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Trigonometric Commutative Caputo Fractional Korovkin Theory for Stochastic Processes

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Abstract: Here we consider and study from the trigonometric point of view expectation commutative stochastic positive linear operators acting on L^1-continuous stochastic processes which are Caputo fractional differentiable. Under some mild, general and natural assumptions on the stochastic processes we produce related trigonometric Caputo fractional stochastic Shisha-Mond type inequalities pointwise and uniform. All convergences are produced with rates and are given by the trigonometric fractional stochastic inequalities involving the first modulus of continuity of the expectation of the α-th right and left fractional derivatives of the engaged stochastic process, $\alpha > 0$, $\alpha \notin \mathbb{N}$. The amazing fact here is that the basic non-stochastic real Korovkin test functions assumptions impose the conclusions of our trigonometric Caputo fractional stochastic Korovkin theory. We include also a detailed trigonometric application to stochastic Bernstein operators.

Keywords: Stochastic positive linear operator, trigonometric Caputo fractional stochastic Korovkin theory and trigonometric fractional inequalities, trigonometric Caputo fractional stochastic Shisha-Mond inequality, modulus of continuity, stochastic process, expectation commutative operator.

1 Introduction

In this work among others we are motivated by the following results.

Theorem A (P.P. Korovkin [1], (1960)) Let $L_n : C([-\pi,\pi]) \to C([-\pi,\pi])$, $n \in \mathbb{N}$, be a sequence of positive linear operators. Assume $L_n(1) \xrightarrow{u} 1$ (uniformly), $L_n(\cos t) \xrightarrow{u} \cos t$, $L_n(\sin t) \xrightarrow{u} \sin t$, as $n \to \infty$. Then $L_n f \xrightarrow{u} f$, for every $f \in C([-\pi,\pi])$ that is 2π-periodic.

Let $f \in C([a,b])$ and $0 \le \delta \le b-a$. The first modulus of continuity of f at δ is given by

$$\omega_1(f,\delta) = \sup\{|f(x)-f(y)|;\, x,y \in [a,b],\ |x-y| \le \delta\}.$$

If $\delta > b-a$, then we define

$$\omega_1(f,\delta) = \omega_1(f,b-a).$$

Another motivation is the following.

Theorem B (Shisha and Mond [2], (1968)). Let $L_1, L_2, \ldots,$ be linear positive operators, whose common domain D consists of real functions with domain $(-\infty,\infty)$. Suppose $1, \cos x, \sin x, f$ belong to D, where f is an everywhere continuous, 2π-periodic function, with modulus of continuity ω_1. Let $-\infty < a < b < \infty$, and suppose that for $n = 1,2,\ldots, L_n(1)$ is bounded in $[a,b]$. Then for $n = 1,2,\ldots,$

$$\|L_n(f) - f\|_\infty \le \|f\|_\infty \|L_n(1) - 1\|_\infty + \|L_n(1)+1\|_\infty \,\omega_1(f,\mu_n), \tag{*}$$

where

$$\mu_n = \pi \left\| \left(L_n\left(\sin^2\left(\frac{t-x}{2}\right)\right)\right)(x) \right\|_\infty^{\frac{1}{2}},$$

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and $\|\cdot\|_\infty$ stands for the sup norm over $[a,b]$.

In particular, if $L_n(1)=1$, then (*) reduces to

$$\|L_n(f)-f\|_\infty \le 2\omega_1(f,\mu_n).$$

One can easily see that, for $m=1,2,\ldots,$

$$\mu_n^2 \le \left(\frac{\pi^2}{2}\right)[\|L_n(1)-1\|_\infty +$$

$$\|(L_n(\cos t))(x)-\cos x\|_\infty + \|(L_n(\sin t))(x)-\sin x\|_\infty],$$

so the last along with (*) prove Korovkin's Theorem A in a quantitative way and with rates of convergence.

One more motivation follows.

Theorem C (see [3], p. 217). Let $f\in C^n([-\pi,\pi])$, $n\ge 1$, and μ a measure on $[-\pi,\pi]$ of mass $m>0$. Put

$$\beta := \left(\int\left(\sin\frac{|t|}{2}\right)^{n+1}\cdot\mu(dt)\right)^{\frac{1}{n+1}}$$

and denote by $\omega := \omega_1\left(f^{(n)},\beta\right)$ the modulus of continuity of $f^{(n)}$ at β. Then

$$\left|\int f d\mu - f(0)\right| \le |f(0)|\cdot|m-1| + \sum_{k=1}^{n}\frac{\left|f^{(k)}(0)\right|}{k!}\cdot\left|\int t^k\mu(dt)\right| +$$

$$w\left[m^{\frac{1}{n+1}}+\frac{\pi}{n+1}\right]\cdot\frac{\pi^n\beta^n}{n!}.$$

Anastassiou in [4]-[6] established a series of sharp inequalities for various cases of the parameters of the problem. However, Weba in [7]-[10] was the first, among many workers in quantitative results of Shisha-Mond type, to produce inequalities for stochastic processes. He assumed that the positive linear operators L_j are E-commutative (E means expectation) and stochastically simple. According to his work, if a stochastic process $X(t,\omega)$, $t\in Q$ - a compact convex subset of a real normed vector space, $\omega\in Q$ - probability space, is to be approximated by positive linear operators L_j, then the maximal error in the qth mean is ($q\ge 1$)

$$\|L_jX-X\| = \sup_{t\in Q}\left(E\left|(L_jX)(t,\omega)-X(t,\omega)\right|^q\right)^{\frac{1}{q}}.$$

So, Weba established upper bounds for $\|L_jX-X\|$ involving his own natural general first modulus of continuity of X with several interesting applications.

Anastassiou met ([11]) the pointwise case of $q=1$. Without stochastic simplicity of L_j he found nearly best and best upper bounds for $\left|E(L_jX)(x_0)-(EX)(x_0)\right|$, $x_0\in Q$.

The author here continues his above work on the trigonometric approximation of stochastic processes, now at the Caputo stochastic fractional level. He derives pointwise and uniform trigonometric Caputo fractional stochastic Shisha-Mond type inequalities, see the main Theorems 3, 4 and the several related corollaries. He gives an extensive trigonometric application to stochastic Bernstein operators. He finishes with a pointwise and a uniform fractional trigonometric stochastic Korovkin theorem, derived by Theorems 3, 4. The stochastic convergences, about stochastic processes, of our trigonometric fractional Korovkin Theorems 5, 6 are implied only by the convergences of real basic non-stochastic functions.

Our results here are built on [12].

2 Background - I

We need

Definition 1.([13]) *Let non-integer $\alpha > 0$, $n = \lceil \alpha \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $t \in [a,b] \subset \mathbb{R}$, $\omega \in \Omega$, where $(\Omega, \mathscr{F}, P)$ is a general probability space. Here $X(t,\omega)$ stands for a stochastic process. Assume that $X(\cdot,\omega) \in AC^n([a,b])$ (spaces of functions $X(\cdot,\omega)$ with $X^{(n-1)}(\cdot,\omega) \in AC([a,b])$ absolutely continuous functions), $\forall\ \omega \in \Omega$.*

We call stochastic left Caputo fractional derivative

$$D_{*a}^{\alpha} X(x,\omega) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} X^{(n)}(t,\omega)\, dt, \tag{1}$$

$\forall\ x \in [a,b]$, $\forall\ \omega \in \Omega$.

And, we call stochastic right Caputo fractional derivative

$$D_{b-}^{\alpha} X(x,\omega) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (z-x)^{n-\alpha-1} X^{(n)}(z,\omega)\, dz, \tag{2}$$

$\forall\ x \in [a,b]$, $\forall\ \omega \in \Omega$. *Above Γ stands for the gamma function.*

We make

Remark.(to Definition 1) We further assume here that

$$\left| X^{(n)}(t,\omega) \right| \le M, \ \forall\ (t,\omega) \in [a,b] \times \Omega,$$

where $M > 0$.

Then, by (1), we have

$$|D_{*a}^{\alpha} X(x,\omega)| \le \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} \left| X^{(n)}(t,\omega) \right| dt \le$$

$$\frac{M}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} dt = \frac{M(x-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}.$$

That is

$$|D_{*a}^{\alpha} X(x,\omega)| \le \frac{M(x-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}, \quad \forall\ x \in [a,b], \text{ any } \omega \in \Omega. \tag{3}$$

Also, from (2) we get

$$\left|D_{b-}^{\alpha} X(x,\omega)\right| \le \frac{1}{\Gamma(n-\alpha)} \int_x^b (z-x)^{n-\alpha-1} \left| X^{(n)}(z,\omega) \right| dz \le$$

$$\frac{M}{\Gamma(n-\alpha)} \int_x^b (z-x)^{n-\alpha-1} dz = \frac{M(b-x)^{n-\alpha}}{\Gamma(n-\alpha+1)}.$$

That is

$$\left|D_{b-}^{\alpha} X(x,\omega)\right| \le \frac{M(b-x)^{n-\alpha}}{\Gamma(n-\alpha+1)}, \quad \forall\ x \in [a,b], \text{ any } \omega \in \Omega. \tag{4}$$

It is not strange to assume that $D_{*a}^{\alpha} X$, $D_{b-}^{\alpha} X$ are stochastic processes.

By [14], p. 388, we get that $D_{*a}^{\alpha} X(\cdot,\omega) \in C([a,b])$, $\forall\ \omega \in \Omega$. And by [15], we get that $D_{b-}^{\alpha} X(\cdot,\omega) \in C([a,b])$, $\forall\ \omega \in \Omega$.

Similarly, we obtain

$$|D_{*t}^{\alpha} X(x,\omega)| \le \frac{M(x-t)^{n-\alpha}}{\Gamma(n-\alpha+1)} \le \frac{M(b-t)^{n-\alpha}}{\Gamma(n-\alpha+1)}, \tag{5}$$

$\forall\ x \in [t,b]$, any $t \in [a,b]$, $\forall\ \omega \in \Omega$,

and

$$\left|D_{t-}^{\alpha} X(x,\omega)\right| \le \frac{M(t-x)^{n-\alpha}}{\Gamma(n-\alpha+1)} \le \frac{M(t-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}, \tag{6}$$

$\forall\ x \in [a,t]$, any $t \in [a,b]$, $\forall\ \omega \in \Omega$.

Above $D_{*t}^{\alpha}X$, $D_{t-}^{\alpha}X$ are assumed to be stochastic processes for any $t \in [a,b]$, and it holds $D_{*t}^{\alpha}X(\cdot,\omega) \in C([t,b])$, $D_{t-}^{\alpha}X(\cdot,\omega) \in C([a,t])$, $\forall\, \omega \in \Omega$.

Clearly, then

$$\left|E\left(D_{*t}^{\alpha}X\right)(x)\right| \leq \frac{M(b-t)^{n-\alpha}}{\Gamma(n-\alpha+1)}, \tag{7}$$

$\forall\, x \in [t,b]$, any $t \in [a,b]$, where E is the expectation operator $(EX)(t) = \int_{\Omega} X(t,\omega) P(d\omega)$ and similarly,

$$\left|E\left(D_{t-}^{\alpha}X\right)(x)\right| \leq \frac{M(t-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}, \tag{8}$$

$\forall\, x \in [a,t]$, any $t \in [a,b]$.

We observe that the first modulus of continuity $(\delta > 0)$

$$\omega_1\left(E\left(D_{*t}^{\alpha}X\right),\delta\right)_{[t,b]} := \sup_{\substack{x,y\in[t,b]:\\ |x-y|\leq\delta}} \left|E\left(D_{*t}^{\alpha}X\right)(x) - E\left(D_{*t}^{\alpha}X\right)(y)\right|$$

$$\overset{(7)}{\leq} \frac{2M(b-t)^{n-\alpha}}{\Gamma(n-\alpha+1)}, \tag{9}$$

any $t \in [a,b]$.

Hence, it holds $(\delta > 0)$

$$\sup_{t\in[a,b]} \omega_1\left(E\left(D_{*t}^{\alpha}X\right),\delta\right)_{[t,b]} \leq \frac{2M(b-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}. \tag{10}$$

Similarly, it holds $(\delta > 0)$

$$\sup_{t\in[a,b]} \omega_1\left(E\left(D_{t-}^{\alpha}X\right),\delta\right)_{[a,t]} \leq \frac{2M(b-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}. \tag{11}$$

We also set

$$\omega_1\left(E\left(D_t^{\alpha}X\right),\delta\right) := \max\left\{\omega_1\left(E\left(D_{*t}^{\alpha}X\right),\delta\right)_{[t,b]}, \omega_1\left(E\left(D_{t-}^{\alpha}X\right),\delta\right)_{[a,t]}\right\}, \tag{12}$$

where $\delta > 0$.

We make

*Remark.*Let the positive linear operator L mapping $C([a,b])$ into $B([a,b])$ (the bounded functions). By the Riesz representation theorem ([16]) we have that there exists μ_t unique, completed Borel measure on $[a,b]$ with

$$\mu_t([a,b]) = L(1)(t) > 0, \tag{13}$$

such that

$$L(f)(t) = \int_{[a,b]} f(s)\, d\mu_t(s), \quad \forall\, t \in [a,b], \forall\, f \in C([a,b]). \tag{14}$$

We denote $\|\cdot\|_{\infty} = \|\cdot\|_{\infty,[-\pi,\pi]}$ the supremum norm.

Next we specify $[a,b]$ as $[-\pi,\pi]$. Clearly then $L: C([-\pi,\pi]) \to B([-\pi,\pi])$ is the positive linear operator on hand.

Here $n = \lceil\alpha\rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, $k = 1,...,n-1$. By the use of Hölder's inequality we notice that

$$L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^k\right)(t) = \int_{[-\pi,\pi]} \left(\sin\left(\frac{|s-t|}{4}\right)\right)^k d\mu_t(s) \leq$$

$$\left(\int_{[-\pi,\pi]} \left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1} d\mu_t(s)\right)^{\frac{k}{\alpha+1}} \left(\mu_t([-\pi,\pi])\right)^{\frac{\alpha+1-k}{\alpha+1}} = \tag{15}$$

$$\left(L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{k}{\alpha+1}} \left(L(1)(t)\right)^{\frac{\alpha+1-k}{\alpha+1}}.$$

That is

$$L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{k}\right)(t) \le \left(L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{k}{\alpha+1}} (L(1)(t))^{\frac{\alpha+1-k}{\alpha+1}}, \tag{16}$$

for $k = 1, ..., n-1$.

Consequently, it holds

$$\left\|L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{k}\right)(t)\right\|_{\infty} \le \left\|L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{k}{\alpha+1}} \|L(1)\|_{\infty}^{\frac{\alpha+1-k}{\alpha+1}}, \tag{17}$$

for $k = 1, ..., n-1$.

In this work we will use a lot the following well known inequality:

$$|z| \le \pi \sin\left(\frac{|z|}{2}\right), \ \forall z \in [-\pi, \pi]. \tag{18}$$

Furthermore, we observe that

$$\left|L\left((s-t)^{k}\right)(t)\right| = \left|\int_{[-\pi,\pi]} (s-t)^{k} d\mu_t(s)\right| \le \int_{[-\pi,\pi]} |s-t|^{k} d\mu_t(s) =$$

$$2^{k}\int_{[-\pi,\pi]} \left(\frac{|s-t|}{2}\right)^{k} d\mu_t(s) \overset{(18)}{\le} (2\pi)^{k} \int_{[-\pi,\pi]} \left(\sin\left(\frac{|s-t|}{4}\right)\right)^{k} d\mu_t(s) \tag{19}$$

$$= (2\pi)^{k} L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{k}\right)(t).$$

That is

$$\left|L\left((s-t)^{k}\right)(t)\right| \le (2\pi)^{k} L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{k}\right)(t), \tag{20}$$

$\forall t \in [-\pi, \pi]$, and

$$\left\|L\left((s-t)^{k}\right)(t)\right\|_{\infty} \le (2\pi)^{k} \left\|L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{k}\right)(t)\right\|_{\infty}, \tag{21}$$

all $k = 1, ..., n-1$.

Then, by (16) and (20) we get

$$\left|L\left((s-t)^{k}\right)(t)\right| \le (2\pi)^{k} \left(L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{k}{\alpha+1}} (L(1)(t))^{\frac{\alpha+1-k}{\alpha+1}}, \tag{22}$$

$k = 1, ..., n-1$, $\forall t \in [-\pi, \pi]$, and by (17) and (21), we find

$$\left\|L\left((s-t)^{k}\right)(t)\right\|_{\infty} \le (2\pi)^{k} \left\|L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right\|^{\frac{k}{\alpha+1}} \|L(1)\|_{\infty}^{\frac{\alpha+1-k}{\alpha+1}}, \tag{23}$$

$k = 1, ..., n-1$.

We also have

$$L\left(|s-t|^{\alpha+1}\right)(t) = \int_{[-\pi,\pi]} |s-t|^{\alpha+1} d\mu_t(s) =$$

$$2^{\alpha+1}\int_{[-\pi,\pi]}\left(\frac{|s-t|}{2}\right)^{\alpha+1} d\mu_t(s) \overset{(18)}{\leq} (2\pi)^{\alpha+1}\int_{[-\pi,\pi]}\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1} d\mu_t(s) \tag{24}$$

$$= (2\pi)^{\alpha+1} L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t).$$

That is

$$L\left(|s-t|^{\alpha+1}\right)(t) \leq (2\pi)^{\alpha+1} L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t), \tag{25}$$

$\forall\, t \in [-\pi,\pi]$,
and

$$\left\| L\left(|s-t|^{\alpha+1}\right)(t)\right\|_\infty \leq (2\pi)^{\alpha+1}\left\| L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right\|_\infty. \tag{26}$$

Also we have

$$D_{*t}^{\alpha} X(t,\omega) = D_{t-}^{\alpha} X(t,\omega) = 0, \tag{27}$$

$\forall\, \omega \in \Omega$, see [17], pp. 358-359.
We assume that

$$D_{*t}^{\alpha} X(s,\omega) = 0, \text{ for } s < t,$$

and

$$D_{t-}^{\alpha} X(s,\omega) = 0, \text{ for } s > t,$$

$\forall\, \omega \in \Omega$.

3 Preliminaries

Let $(\Omega,\mathscr{F},P)$ be a probabilistic space and $L^1(\Omega,\mathscr{F},P)$ be the space of all real-valued random variables $Y = Y(\omega)$ with

$$\int_\Omega |Y(\omega)|\, P(d\omega) < \infty.$$

Let $X = X(t,\omega)$ denote a stochastic process with index set $[a,b] \subset \mathbb{R}$ and real state space $(\mathbb{R},\mathscr{B})$, where $\mathscr{B}$ is the σ-field of Borel subsets of $\mathbb{R}$. Here $C([a,b])$ is the space of continuous real-valued functions on $[a,b]$ and $\mathscr{B}([a,b])$ is the space of bounded real-valued functions on $[a,b]$. Also $C_\Omega([a,b]) = C\left([a,b], L^1(\Omega,\mathscr{F},P)\right)$ is the space of L^1-continuous stochastic processes in t and $B_\Omega([a,b]) = \left\{X : \sup_{t\in[a,b]} \int_\Omega |X(t,\omega)|\, P(d\omega) < \infty\right\}$, obviously $C_\Omega([a,b]) \subset B_\Omega([a,b])$.

Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $\lceil\alpha\rceil = n$, and consider the subspace of stochastic processes $C_\Omega^{\alpha,n}([a,b]) := \{X : X(\cdot,\omega) \in AC^n([a,b]),\ \forall\ \omega \in \Omega$ and $\left|X^{(n)}(t,\omega)\right| \leq M,\ \forall\ (t,\omega) \in [a,b]\times\Omega$, where $M > 0$; $X^{(k)}(t,\omega) \in C_\Omega([a,b])$, $k = 0,1,\ldots,n-1$; also $D_{*t}^{\alpha}X$, $D_{t-}^{\alpha}X$ are stochastic processes for any $t \in [a,b]\}$. That is, for every $\omega \in \Omega$ we have $X(t,\omega) \in C^{n-1}([a,b])$.

Consider the linear operator

$$L : C_\Omega([a,b]) \hookrightarrow B_\Omega([a,b]).$$

If $X \in C_\Omega([a,b])$ is nonnegative and LX, too, then L is called positive. If $EL = LE$, then L is called E-commutative.

4 Background - II

Following 3. Preliminaries we proved

Theorem 1.([12]) *Consider the positive E-commutative linear operator* $L : C_{\Omega}([a,b]) \hookrightarrow B_{\Omega}([a,b])$, *and* $\alpha > 0$, $\alpha \notin \mathbb{N}$, $\lceil \alpha \rceil = n$, *and let* $X \in C_{\Omega}^{\alpha,n}([a,b])$, *with* $\delta > 0$.

Then

$$|(E(LX))(t) - (EX)(t)| \leq |(EX)(t)| |(L(1))(t) - 1| + \sum_{k=1}^{n-1} \frac{\left|\left(EX^{(k)}\right)(t)\right|}{k!} \left|L\left((s-t)^k\right)(t)\right| + \frac{\omega_1(E(D_t^{\alpha}X), \delta)}{\Gamma(\alpha+1)} \left(L\left(|s-t|^{\alpha+1}\right)(t)\right)^{\frac{\alpha}{\alpha+1}} \left[(L(1)(t))^{\frac{1}{\alpha+1}} + \frac{\left(L\left(|s-t|^{\alpha+1}\right)(t)\right)^{\frac{1}{\alpha+1}}}{\delta(\alpha+1)}\right], \tag{28}$$

$\forall\, t \in [a,b]$.

Above $\omega_1(E(D_t^{\alpha}X), \delta)$ *is as in* (12).

We also mention

Theorem 2.([12]) *All as in Theorem 1. Then*

$$\|E(LX) - EX\|_{\infty} \leq \|EX\|_{\infty} \|L(1) - 1\|_{\infty} + \sum_{k=1}^{n-1} \frac{\left\|EX^{(k)}\right\|_{\infty}}{k!} \left\|L\left((s-t)^k\right)(t)\right\|_{\infty} + \sup_{t\in[a,b]} \frac{\omega_1(E(D_t^{\alpha}X), \delta)}{\Gamma(\alpha+1)} \left\|L\left(|s-t|^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{\alpha}{\alpha+1}} \left[\|L(1)\|_{\infty}^{\frac{1}{\alpha+1}} + \frac{\left\|L\left(|s-t|^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{1}{\alpha+1}}}{\delta(\alpha+1)}\right] < \infty. \tag{29}$$

We specify:

Definition 2.*If* $0 < \alpha < 1$, *then* $n = 1$, *and* $C_{\Omega}^{\alpha,1}([a,b]) := \{X : X(\cdot,\omega) \in AC([a,b]), \forall\, \omega \in \Omega$ *and* $\left|X^{(1)}(t,\omega)\right| \leq M$, $\forall$ $(t,\omega) \in [a,b] \times \Omega$, *where* $M > 0$; $X(t,\omega) \in C_{\Omega}([a,b])$; *also* $D_{*t}^{\alpha}X$, $D_{t-}^{\alpha}X$ *are stochastic processes for any* $t \in [a,b]\}$. *We will specialize on* $C_{\Omega}^{\alpha,1}([-\pi,\pi])$.

*Note 1.*From [9, pp. 3-5] we have the following results

(i) $C([a,b]) \subset C_{\Omega}([a,b])$,

(ii) if $X \in C_{\Omega}([a,b])$, then $EX \in C([a,b])$,

and

(iii) if L is E-commutative, then L maps the subspace $C([a,b])$ into $B([a,b])$.

5 Main Results

We present the following pointwise result over $[-\pi,\pi]$.

Theorem 3.*Consider the 3. Preliminaries for* $[a,b] = [-\pi,\pi]$ *and the positive E-commutative linear operator* $L : C_{\Omega}([-\pi,\pi]) \hookrightarrow B_{\Omega}([-\pi,\pi])$, *and* $\alpha > 0$, $\alpha \notin \mathbb{N}$, $\lceil \alpha \rceil = n$, *and let* $X \in C_{\Omega}^{\alpha,n}([-\pi,\pi])$, *with* $\delta > 0$.

Then

$$|(E(LX))(t) - (EX)(t)| \leq |(EX)(t)| |(L(1))(t) - 1| + \sum_{k=1}^{n-1} \frac{\left|\left(EX^{(k)}\right)(t)\right|}{k!} (2\pi)^k L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^k\right)(t) + \frac{\omega_1(E(D_t^{\alpha}X), \delta)}{\Gamma(\alpha+1)} (2\pi)^{\alpha} \left(L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{\alpha}{\alpha+1}}$$

$$\left[(L(1)(t))^{\frac{1}{\alpha+1}}+\frac{2\pi}{\delta(\alpha+1)}\left(L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{1}{\alpha+1}}\right], \quad (30)$$

$\forall\, t\in[-\pi,\pi]$.

Above $\omega_1(E(D_t^{\alpha}X),\delta)$ *is over* $[-\pi,\pi]$.

Proof. From (28) we have ($\forall\, t\in[-\pi,\pi]$)

$$|(E(LX))(t)-(EX)(t)|\leq|(EX)(t)|\,|(L(1))(t)-1|+$$

$$\sum_{k=1}^{n-1}\frac{\left|\left(EX^{(k)}\right)(t)\right|}{k!}\left|L\left((s-t)^k\right)(t)\right|+\frac{\omega_1(E(D_t^{\alpha}X),\delta)}{\Gamma(\alpha+1)}$$

$$\left(L\left(|s-t|^{\alpha+1}\right)(t)\right)^{\frac{\alpha}{\alpha+1}}\left[(L(1)(t))^{\frac{1}{\alpha+1}}+\frac{\left(L\left(|s-t|^{\alpha+1}\right)(t)\right)^{\frac{1}{\alpha+1}}}{\delta(\alpha+1)}\right]$$

$$\overset{\text{(by (20), (25))}}{\leq}|(EX)(t)|\,|(L(1))(t)-1|+$$

$$\sum_{k=1}^{n-1}\frac{\left|\left(EX^{(k)}\right)(t)\right|}{k!}(2\pi)^kL\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^k\right)(t)+$$

$$\frac{\omega_1(E(D_t^{\alpha}X),\delta)}{\Gamma(\alpha+1)}(2\pi)^{\alpha}\left(L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{\alpha}{\alpha+1}}$$

$$\left[(L(1)(t))^{\frac{1}{\alpha+1}}+\frac{2\pi}{\delta(\alpha+1)}\left(L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{1}{\alpha+1}}\right], \quad (31)$$

proving the claim.

By (30) we obtain the following uniform estimate

Theorem 4. *All as in Theorem 3. Then*

$$\|E(LX)-EX\|_{\infty}\leq\|EX\|_{\infty}\|L(1)-1\|_{\infty}+$$

$$\sum_{k=1}^{n-1}\frac{\left\|EX^{(k)}\right\|_{\infty}}{k!}(2\pi)^k\left\|L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^k\right)(t)\right\|_{\infty}+$$

$$\frac{\sup\limits_{t\in[-\pi,\pi]}\omega_1(E(D_t^{\alpha}X),\delta)}{\Gamma(\alpha+1)}(2\pi)^{\alpha}\left\|L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{\alpha}{\alpha+1}}$$

$$\left[\|L(1)\|_{\infty}^{\frac{1}{\alpha+1}}+\frac{2\pi}{\delta(\alpha+1)}\left\|L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{1}{\alpha+1}}\right]<\infty. \quad (32)$$

We give

Corollary 1. *Consider the positive E-commutative linear operator $L : C_{\Omega}([-\pi,\pi]) \hookrightarrow B_{\Omega}([-\pi,\pi])$, and $0<\alpha<1$ and let $X \in C_{\Omega}^{\alpha,1}([-\pi,\pi])$, with $\delta>0$.*
Then

$$|(E(LX))(t)-(EX)(t)| \leq |(EX)(t)|\,|(L(1))(t)-1| +$$

$$\frac{\omega_1(E(D_t^{\alpha}X),\delta)}{\Gamma(\alpha+1)}(2\pi)^{\alpha}\left(L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{\alpha}{\alpha+1}}$$

$$\left[(L(1)(t))^{\frac{1}{\alpha+1}}+\frac{2\pi}{\delta(\alpha+1)}\left(L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right)^{\frac{1}{\alpha+1}}\right], \tag{33}$$

$\forall\, t \in [-\pi,\pi]$.

Proof. By (30).

We continue with

Corollary 2. *All as in Corollary 1. Then*

$$\|E(LX)-EX\|_{\infty} \leq \|EX\|_{\infty}\|L(1)-1\|_{\infty}+$$

$$\frac{\sup_{t\in[-\pi,\pi]}\omega_1(E(D_t^{\alpha}X),\delta)}{\Gamma(\alpha+1)}(2\pi)^{\alpha}\left\|L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{\alpha}{\alpha+1}}$$

$$\left[\|L(1)\|_{\infty}^{\frac{1}{\alpha+1}}+\frac{2\pi}{\delta(\alpha+1)}\left\|L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{1}{\alpha+1}}\right]. \tag{34}$$

Proof. From (33).

Corollary 3. *All as in Corollary 1 and $L(1)=1$. Then*

$$\|E(LX)-EX\|_{\infty} \leq$$

$$\frac{\sup_{t\in[-\pi,\pi]}\omega_1(E(D_t^{\alpha}X),\delta)}{\Gamma(\alpha+1)}(2\pi)^{\alpha}\left\|L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{\alpha}{\alpha+1}}$$

$$\left[1+\frac{2\pi}{\delta(\alpha+1)}\left\|L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{1}{\alpha+1}}\right]. \tag{35}$$

Proof. From (34).

In particular we give

Corollary 4. *All as in Corollary 1 and $L(1)=1$. Then*

$$\|E(LX)-EX\|_{\infty} \leq$$

$$\frac{2^{\alpha+1}\pi^{\alpha}}{\Gamma(\alpha+1)}\sup_{t\in[-\pi,\pi]}\omega_1\left(E(D_t^{\alpha}X),\frac{2\pi}{\alpha+1}\left\|L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{1}{\alpha+1}}\right)$$

$$\left\|L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{\alpha}{\alpha+1}}. \tag{36}$$

Proof. By (35): we take there

$$\delta = \frac{2\pi}{\alpha+1}\left\| L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty}^{\frac{1}{\alpha+1}} > 0.$$

In case of $\left\| L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t)\right\|_{\infty} = 0$, we have that $L\left(\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1}\right)(t) = 0$, $\forall\, t \in [-\pi,\pi]$. That is, by (14), $\int_{[-\pi,\pi]}\left(\sin\left(\frac{|s-t|}{4}\right)\right)^{\alpha+1} d\mu_t(s) = 0$, $\forall\, t \in [-\pi,\pi]$, where μ_t is a probability measure on $[-\pi,\pi]$.

Since $\sin\left(\frac{|s-t|}{4}\right) \geq 0$, $\forall\, s \in [-\pi,\pi]$, we get $\sin\left(\frac{|s-t|}{4}\right) = 0$, a.e., that is $|s-t| = 0$, a.e., and $s = t$, a.e., which means $\mu_t\{s \in [-\pi,\pi] : s \neq t\} = 0$, i.e. $\mu_t = \delta_t$, $\forall\, t \in [-\pi,\pi]$, where δ_t is the unit Dirac measure.

Consequently we have

$$E(LX)(t) = L(EX)(t) = \int_{[-\pi,\pi]}(EX)(s)\,d\delta_t(s) = (EX)(t),$$

$\forall\, t \in [-\pi,\pi]$.

That is $E(LX) = EX$ over $[-\pi,\pi]$. Therefore both sides of inequality (36) equal to zero.

Hence (36) is always true.

6 Application

Consider the Bernstein polynomials on $[-\pi,\pi]$ for $f \in C([-\pi,\pi])$:

$$B_N(f)(x) = \sum_{k=0}^{N}\binom{N}{k} f\left(-\pi+\frac{2\pi k}{N}\right)\left(\frac{x+\pi}{2\pi}\right)^k\left(\frac{\pi-x}{2\pi}\right)^{N-k}, \quad (37)$$

$N \in \mathbb{N}$, any $x \in [-\pi,\pi]$. There are positive linear operators from $C([-\pi,\pi])$ into itself.

Setting $g(t) = f(2\pi t - \pi)$, $t \in [0,1]$, we have $g(0) = f(-\pi)$, $g(1) = f(\pi)$, and

$$(B_N g)(t) = \sum_{k=0}^{N}\binom{N}{k} g\left(\frac{k}{N}\right) t^k (1-t)^{N-k} = (B_N f)(x), \quad x \in [-\pi,\pi]. \quad (38)$$

Here $x = \varphi(t) = 2\pi t - \pi$ is an $1-1$ and onto map from $[0,1]$ onto $[-\pi,\pi]$. Clearly here $g \in C([0,1])$.

Notice also that

$$\left(B_N\left((\cdot - x)^2\right)\right)(x) = \left[\left(B_N\left((\cdot - t)^2\right)\right)(t)\right](2\pi)^2 = \frac{(2\pi)^2}{N}t(1-t)$$

$$= \frac{(2\pi)^2}{N}\left(\frac{x+\pi}{2\pi}\right)\left(\frac{\pi-x}{2\pi}\right) = \frac{1}{N}(x+\pi)(\pi-x) \leq \frac{\pi^2}{N}, \quad \forall\, x \in [-\pi,\pi].$$

I.e.

$$\left(B_N\left((\cdot - x)^2\right)\right)(x) \leq \frac{\pi^2}{N}, \quad \forall\, x \in [-\pi,\pi]. \quad (39)$$

In particular

$$(B_N 1)(x) = 1, \quad \forall\, x \in [-\pi,\pi]. \quad (40)$$

Define the corresponding stochastic application of B_N as follows:

$$B_N(X(\cdot,\omega))(t) = \sum_{k=0}^{N}\binom{N}{k} X\left(-\pi+\frac{2\pi k}{N},\omega\right)\left(\frac{t+\pi}{2\pi}\right)^k\left(\frac{\pi-t}{2\pi}\right)^{N-k}, \quad (41)$$

$\forall\, N \in \mathbb{N}$, $\forall\, t \in [-\pi,\pi]$, $\forall\, \omega \in \Omega$, where X is a stochastic process. Clearly $B_N X$ is a stochastic process.

Notice that

$$E(B_N X)(t) = \sum_{k=0}^{N}\binom{N}{k}(EX)\left(-\pi+\frac{2\pi k}{N}\right)\left(\frac{t+\pi}{2\pi}\right)^k\left(\frac{\pi-t}{2\pi}\right)^{N-k}$$

$$= B_N(EX)(t), \quad (42)$$

i.e. $EB_N = B_N E$, that is B_N is an E-commutative positive linear operator from $C_\Omega([-\pi,\pi])$ into itself.

We give

Corollary 5. *Let $0 < \alpha < 1$ and $X \in C_{\Omega}^{\alpha,1}([-\pi,\pi])$. Then*

$$\|E(B_N X) - EX\|_{\infty} \leq \frac{2^{\alpha+1}\pi^{\alpha}}{\Gamma(\alpha+1)}$$

$$\sup_{t\in[-\pi,\pi]} \omega_1 \left(E\left(D_t^{\alpha} X\right), \frac{2\pi}{\alpha+1} \left\| B_N \left(\left(\sin\left(\frac{|s-t|}{4} \right) \right)^{\alpha+1} \right)(t) \right\|_{\infty}^{\frac{1}{\alpha+1}} \right)$$

$$\left\| B_N \left(\left(\sin\left(\frac{|s-t|}{4} \right) \right)^{\alpha+1} \right)(t) \right\|_{\infty}^{\frac{\alpha}{\alpha+1}}, \tag{43}$$

$\forall N \in \mathbb{N}$.

Proof. By (36).

In particular we get:

Corollary 6. *Let $X \in C_{\Omega}^{\frac{1}{2},1}([-\pi,\pi])$. Then*

$$\|E(B_N X) - EX\|_{\infty} \leq 4\sqrt{2}$$

$$\sup_{t\in[-\pi,\pi]} \omega_1 \left(E\left(D_t^{\frac{1}{2}} X\right), \frac{4\pi}{3} \left\| B_N \left(\left(\sin\left(\frac{|s-t|}{4} \right) \right)^{\frac{3}{2}} \right)(t) \right\|_{\infty}^{\frac{2}{3}} \right)$$

$$\left\| B_N \left(\left(\sin\left(\frac{|s-t|}{4} \right) \right)^{\frac{3}{2}} \right)(t) \right\|_{\infty}^{\frac{1}{3}}, \tag{44}$$

$\forall N \in \mathbb{N}$.

Proof. By (43) for $\alpha = \frac{1}{2}$.

We make

Remark. By $|\sin x| < |x|$, $\forall x \in \mathbb{R} - \{0\}$, in particular $\sin x \leq x$, for $x \geq 0$, we get

$$\left(\sin\left(\frac{|\cdot - t|}{4} \right) \right)^{\frac{3}{2}} \leq \left(\frac{|\cdot - t|}{4} \right)^{\frac{3}{2}} = \frac{1}{8} |\cdot - t|^{\frac{3}{2}}.$$

Hence

$$\left\| B_N \left(\left(\sin\left(\frac{|\cdot - t|}{4} \right) \right)^{\frac{3}{2}} \right)(t) \right\|_{\infty} \leq \frac{1}{8} \left\| B_N \left(|\cdot - t|^{\frac{3}{2}} \right)(t) \right\|_{\infty}. \tag{45}$$

We observe that

$$B_N\left(|\cdot - t|^{\frac{3}{2}}\right)(t) = \sum_{k=0}^{N} \left| t + \pi - \frac{2\pi k}{N} \right|^{\frac{3}{2}} \binom{N}{k} \left(\frac{t+\pi}{2\pi} \right)^k \left(\frac{\pi - t}{2\pi} \right)^{N-k}$$

(by discrete Hölder's inequality)

$$\leq \left[\sum_{k=0}^{N} \left(t + \pi - \frac{2\pi k}{N} \right)^2 \binom{N}{k} \left(\frac{t+\pi}{2\pi} \right)^k \left(\frac{\pi - t}{2\pi} \right)^{N-k} \right]^{\frac{3}{4}} \tag{46}$$

$$= \left(B_N\left((\cdot - t)^2 \right)(t) \right)^{\frac{3}{4}} \overset{(39)}{\leq} \frac{\pi^{\frac{3}{2}}}{N^{\frac{3}{4}}}, \ \forall t \in [-\pi,\pi].$$

Consequently it holds

$$\left\| B_N \left(|\cdot - t|^{\frac{3}{2}} \right)(t) \right\|_\infty \leq \frac{\pi^{\frac{3}{2}}}{N^{\frac{3}{4}}}, \tag{47}$$

and

$$\left\| B_N \left(\left(\sin \left(\frac{|\cdot - t|}{4} \right) \right)^{\frac{3}{2}} \right)(t) \right\|_\infty \leq \frac{\pi^{\frac{3}{2}}}{8N^{\frac{3}{4}}}, \quad \forall\, N \in \mathbb{N}. \tag{48}$$

We derive

Proposition 1. *Let $X \in C_{\Omega}^{\frac{1}{2},1}([-\pi,\pi])$. Then*

$$\|E(B_N X) - EX\|_\infty \leq \frac{2\sqrt{2\pi}}{\sqrt[4]{N}} \sup_{t\in[-\pi,\pi]} \omega_1 \left(E\left(D_t^{\frac{1}{2}} X \right), \frac{\pi^2}{3\sqrt{N}} \right), \tag{49}$$

$\forall\, N \in \mathbb{N}$.

Hence $\lim_{N\to\infty} E(B_N X) = EX$, uniformly.

Proof. By (44) and (48).

7 Commutative Trigonometric Caputo fractional Stochastic Korovkin Results

Here L is meant as a sequence of positive E-commutative linear operators and all assumptions are as in Theorem 3.

We give

Theorem 5. *We further assume that $L(1)(t) \to 1$ and $L\left(\left(\sin \left(\frac{|s-t|}{4} \right) \right)^{\alpha+1} \right)(t) \to 0$, then $(E(LX))(t) \to (EX)(t)$, for any $X \in C_{\Omega}^{\alpha,n}([-\pi,\pi])$, $\forall\, t \in [-\pi,\pi]$, a pointwise convergence; where $\alpha > 0$, $\alpha \notin \mathbb{N}$, $\lceil \alpha \rceil = n$.*

Proof. Based on (30) and (16), and that $L(1)(t)$ is bounded as a sequence of functions. Also $\omega_1(E(D_t^{\alpha} X), \delta)$ over $[-\pi,\pi]$ is bounded, see (10), (11) and (12).

We continue with

Theorem 6. *We further assume that $L(1)(t) \to 1$, uniformly and $\left\| L\left(\left(\sin \left(\frac{|s-t|}{4} \right) \right)^{\alpha+1} \right)(t) \right\|_\infty \to 0$, then $E(LX) \to EX$, uniformly over $[-\pi,\pi]$, for any $X \in C_{\Omega}^{\alpha,n}([-\pi,\pi])$; where $\alpha > 0$, $\alpha \notin \mathbb{N}$, $\lceil \alpha \rceil = n$.*

Proof. Based on (32) and (17), and that $\|L(1)\|_\infty$ is bounded. Also it is $\sup_{t\in[-\pi,\pi]} \omega_1(E(D_t^{\alpha} X), \delta) < \infty$, by (10), (11) and (12).

We finish with

Remark. The stochastic convergences of Theorems 5, 6 are derived by the convergences of the basic and simple real non-stochastic functions $\left\{ 1, \left(\sin \left(\frac{|s-t|}{4} \right) \right)^{\alpha+1} \right\}$, an amazing fact!

Conflict of Interest

The authors declare that they have no conflict of interest.

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