}(\alpha) - \delta_{\psi}(\alpha)|. \end{aligned}$$

Then

$$|\delta_{\nu}(\alpha) - \delta_{\psi}(\alpha)| \leq \rho_3|\nu(\alpha) - \psi(\alpha)|. \tag{33}$$

Thus, by (31), (32), and (33)

$$|\nu(\alpha) - \psi(\alpha)| \leq \bar{\varepsilon}\lambda_{\chi}\chi(\alpha) + \frac{\rho_3}{\Gamma(\zeta)}\int_0^{\alpha}(e^{\alpha} - e^{s})^{\zeta-1}|\nu(s) - \psi(s)|e^{s}ds.$$

$$\begin{aligned} |\nu(\alpha) - \psi(\alpha)| &\leq \bar{\varepsilon}\lambda_{\chi}\chi(\alpha) + \frac{\rho_3 \parallel \nu - \psi \parallel_{\infty}}{\Gamma(\zeta)}\int_0^{\alpha}(e^{\alpha} - e^{s})^{\zeta-1}e^{s}ds \\ &\leq \bar{\varepsilon}\lambda_{\chi}\chi(\alpha) + \frac{\rho_3 \parallel \nu - \psi \parallel_{\infty}}{\Gamma(\zeta+1)}(e^{b} - 1)^{\zeta}. \end{aligned}$$

Thus, we have

$$\| \nu - \psi \|_\infty \left[1 - \frac{\rho_3 (e^b - 1)^\zeta}{\Gamma(\zeta+1)}\right] \leq \bar{\varepsilon} \lambda_\chi \chi(\alpha),$$

and

$$\| \nu - \psi \|_\infty \leq \left[1 - \frac{\rho_3 (e^b - 1)^\zeta}{\Gamma(\zeta+1)}\right]^{-1} \bar{\varepsilon} \lambda_\chi \chi(\alpha).$$

Then for each $\alpha \in \Theta$

$$|\nu(\alpha) - \psi(\alpha)| \leq \left[1 - \frac{\rho_3 (e^b - 1)^\zeta}{\Gamma(\zeta+1)}\right]^{-1} \bar{\varepsilon} \lambda_\chi \chi(\alpha) := c \bar{\varepsilon} \chi(\alpha). \quad (34)$$

Thus, Equation (1) is (U-H-R) stable.

6 Examples

Example 1. Consider the following Cauchy problem

$${}^c D^{\frac{1}{2}} \psi(\alpha) = \frac{2 + |\psi(\alpha)| + |{}^c D^{\frac{1}{2}} \psi(\alpha)|}{150 e^{\alpha+10} (1 + |\psi(\alpha)| + |{}^c D^{\frac{1}{2}} \psi(\alpha)|)}, \quad \alpha \in [0,1], \quad (35)$$

$$\psi(0) = 1. \quad (36)$$

Set

$$\vartheta(\alpha, \psi, \nu) = \frac{2 + |\psi| + |\nu|}{150 e^{\alpha+10} (1 + |\psi| + |\nu|)}, \quad \alpha \in [0,1], \ \psi, \nu \in \mathbb{R}.$$

It is obvious that the function ϑ is jointly continuous.
For any $\psi, \nu, \bar{\psi}, \bar{\nu} \in \mathbb{R}$ and $\alpha \in [0,1]$

$$|\vartheta(\alpha, \psi, \nu) - \vartheta(\alpha, \bar{\psi}, \bar{\nu})| \leq \frac{1}{150 e^{10}} (|\psi - \bar{\psi}| + |\nu - \bar{\nu}|).$$

Hence condition $(\mathscr{A}_2)$ is satisfied with $\rho_1 = \rho_2 = \frac{1}{150 e^{10}}$.
Since

$$|\vartheta(\alpha, \psi, \nu)| \leq \frac{1}{150 e^{\alpha+10}} (2 + |\psi| + |\nu|),$$

then the condition $(\mathscr{A}_3)$ is verified with

$$\theta_1(\alpha) = \frac{1}{75 e^{\alpha+10}} \text{ and } \theta_2(\alpha) = \theta_3(\alpha) = \frac{1}{150 e^{\alpha+10}}.$$

And condition

$$\frac{\theta_2{}^* b^\zeta}{(1 - \theta_3{}^*) \Gamma(\zeta+1)} = \frac{(e-1)^{\frac{1}{2}}}{(150 e^{10} - 1) \Gamma(\frac{3}{2})} = \frac{1}{(150 e^{10} - 1)} \sqrt{\frac{e-1}{\pi}} < 1,$$

is verified with $b = 1$, $\zeta = \frac{1}{2}$, and $\theta_2{}^* = \theta_3{}^* = \frac{1}{150 e^{10}}$. As consequence of Theorem 4, we can say that problem (35)–(36) has at least one solution.

Example 2. Consider the following Cauchy problem

$${}^e_c D^{\frac{1}{2}} \psi(\alpha) = \frac{1}{300} (\alpha \cos \psi(\alpha) - \psi(\alpha) \sin(\alpha)) + \frac{1}{150} {}^e_c D^{\frac{1}{2}} \psi(\alpha), \text{ for each } \alpha \in [0,1], \quad (37)$$

$$\psi(0) = 1. \quad (38)$$

Set

$$\vartheta(\alpha,\psi,\nu)=\frac{1}{300}(\alpha\cos\psi-\psi\sin(\alpha))+\frac{1}{150}\nu,\ \ \alpha\in[0,1],\ \psi,\nu\in\mathbb{R}$$

It is obvious that the function ϑ is jointly continuous.
For any $\psi,\nu,\bar{\psi},\bar{\nu}\in\mathbb{R}$ and $\alpha\in[0,1]$:

$$\begin{aligned}|\vartheta(\alpha,\psi,\nu)-\vartheta(\alpha,\bar{\psi},\bar{\nu})| &\leq \frac{1}{300}|\alpha||\cos\psi-\cos\bar{\psi}|+\frac{1}{300}|\sin\alpha||\psi-\bar{\psi}|+\frac{1}{150}|\nu-\bar{\nu}|\\ &\leq \frac{1}{300}|\psi-\bar{\psi}|+\frac{1}{300}|\psi-\bar{\psi}|+\frac{1}{150}|\nu-\bar{\nu}|.\\ &=\frac{1}{150}(|\psi-\bar{\psi}|+|\nu-\bar{\nu}|).\end{aligned}$$

Hence condition $(\mathscr{A}_2)$ is satisfied with $\rho_1=\rho_2=\frac{1}{150}$.
Thus condition

$$\frac{\rho_3(e^b-1)^{\zeta}}{\Gamma(\zeta+1)}=\frac{\frac{1}{150}}{(1-\frac{1}{150})\Gamma(\frac{3}{2})}=\frac{2}{149\sqrt{\pi}}\approx 0.00757<1,$$

is satisfied with $\rho_1=\rho_2=\frac{1}{150}$, $b=1$, and $\zeta=\frac{1}{2}$. It follows from Theorem 3 that the problem (37)–(38) as a unique solution on Θ. And it follows from Theorem 6 that the problem (37)–(38) is (U-H) stable.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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