Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/100116

A New Three Parameter Weighted Distribution Applicable To Relief Times, Waiting Times and Carbon Fiber

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Received: 19 Mar. 2020, Revised: 20 Oct. 2020, Accepted: 11 Nov. 2020
Published online: 1 Mar. 2021

Abstract: We have obtained a new generalization of two parameter Pranav distributions by weighting technique and formulated three parameter Pranav distribution. The statistical properties which are of prime importance have been derived for proposed model. Reliability measures of proposed model have also been obtained. Likelihood ratio statistic has been used for testing significance of weight parameter. Unknown parameters of proposed model are estimated by maximum likelihood method of estimation. For examining the suitability of proposed model in the real life, we fitted the proposed model and its related models to three real life data sets.
Keywords: Weighting technique, statistical properties, maximum likelihood estimation, likelihood ratio, curve fitting, application.

1. Introduction

There are many real life situations where we analyze the data for better decision making. For analyzing the data, we fit the suitable probability models to the data. Depending on the nature of data we deal with, researchers have fitted different types of models to the real life data. Models with large number of parameters have always found greater applicability in real life, as presence of large number of parameters will increase the capability of the model to cover large dispersion from the data and provide more flexibility in applying probability models to real life situations. There are many methods by which an extra parameter can be added to existing models, one of such methods is weighted method. Sometimes, data obtained by sampling mechanism is not equally probable (i.e., observations are not recorded with equal probability), like area of agricultural holdings, size of family, etc.; in such situations we assign the appropriate weights (c) to these observations by weighting technique. By weighting technique, we add the weight parameter value of which depends on the nature of data. Fisher (1934) introduced and unified the concept of weighted distributions [1]. Hassan, Wani & Para (2018) obtained three parameter Quasi Lindley distribution by using weighting technique and obtained various properties of that model [2]. Mudasir & Ahmad (2015) studied structural properties of length biased Nakagami distribution [3]. Shukla (2018) obtained Pranav distribution as a mixture of exponential distribution and gamma distribution with shape parameter 4 and obtained its properties [4]. Patil & Rao (1978) developed weighted distributions and size biased sampling with applications to wild life populations and human families and obtained its properties [5]. Rezaeia, Nadarajah & Tahghighniac (2013) obtained a new three parameter life time distribution and studied its properties [6]. So, generalizing probability distributions to obtain more flexible probability distributions is practically needed in many situations. Hassan et al. (2019) obtained generalized Ishita distribution [7]. Hassan et al. (2019) obtained two parameter Pranav distribution [8]. Here, we have incorporated an extra parameter to two-parameter Pranav distribution which, is useful model introduced by Umeh & Ibenegbu (2019) [9].

Continuous random variable X is said to follow two parameter Pranav distribution (TPPD) if its probability density function is of the form

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$$f(x)=\frac{\theta^4}{(\alpha\theta^4+6)}\left(\alpha\theta+x^3\right)e^{-\theta x}\quad x>0,\theta>0,\alpha>0 \tag{1.1}$$

with the corresponding c.d.f $F(x)$ given below:

$$F(x)=1-\left[1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{(\alpha\theta^4+6)}\right]e^{-\theta x}\quad x>0,\theta>0,\alpha>0, \tag{1.2}$$

denoted by X~TPPD (α,θ).

In this paper we have obtained the generalization of two-parameter Pranav distribution by using weighting technique.

2. Three Parameter Pranav Distribution (THPPD)

Assume X is a non-negative random variable following two-Parameter Pranav distribution with probability density function (p.d.f) $f(x)$. Let $W(x)$ be the weight function, which is a non-negative function, then the probability density function of the weighted random variable, X_w, is given by:

$$f_w(x)=\frac{W(x)f(x)}{E(w(x))},\quad x>0$$

Where $w(x)$ is a non-negative weight function, and $E(w(x))=\int w(x)f(x)dx<\infty$.

In this paper, we have considered the weight function as $w(x)=x^c$ to obtain the three parameter Pranav model with p.d.f $f_w(x,\alpha,\theta,c)$ from two parameter Pranav distribution with p.d.f $f(x)$ given in (1.1). The probability density of three parameter Pranav distribution is given as:

$$f_w(x,\alpha,\theta,c)=\frac{x^c f(x)}{E\left[x^c\right]},$$

$$f_w(x,\alpha,c,\theta)=\frac{x^c\theta^{c+4}[\alpha\theta+x^3]e^{-\theta x}}{c![\alpha\theta^4+(c+1)(c+2)(c+3)]} \tag{2.1}$$

$$x>0,\theta>0,c>0,\alpha>0$$

The graphs of probability density function for different parameter values are given below, indicating positive skewed nature of three parameter Pranav distribution

The corresponding c.d.f of THPPD is obtained as

$$F_w(x,\alpha,\theta,c)=\int_0^x\frac{x^c\theta^{c+4}[\alpha\theta+x^3]e^{-\theta x}}{c![\alpha\theta^4+(c+1)(c+2)(c+3)]}dx \tag{2.2}$$

Put $\theta x=t$ in (2.2)

$\theta\,dx=dt$

$as\,x\to0,t\to0\;\; and\;\; x\to x,t\to\theta x$

$$F_w(x,\alpha,\theta,c)=\left\{\frac{1}{c![\alpha\theta^4+(c+1)(c+2)(c+3)]}\left(\alpha\theta^4\gamma\{(c+1),\theta x\}+\gamma\{(c+4),\theta x\}\right)\right\} \tag{2.3}$$

Where θ,α and c are positive parameters, and $\gamma(s,x)=\int_0^x t^{s-1}e^{-t}dt$ is a lower incomplete gamma function.

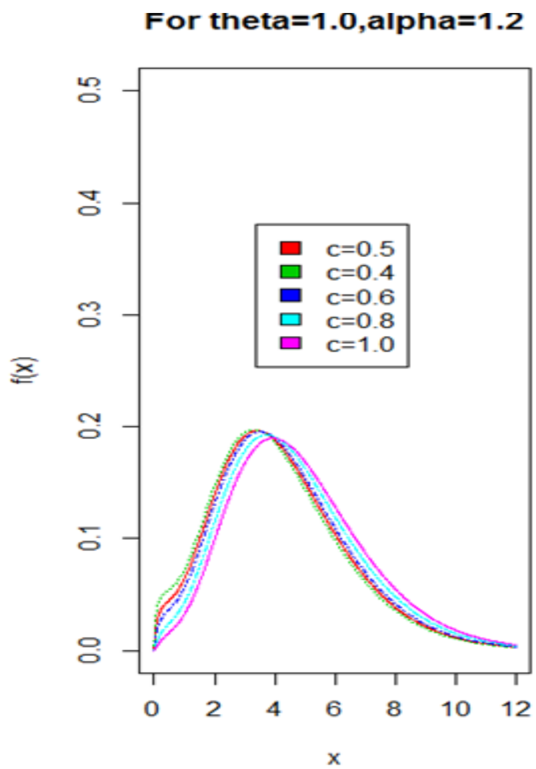

Fig. 1(a): Graph of density function

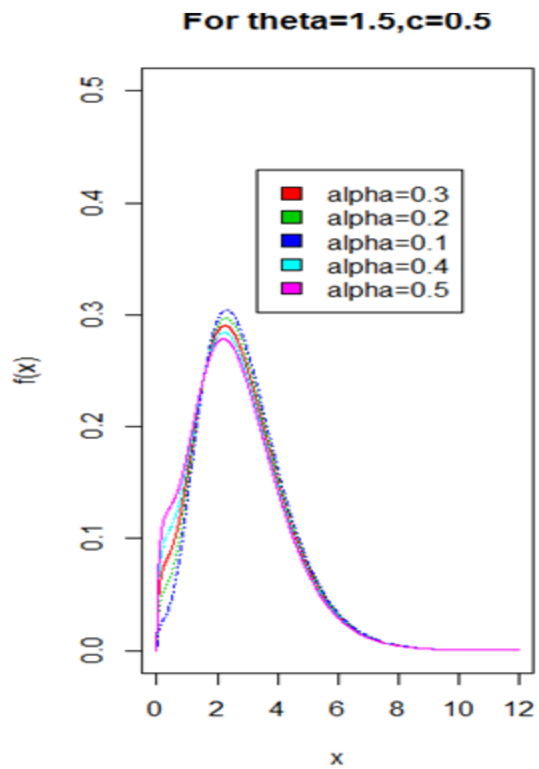

Fig. 1(b): Graph of density function

The graphs of cumulative distribution function of THPPD are given below:

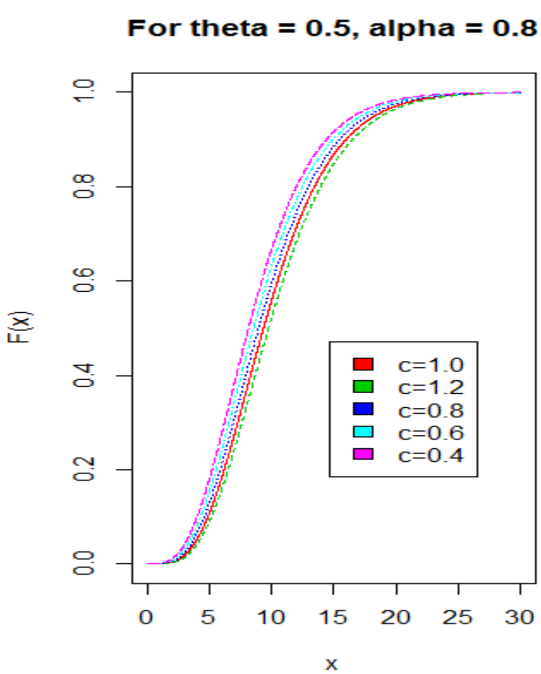

Fig. 2(a): Graph of distribution function

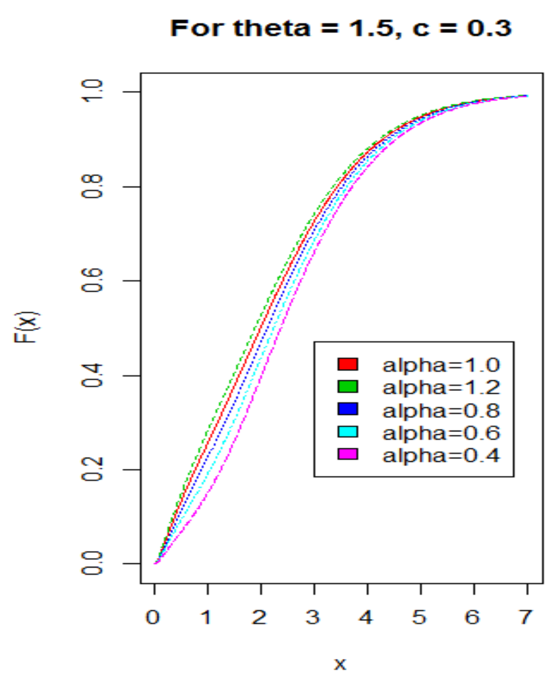

Fig. 2(b): Graph of distribution function

3. Reliability Analysis

This section explores important reliability measures of random variable X denoting the lifetime of a system following three parameter Pranav distribution.

3.1 Reliability Function R(x)

Reliability function or survival analysis $R_w(x,\alpha,\theta,c)$ gives the numerical value of the chance of surviving a system with lifetime X beyond a specified time't'. It can be computed as a complement of the cumulative distribution function of the model. The reliability function or the survival function of three parameter Pranav distribution is calculated as:

$$R_w(x,\alpha,\theta,c)=1-F_w(x,\alpha,\theta,c)$$

$$R_w(x,\alpha,\theta,c)=1-\left\{\frac{1}{\left(c![\alpha\theta^4+(c+1)(c+2)(c+3)]\right)}[\alpha\theta^4\gamma\{(c+1),\theta x\}+\gamma\{(c+4),\theta x\}]\right\}$$

The below graph represents the reliability function of our proposed model.

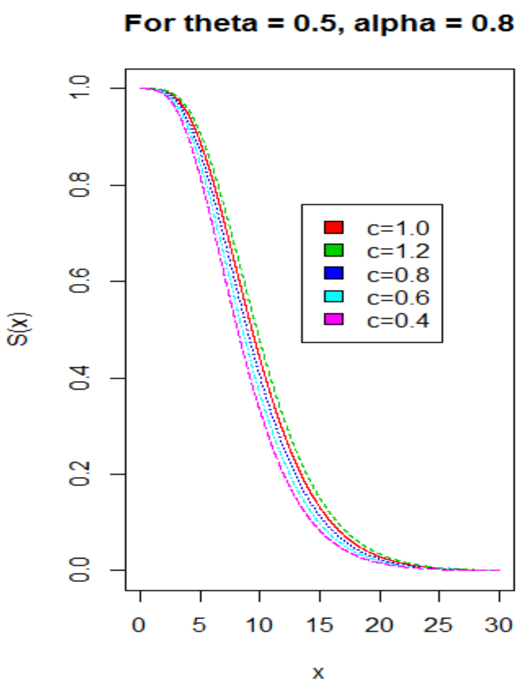

Fig. 3(a): Graph of survival function

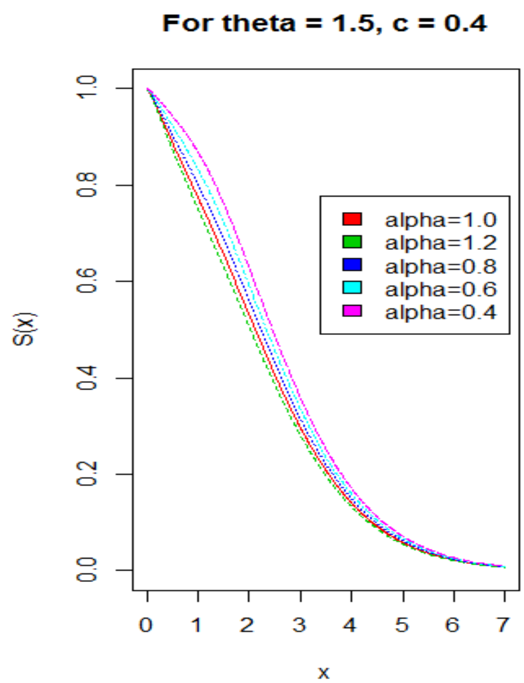

Fig. 3(b): Graph of survival function

3.2 Hazard Function

The hazard function is given as:

$$H.R=h(x;\alpha,\theta,c)=\frac{f_w(x,\alpha,\theta,c)}{R_w(x,\alpha,\theta,c)}$$

$$=\frac{x^c\theta^{c+4}[\alpha\theta+x^3]e^{-\theta x}}{\left(c![\alpha\theta^4+(c+1)(c+2)(c+3)]\right)-\left(\alpha\theta^4\gamma\{(c+1),\theta x\}+\gamma\{(c+4),\theta x\}\right)}$$

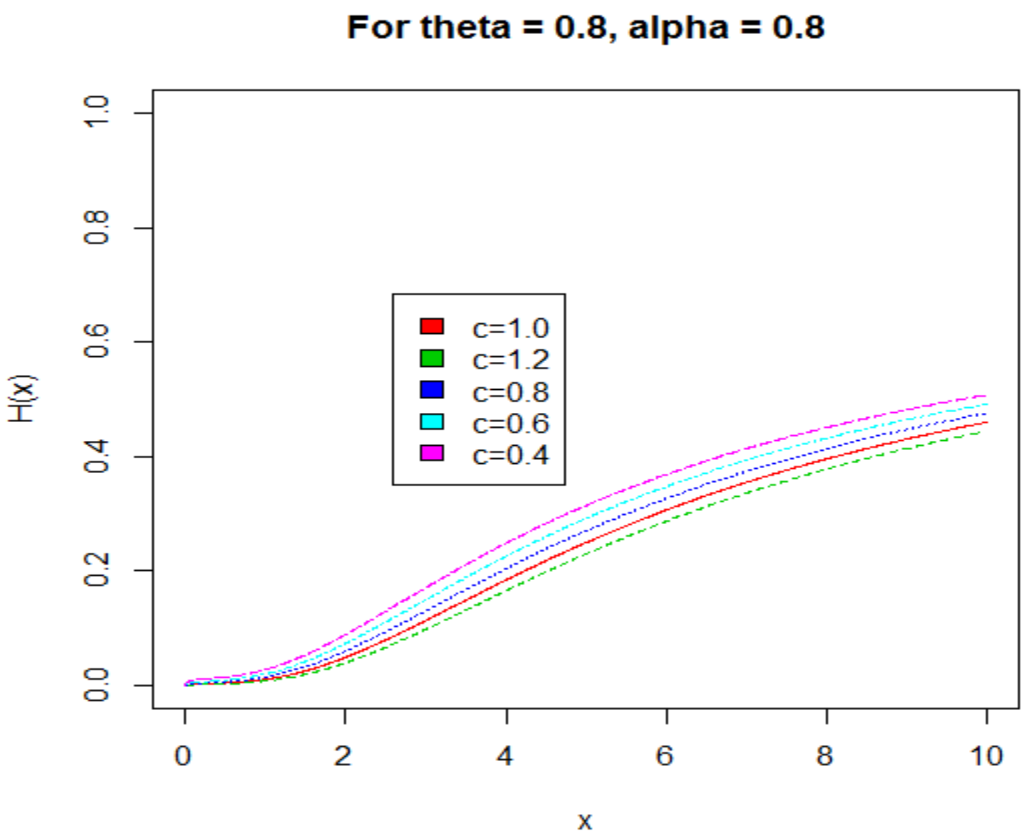

Fig. 4: Graph of hazard function

3.3 Reverse Hazard Rate

The reverse hazard rate of the three parameter Pranav distribution is given as:

$$R.H.R = h_{rw}(x,\alpha,\theta,c) = \frac{f_w(x,\alpha,\theta,c)}{F_w(x,\alpha,\theta,c)} = \frac{x^c\theta^{c+4}[\alpha\theta + x^3]e^{-\theta x}}{\left(\alpha\theta^4\gamma\{(c+1),\theta x\} + \gamma\{(c+4),\theta x\}\right)}$$

4. Statistical Properties

Here, we have obtained some crucial properties like moments, index of dispersion, coefficient of variation, harmonic mean, moment generating function and characteristic function of the three parameter Pranav distribution.

4.1 Moments

Suppose X is a random variable following three parameter Pranav distribution with parameters θ, α and c. Then, the r^{th} moment for a given probability distribution is given by

$$\mu_r' = E\left(X^r\right) = \int_0^\infty x^r f_w(x,c,\alpha,\theta)dx$$

$$= \int_0^\infty x^r \frac{x^c\theta^{c+4}[\alpha\theta + x^3]e^{-\theta x}}{c![\alpha\theta^4 + (c+1)(c+2)(c+3)]}dx$$

$$\mu_r{}' = \frac{(r+c)!\left\{\alpha\theta^4 + (c+r+3)(c+r+2)(c+r+1)\right\}}{\theta^r c!\left[\alpha\theta^4 + (c+3)(c+1)(c+2)\right]} \tag{4.1.1}$$

Putting r=1 in equation (4.1.1), we get

$$\mu_1{}' = \frac{(c+1)\left(\alpha\theta^4 + (c+4)(c+3)(c+2)\right)}{\theta\{\alpha\theta^4 + (c+1)(c+2)(c+3)\}},$$

which is the mean of the three parameter Pranav distribution

Putting r=2 in equation (4.1.1), we get

$$\mu_2{}' = \frac{(c+1)(c+2)\left(\alpha\theta^4 + (c+5)(c+4)(c+3)\right)}{\theta^2\{\alpha\theta^4 + (c+1)(c+2)(c+3)\}}$$

And variance of three parameter Pranav distribution is

$$V(x) = \mu_2 = \frac{(c+1)\left\{\begin{array}{l}(c+2)\left(\alpha\theta^4 + (c+1)(c+2)(c+3)\right)\left(\alpha\theta^4 + (c+5)(c+4)(c+3)\right)\\ -(c+1)\left(\alpha\theta^4 + (c+4)(c+3)(c+2)\right)^2\end{array}\right\}}{\theta^2\left(\alpha\theta^4 + (c+1)(c+2)(c+3)\right)^2}$$

Putting r=3 in equation (4.1.1), we get

$$\mu_3{}' = \frac{(c+1)(c+2)(c+3)\left(\alpha\theta^4 + (c+6)(c+5)(c+4)\right)}{\theta^3\{\alpha\theta^4 + (c+1)(c+2)(c+3)\}}$$

Putting r=4 in equation (4.1.1), we get

$$\mu_4{}' = \frac{(c+1)(c+2)(c+3)(c+4)\left(\alpha\theta^4 + (c+7)(c+6)(c+5)\right)}{\theta^4\{\alpha\theta^4 + (c+1)(c+2)(c+3)\}}$$

The central moments are

$$\mu_3=\frac{(c+1)\left\{\begin{array}{l}(c+3)(c+2)\left(\alpha\theta^4+(c+1)(c+2)(c+3)\right)^2\left(\alpha\theta^4+(c+6)(c+5)(c+4)\right)-\\3(c+1)(c+2)\left(\alpha\theta^4+(c+1)(c+2)(c+3)\right)\\\left(\alpha\theta^4+(c+5)(c+4)(c+3)\right)\left(\alpha\theta^4+(c+4)(c+3)(c+2)\right)\\+2(c+1)^2\left(\alpha\theta^4+(c+4)(c+3)(c+2)\right)^3\end{array}\right\}}{\theta^3\left(\alpha\theta^4+(c+1)(c+2)(c+3)\right)^3}$$

$$\mu_4=\frac{(c+1)\left\{\begin{array}{l}(c+4)(c+3)(c+2)\left(\alpha\theta^4+(c+1)(c+2)(c+3)\right)^3\left(\alpha\theta^4+(c+7)(c+6)(c+5)\right)-\\4(c+1)(c+2)(c+3)\left(\alpha\theta^4+(c+1)(c+2)(c+3)\right)^2\\\left(\alpha\theta^4+(c+2)(c+3)(c+4)\right)\left(\alpha\theta^4+(c+6)(c+5)(c+4)\right)+\\6(c+1)^2(c+2)\left(\alpha\theta^4+(c+1)(c+2)(c+3)\right)\\\left(\alpha\theta^4+(c+5)(c+4)(c+3)\right)\left(\alpha\theta^4+(c+4)(c+3)(c+2)\right)^2\\-3(c+1)^3\left(\alpha\theta^4+(c+4)(c+3)(c+2)\right)^4\end{array}\right\}}{\theta^4\left(\alpha\theta^4+(c+1)(c+2)(c+3)\right)^4}$$

4.2 Coefficient of Variation, Kurtosis, Index of Dispersion and Skewness of THPPD

The coefficient of variation (C.V), coefficient of skewness $(\sqrt{\beta_1})$, coefficient of kurtosis (β_2) and index of dispersion (γ) of the THPPD are determined as

$$C.V=\frac{(\mu_2)^{\frac{1}{2}}}{\mu_1'}=\frac{\left\{(c+1)\left\{\begin{array}{l}(c+2)\left(\alpha\theta^4+(c+1)(c+2)(c+3)\right)\left(\alpha\theta^4+(c+5)(c+4)(c+3)\right)\\-(c+1)\left(\alpha\theta^4+(c+4)(c+3)(c+2)\right)^2\end{array}\right\}\right\}^{\frac{1}{2}}}{\left\{(c+1)\left(\alpha\theta^4+(c+4)(c+3)(c+2)\right)\right\}}$$

$$\sqrt{\beta_1}=\frac{\mu_3}{\mu_2^{3/2}}=\left\{\frac{\left[(c+1)\left\{\begin{array}{l}(c+3)(c+2)\left(\alpha\theta^4+(c+1)(c+2)(c+3)\right)^2\left(\alpha\theta^4+(c+6)(c+5)(c+4)\right)-\\3(c+1)(c+2)\left(\alpha\theta^4+(c+1)(c+2)(c+3)\right)\\\left(\alpha\theta^4+(c+5)(c+4)(c+3)\right)\left(\alpha\theta^4+(c+4)(c+3)(c+2)\right)\\+2(c+1)^2\left(\alpha\theta^4+(c+4)(c+3)(c+2)\right)^3\end{array}\right\}\right]}{\left\{(c+1)\left\{\begin{array}{l}(c+2)\left(\alpha\theta^4+(c+1)(c+2)(c+3)\right)\left(\alpha\theta^4+(c+5)(c+4)(c+3)\right)\\-(c+1)\left(\alpha\theta^4+(c+4)(c+3)(c+2)\right)^2\end{array}\right\}\right\}^{3/2}}\right\}$$

$$\beta_2=\frac{\mu_4}{\mu_2^2}=\frac{(c+1)\left\{\begin{array}{l}(c+4)(c+3)(c+2)\left(\alpha\theta^4+(c+1)(c+2)(c+3)\right)^3\left(\alpha\theta^4+(c+7)(c+6)(c+5)\right)-\\4(c+1)(c+2)(c+3)\left(\alpha\theta^4+(c+1)(c+2)(c+3)\right)^2\\\left(\alpha\theta^4+(c+2)(c+3)(c+4)\right)\left(\alpha\theta^4+(c+6)(c+5)(c+4)\right)+\\6(c+1)^2(c+2)\left(\alpha\theta^4+(c+1)(c+2)(c+3)\right)\\\left(\alpha\theta^4+(c+5)(c+4)(c+3)\right)\left(\alpha\theta^4+(c+4)(c+3)(c+2)\right)^2\\-3(c+1)^3\left(\alpha\theta^4+(c+4)(c+3)(c+2)\right)^4\end{array}\right\}}{\left\{(c+1)\left\{\begin{array}{l}(c+2)\left(\alpha\theta^4+(c+1)(c+2)(c+3)\right)\left(\alpha\theta^4+(c+5)(c+4)(c+3)\right)\\-(c+1)\left(\alpha\theta^4+(c+4)(c+3)(c+2)\right)^2\end{array}\right\}\right\}^2}$$

$$\gamma=\frac{\mu_2}{\mu_1'}=\frac{\left\{(c+1)\left\{\begin{array}{l}(c+2)\left(\alpha\theta^4+(c+1)(c+2)(c+3)\right)\left(\alpha\theta^4+(c+5)(c+4)(c+3)\right)\\-(c+1)\left(\alpha\theta^4+(c+4)(c+3)(c+2)\right)^2\end{array}\right\}\right\}}{\theta\left(\alpha\theta^4+(c+1)(c+2)(c+3)\right)\left\{(c+1)\left(\alpha\theta^4+(c+4)(c+3)(c+2)\right)\right\}}$$

4.3 Harmonic Mean (H.M)

The harmonic mean for the proposed model is computed as:

$$\frac{1}{H.M.} = E\left[\frac{1}{X}\right] = \int_0^\infty \frac{1}{x} f_w(x;c,\alpha,\theta)dx$$

$$= \int_0^\infty \frac{1}{x} \frac{x^c \theta^{c+4} [\alpha\theta + x^3] e^{-\theta x}}{c![\alpha\theta^4 + (c+1)(c+2)(c+3)]} \, dx$$

$$H.M = \frac{c\left(\alpha\theta^4 + (c+1)(c+2)(c+3)\right)}{\theta\left(\alpha\theta^4 + (c+1)(c+2)c\right)}.$$

4.4 Moment Generating Function and Characteristic Function of THPPD

We derived the moment generating function and characteristic function of THPPD in this section.

Theorem 1.1: If X has the THPPD $(\alpha, \theta and c)$, then, the moment generating function $M_X(t)$ and characteristic generating function $\phi_X(t)$ are

$$M_X(t) = \frac{\theta^{c+4}\left\{\alpha\theta(\theta - t)^3 + (c+3)(c+2)(c+1)\right\}}{\left(\alpha\theta^4 + (c+3)(c+1)(c+2)\right)\left(\theta - t\right)^{c+4}}$$

And

$$\phi_X(t) = \frac{\theta^{c+4}\left\{\alpha\theta(\theta - it)^3 + (c+3)(c+2)(c+1)\right\}}{\left(\alpha\theta^4 + (c+3)(c+1)(c+2)\right)\left(\theta - it\right)^{c+4}},$$

respectively.

Proof: We begin with the well-known definition of the moment generating function given by

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f_w(x;\alpha,\theta,c)\,dx$$

$$= \int_0^\infty \frac{e^{tx} x^c \theta^{c+4} [\alpha\theta + x^3] e^{-\theta x}}{c![\alpha\theta^4 + (c+1)(c+2)(c+3)]} \, dx$$

$$= \frac{\theta^{c+4}}{c![\alpha\theta^4 + (c+1)(c+2)(c+3)]} \int_0^\infty x^c (\alpha\theta + x^3) e^{-x(\theta - t)} \, dx$$

$$M_X(t) = \frac{\theta^{c+4}\left\{\alpha\theta(\theta - t)^3 + (c+3)(c+2)(c+1)\right\}}{\left(\alpha\theta^4 + (c+3)(c+1)(c+2)\right)\left(\theta - t\right)^{c+4}}, \tag{4.4.1}$$

which is the m.g.f of three parameter Pranav distribution.

Also, we know that $\phi_X(t) = M_X(it)$.

Therefore,

$$\phi_X(t)=\frac{\theta^{c+4}\left\{\alpha\theta(\theta-it)^3+(c+3)(c+2)(c+1)\right\}}{\left(\alpha\theta^4+(c+3)(c+1)(c+2)\right)\left(\theta-it\right)^{c+4}}, \tag{4.4.2}$$

which is the characteristic function of three parameter Pranav distribution.

5. Order Statistics

Let $X_{(1)}, X_{(2)}, X_{(3)}...., X_{(n)}$ be the ordered statistics of random sample $x_1, x_2, x_3,....x_n$ obtained from the three parameter Pranav distribution with cumulative distribution function $F_w(x;\alpha,\theta,c)$ and probability density function $f_w(x:\alpha,\theta,c)$, then the probability density function of r^{th} order statistics $X_{(r)}$ is given by:

$$f_{w(r)}(x,\alpha,\theta,c)=\frac{n!}{(r-1)!(n-r)!}f_w(x:\alpha,\theta,c)[F(x:\alpha,\theta,c)]^{r-1}[1-F(x:\alpha,\theta,c)]^{n-r}.r=1,2,3\ldots n$$

Using Eqs.(2.1) and (2.3), the probability density function of rth order statistics of THPPD is given by:

$$f_{w(r)}(x,\alpha,\theta,c)=\left\{\begin{array}{l}\dfrac{n!}{(r-1)!(n-r)!}\dfrac{x^c\theta^{c+4}[\alpha\theta+x^3]e^{-\theta x}}{c![\alpha\theta^4+(c+1)(c+2)(c+3)]}\\ \left[\left\{\dfrac{1}{c![\alpha\theta^4+(c+1)(c+2)(c+3)]}\left(\alpha\theta^4\gamma\{(c+1),\theta x\}+\gamma\{(c+4),\theta x\}\right)\right\}\right]^{r-1}\\ \left[1-\left\{\dfrac{1}{c![\alpha\theta^4+(c+1)(c+2)(c+3)]}\left(\alpha\theta^4\gamma\{(c+1),\theta x\}+\gamma\{(c+4),\theta x\}\right)\right\}\right]^{n-r}.\end{array}\right\}$$

Then, the pdf of first order statistic $X_{(1)}$ of THPPD is given by:

$$f_{w(1)}(x,\alpha,\theta,c)=\left\{\begin{array}{l}n\dfrac{x^c\theta^{c+4}[\alpha\theta+x^3]e^{-\theta x}}{c![\alpha\theta^4+(c+1)(c+2)(c+3)]}\\ \left[1-\left\{\dfrac{1}{c![\alpha\theta^4+(c+1)(c+2)(c+3)]}\left(\alpha\theta^4\gamma\{(c+1),\theta x\}+\gamma\{(c+4),\theta x\}\right)\right\}\right]^{n-1}.\end{array}\right\}$$

and the pdf of n^{th} order statistic $X_{(n)}$ of THPPD is given as:

$$f_{w(n)}(x,\alpha,\theta,c)=\left\{\begin{array}{l}n\dfrac{x^c\theta^{c+4}[\alpha\theta+x^3]e^{-\theta x}}{c![\alpha\theta^4+(c+1)(c+2)(c+3)]}\\ \left[\left\{\dfrac{1}{c![\alpha\theta^4+(c+1)(c+2)(c+3)]}\left(\alpha\theta^4\gamma\{(c+1),\theta x\}+\gamma\{(c+4),\theta x\}\right)\right\}\right]^{n-1}.\end{array}\right\}$$

6. Estimation of Parameters of THPPD using Maximum Likelihood Estimation Method

Let $x_1, x_2, x_3.....x_n$ be the random sample of size n drawn from THPPD, then the likelihood function is given as:

$$L(x \mid c,\theta) = \prod_{i=1}^{n} \left\{ \frac{x^c \theta^{c+4} [\alpha\theta + x^3] e^{-\theta x}}{c! [\alpha\theta^4 + (c+1)(c+2)(c+3)]} \right\}$$

The log likelihood function becomes:

$$\log L = \begin{Bmatrix} c \log \sum_{i=1}^{n} x_i + n(c+4)\log(\theta) + \sum_{i=1}^{n} \log\left(\alpha\theta + x_i^3\right) - \theta \sum_{i=1}^{n} x_i - n \log(c!) \\ - n \log(\alpha\theta^4 + (c+1)(c+2)(c+3)) \end{Bmatrix} \tag{6.1}$$

Differentiating the log-likelihood function (6.1) with respect to α, θ and c, equating the results to zero; we obtain the following normal equations:

$$\frac{\partial \log L}{\partial \theta} = \frac{n(c+4)}{\theta} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \left(\frac{\alpha}{\alpha\theta + x_i^3} \right) - n \left(\frac{4\alpha\theta^3}{\alpha\theta^4 + (c+1)(c+2)(c+3)} \right) = 0 \tag{6.2}$$

$$\frac{\partial \log L}{\partial \alpha} = \sum_{i=1}^{n} \left(\frac{\theta}{\alpha\theta + x_i^3} \right) - n \left(\frac{\theta^4}{\alpha\theta^4 + (c+1)(c+2)(c+3)} \right) = 0 \tag{6.3}$$

$$\frac{\partial \log L}{\partial c} = \sum_{i=1}^{n} \log x_i + n \log\theta - n\left(\log(c+1) - \frac{1}{2(c+1)} \right) - n \left(\frac{2c^2 + 6c + 11}{\alpha\theta^4 + (c+1)(c+2)(c+3)} \right) \tag{6.4}$$

Since MLEs of α, θ, c cannot be obtained by solving the above equations as these equations are not in closed form, so we use Newton Raphson method to obtain MLEs of α, θ, c through R software.

7. Special Cases of Three Parameter Pranav Distribution

Case I: If we put $c = 0$, then THPPD (2.1) reduces to two parameter Pranav distribution with probability density function as:

$$f_1(x) = \frac{\theta^4}{(\alpha\theta^4 + 6)} \left(\alpha\theta + x^3 \right) e^{-\theta x} \quad x > 0, \alpha > 0, \theta > 0$$

Case II: For $c = 0, \alpha = 1$, THPPD (2.1) reduces to one parameter Pranav distribution with probability density function given as

$$f_2(x) = \frac{\theta^4}{(\theta^4 + 6)} \left(\theta + x^3 \right) e^{-\theta x} \quad x > 0, \theta > 0$$

8. Applications of Three Parameter Pranav Distribution

We fitted our proposed model THPPD and its base model to three real life data sets for testing the superiority of proposed model as compared to its sub models.

Data Set 1: The data set given in Table 1 represents the relief times (in minutes) of twenty patients receiving an analgesic used by Gross and Clark (1975). This data set was used by Shanker, Hagos and Sujatha (2015). We would like to emphasize that the aim here is not to provide a complete statistical modelling or inferences for the data set involved.

Table 1: Relief time of 20 patients receiving an analgesic

1.1	1.4	1.3	1.7	1.9	1.8	1.6	2.2	1.7	2.7
4.1	1.8	1.5	1.2	1.4	3.0	1.7	2.3	1.6	2.0

Data set 2: The second data set is about the breaking stress of carbon fibres of 50 mm length (GPa). The data has been previously used by Nichols and Padgett (2006), Cordeiro and Lemonte (2011) and Al-Aqtash*et al*. (2014). The data is as follows:

Table 2: Breaking stress of carbon fibres of 50 mm length (Gpa)

0.39	0.85	1.08	1.25	1.47	1.57	1.61	1.61	1.69	1.80	1.84
1.87	1.89	2.03	2.03	2.05	2.12	2.35	2.41	2.43	2.48	2.50
2.53	2.55	2.55	2.56	2.59	2.67	2.73	2.74	2.79	2.81	2.82
2.85	2.87	2.88	2.93	2.95	2.96	2.97	3.09	3.11	3.11	3.15
3.15	3.19	3.22	3.22	3.27	3.28	3.31	3.31	3.33	3.39	3.39
3.56	3.60	3.65	3.68	3.70	3.75	4.20	4.38	4.42	4.70	4.90

Data Set 3: This data set represents the waiting times (in minutes) before serving 100 bank customers, used by Ghitany et al. (2008).

Table 3: Waiting times (in minutes) before serving 100 bank customers

0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1	2.6	2.7
2.9	3.1	3.2	3.3	3.5	3.6	4.0	4.1	4.2	4.2
4.3	4.3	4.4	4.4	4.6	4.7	4.7	4.8	4.9	4.9
5.0	5.3	5.5	5.7	5.7	6.1	6.2	6.2	6.2	6.3
6.7	6.9	7.1	7.1	7.1	7.1	7.4	7.6	7.7	8.0

8.2	8.6	8.6	8.6	8.8	8.8	8.9	8.9	9.5	9.6
9.7	9.8	10.7	10.9	11.0	11.0	11.1	11.2	11.2	11.5
11.9	12.4	12.5	12.9	13.0	13.1	13.3	13.6	13.7	13.9
14.1	15.4	15.4	17.3	17.3	18.1	18.2	18.4	18.9	19.0
19.9	20.6	21.3	21.4	21.9	23.0	27.0	31.6	33.1	38.5

These data sets are used here only for illustrative purposes. The required numerical evaluations are carried out using R software 3.3.2. We have fitted THPPD & TPPD to these three real life data sets. The summary statistic of these three data sets is given in Table 4. The MLEs of the parameters with standard errors in parentheses, model functions are displayed in Table 5 for these three data sets. The corresponding log-likelihood values, AIC, AICC, HQIC, BIC & Shannon's entropy are given in Tables 6, 7 & 8 for data sets 1, 2 & 3, respectively.

Table 4: Summary statistic of Data Sets 1, 2 & 3.

Data set	No. of observations	Min.	First quartile	Median	Mean	Third quartile	Max.
Data set 1	20	1.10	1.475	1.70	1.90	2.05	4.10
Data set 2	66	0.390	2.178	2.835	2.760	3.278	4.900
Data set 3	100	0.800	4.67	8.10	9.87	13.02	38.50

Table 5: ML estimates, standard error of estimates in parenthesis, model function of base model and proposed model for Data Sets 1, 2 and 3.

Data set	**Distribution**	**ML estimates (standard errors)**	**Model function**
Data Set 1	**Three parameter Pranav distribution** **(THPPD)**	$\hat{\theta}=5.522$ (1.861) $\hat{\alpha}=10.629$ (35.040) $\hat{c}=9.128$ (3.1275)	$\frac{x^c\theta^{c+4}[\alpha\theta+x^3]e^{-\theta x}}{c![\alpha\theta^4+(c+1)(c+2)(c+3)]}$
	Two parameter Pranav distribution **(TPPD)**	$\hat{\theta}=2.098$ (0.106) $\hat{\alpha}=0.0010$	$\frac{\theta^4}{(\alpha\theta^4+6)}\left(\alpha\theta+x^3\right)e^{-\theta x}$

Data Set 2	**Three parameter Pranav distribution** **(THPPD)**	$\hat{\theta} = 3.115$ (0.536) $\hat{\alpha} = 0.471$ (0.584) $\hat{c} = 4.95$ (1.619)	$\frac{x^c \theta^{c+4}[\alpha\theta + x^3]e^{-\theta x}}{c![\alpha\theta^4 + (c+1)(c+2)(c+3)]}$
	Two parameter Pranav distribution **(TPPD)**	$\hat{\theta} = 1.448$ (0.098) $\hat{\alpha} = 0.0010$ (0.047)	$\frac{\theta^4}{(\alpha\theta^4 + 6)}\left(\alpha\theta + x^3\right)e^{-\theta x}$
Data Set 3	**Three parameter Pranav distribution** **(THPPD)**	$\hat{\theta} = 2.733e-01$ $\hat{\alpha} = 3.39216e+04$ $\hat{c} = 1.270029e+00$	$\frac{x^c \theta^{c+4}[\alpha\theta + x^3]e^{-\theta x}}{c![\alpha\theta^4 + (c+1)(c+2)(c+3)]}$
	Two parameter Pranav distribution **(TPPD)**	$\hat{\theta} = 0.353$ $\hat{\alpha} = 78.514$	$\frac{\theta^4}{(\alpha\theta^4 + 6)}\left(\alpha\theta + x^3\right)e^{-\theta x}$

Table 6: Model comparison, likelihood ratio of the proposed model and its base model for Data Set 1.

Distribution	$-\log L$	**AIC**	**BIC**	**AICC**	**HQIC**	**Shannon entropy** ***H(X)***	**Likelihood ratio**
THPPD	17.709	41.419	44.406	42.919	42.002	0.88	6.54
TPPD	20.979	45.959	47.950	46.665	46.3478	1.04	

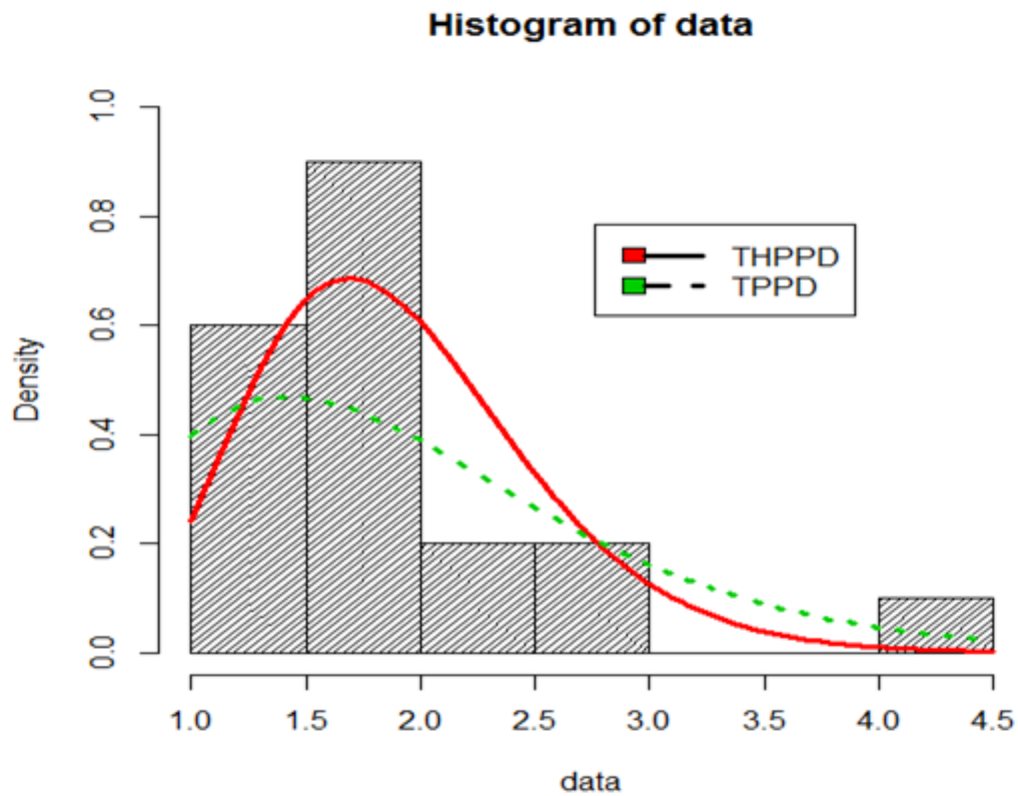

Fig. 5: Graph of Data Set 1 fitted by the proposed model and base model

Table 7: Model comparison, likelihood ratio of the proposed model and base model for Data Set 2.

Distribution	$-\log L$	AIC	BIC	AICC	HQIC	Shannon entropy $H(X)$	Likelihood ratio
THPPD	89.132	184.2652	190.83	184.652	186.860	1.35	15.33
TPPD	96.797	197.595	201.97	197.785	199.325	1.46	

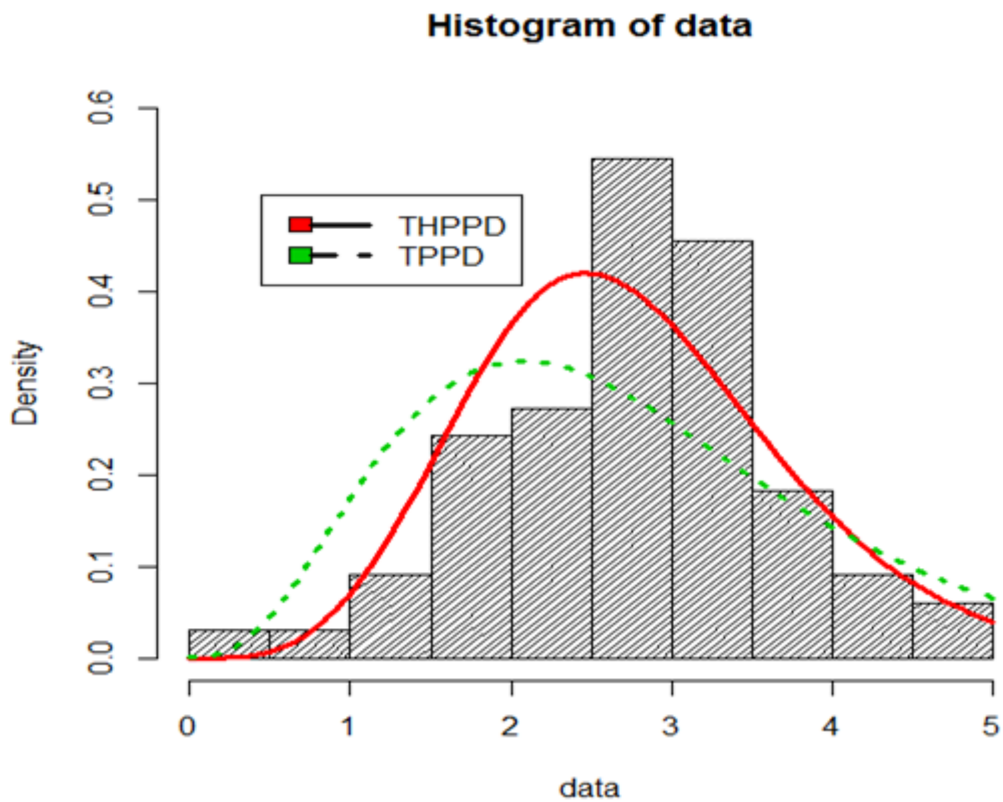

Fig. 6: Graph of Data Set 2 fitted by proposed model and base model

Table 8: Model comparison, likelihood ratio of proposed model and its base model for Data Set 3.

Distribution	$-\log L$	**AIC**	**BIC**	**AICC**	**HQIC**	**Shannon entropy** *H(X)*	**Likelihood ratio**
Three Parameter Pranav Distribution (THPPD)	316.980	639.960	647.77	640.21	643.123	3.16	16.34
Two Parameter Pranav Distribution (TPPD)	325.150	654.301	659.51	654.425	656.410	3.25	

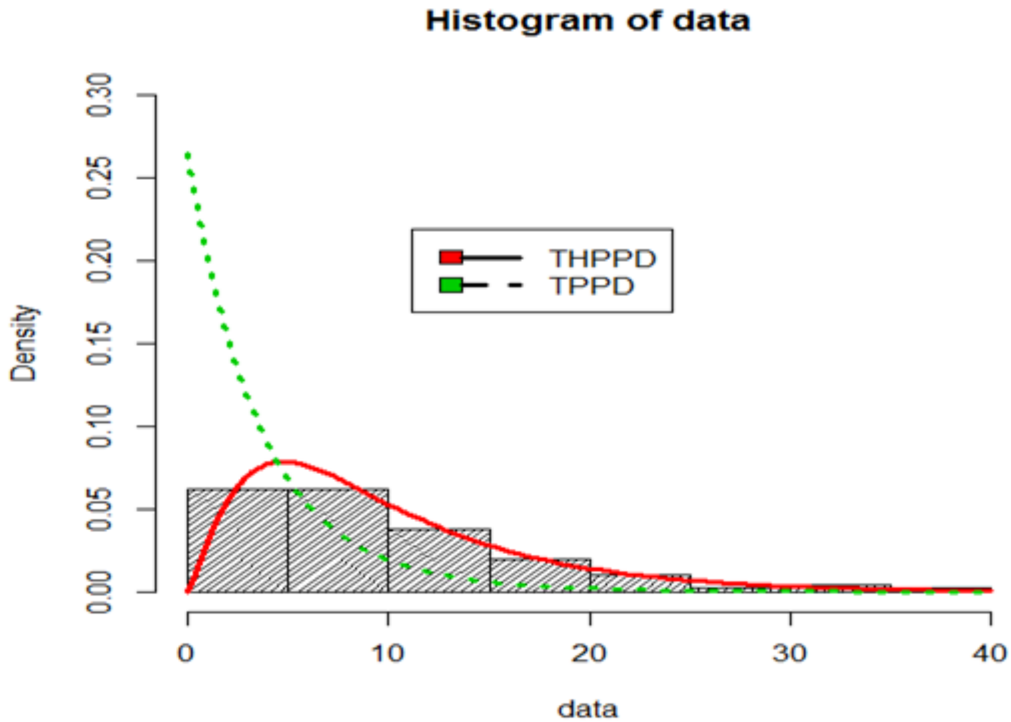

Fig. 7: Graph of Data Set 3 fitted by proposed model and base model

To test whether weight parameter c plays a significant role or not and for checking superiority of THPPD over TPPD for Data Sets 1, 2 & 3, we computed likelihood ratio (LR) statistic for Data Sets 1, 2 & 3 in Tables 6, 7 & 8, respectively. For testing $H_0 : c = 0$ versus $H_1 : c \neq 0$, the LR statistic for testing H_0 is $\omega_1 = 2\{L(\hat{\Theta}) - L(\hat{\Theta}_0)\} = 6.54$ for Data Set 1, $\omega_2 = 2\{L(\hat{\Theta}) - L(\hat{\Theta}_0)\} = 15.33$ for Data Set 2 & $\omega_3 = 2\{L(\hat{\Theta}) - L(\hat{\Theta}_0)\} = 16.34$ for Data Set 3, where $\hat{\Theta}$ and $\hat{\Theta}_0$ are MLEs under H_1 and H_0, respectively. LR statistic $\omega \sim ({\chi_{(1)}}^2 (\alpha = 0.05) = 3.841)$ as $n \to \infty$, where degrees of freedom is the difference in dimensionality. From Tables 6, 7 & 8 $\omega_1 = 6.54 > 3.841$, $\omega_2 = 15.33 > 3.841$ & $\omega_3 = 16.34 > 3.841$ at 5% level of significance for all the three data sets, so we reject H_0 and conclude that weight parameter c plays statistically a significant role.

In order to compare the THPPD with TPPD, we compute the criteria like AIC (Akaike information criterion), AICC (corrected Akaike information criterion), BIC (Bayesian information criterion) & HQIC which represent the loss of information resulting from fitting probability models to data. The better distribution corresponds to lesser AIC, AICC, BIC & HQIC values. Also, we computed the Shannon's entropy *(H(X))* which represents the average uncertainty. The better model possesses lesser Shannon's entropy value.

AIC = 2k-2log$L$ AICC = AIC+$\frac{2k(k+1)}{n-k-1}$

BIC = k log$n$-2log$L$ HQIC=2k log(log(n))+2 log L

$$H(X) = -\frac{\log L}{n},$$

where k is the number of parameters in the statistical model, n is the sample size and -2log$L$ is the maximized value of the log-likelihood function under the considered model. From Tables 6, 7 & 8, it has been observed that the THPPD possesses the lesser AIC, AICC, BIC, HQIC and *H(X)* values as compared to TPPD for Data Sets 1, 2 & 3, respectively. Hence we can conclude that the THPPD leads to a better fit than the TPPD for Data Sets 1, 2 & 3, respectively.

9. Conclusions

We have incorporated three parameter Pranav distribution by using weighting technique from two parameter Pranav distribution. Some crucial properties of proposed model are obtained. The estimates of unknown parameters of proposed model have been obtained by using maximum likelihood method of estimation. The significance of the weight parameter has also been tested. Finally, we fitted proposed model and base model to three real life data sets and concluded that proposed model provides better fit to these data sets as compared to its base model, and hence, proposed model has greater applicability in real life, so it can be of vital importance in the future to decision makers.

Acknowledgement: The authors are grateful for the anonymous referees for the careful checking of the details and for the helpful comments that improved this paper. Authors also acknowledge the timely response of this journal.

Conflict of Interest: The authors declare that they have no conflict of interest.

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