Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/140211

On Ostrowski-Type Inequalities and Strongly h-Convex Functions

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Received: 14 Nov. 2018, Revised: 28 Nov. 2019, Accepted: 2 Dec. 2019
Published online: 1 Mar. 2020

Abstract: We establish some new Ostrowski-type inequalities for the class of functions with the property that the absolute value of their derivatives is strongly h-convex with modulus $c > 0$. Moreover, we give some applications of the results obtained to cumulative distribution functions, special means, and approximation of the Riemann integral.

Keywords: Ostrowski-type inequality, Holder inequality, strongly h-convex function, special means.

1 Introduction

The classical Ostrowski inequality was established in 1938 by Ostrowski [1], and is given as follows: For every $x \in [a,b]$,

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2} \right](b-a)\left\| f' \right\|_\infty, \quad (1)$$

where f is a differentiable function defined on a finite interval $[a,b]$, whose derivative is integrable and bounded over $[a,b]$. Moreover the constant $1/4$ is the best possible.

During the past few years many researchers have given considerable attention to Ostrowski's type inequalities for their applications in different aspects. In particular, such inequalities can be used to estimate errors in approximating values of a function by its integral mean. Also, they can be used in numerical analysis to obtain error bounds for some special means, and for quadrature rules of approximating the Riemann integral. Several generalizations and variants of Ostrowski-type inequalities have been obtained, see for example [2]-[6].

In [7], Set et al. have introduced new Ostrowski-type inequalities for strongly convex functions. Also, they have established several integral inequalities which involve product of strongly-convex and convex functions. In [8], Tunc has introduced new Ostrowski-type inequalities for absolutely continuous mappings whose first derivatives in absolute value are h-convex.

In [9], Angulo et al. have introduced the notion of strongly h-convex functions defined on real normed spaces and presented some properties and representations of such functions. They have developed a characterization of inner product spaces involving the new notion of strongly h-convex functions, and obtained Hermite–Hadamard–type inequality for such functions. For further information about strongly convex functions, we refer the reader to [10].

The main purpose of this paper is to generalize the results obtained in [7] and [8], and establish some new Ostrowski-type inequalities for the class of functions with the property that the absolute value of their derivatives are strongly h-convex. Moreover, we discuss some applications of the results obtained.

Throughout this paper, $h : [0,1] \to [0,\infty)$ is a function, $\mathbb{R}$ is the set of real numbers, I is any interval and I° is the interior of I.

Recall that for $a < b$, a function $f : [a,b] \to \mathbb{R}$ is strongly h-convex with modulus $c > 0$ if

$$f((1-t)x+ty) \le h((1-t))f(x) + h(t)f(y) - ct(1-t)(x-y)^2 \quad (2)$$

for all $x,y \in [a,b]$ and $t \in [0,1]$.

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2 Main Results

We start this section with the following lemma which will be used in obtaining our results.

Lemma 2.1. [3] Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, and $a, b \in I$ with $a < b$. If $f' \in L^1[a,b]$ then

$$f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \quad (3)$$
$$= \frac{(x-a)^2}{b-a}\int_0^1 t f'(tx+(1-t)a)\,dt$$
$$- \frac{(b-x)^2}{b-a}\int_0^1 t f'(tx+(1-t)b)\,dt$$

for each $x \in [a,b]$.

Theorem 2.1. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, and $a, b \in I$ with $a < b$. Let $f' \in L^1[a,b]$. If $|f'|$ is strongly h-convex on $[a,b]$ with modulus $c > 0$ and $|f'| \le M$, where $M > 0$, then

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \quad (4)$$
$$\le \frac{(x-a)^2}{b-a}\left[M\int_0^1 h(t)\,dt - \frac{c(x-a)^2}{12}\right]$$
$$+ \frac{(b-x)^2}{b-a}\left[M\int_0^1 h(t)\,dt - \frac{c(b-x)^2}{12}\right]$$

for each $x \in [a,b]$.

Proof. Using Identity (3) and the triangle inequality, we get that

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\le \frac{(x-a)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)a)\right|dt$$
$$+ \frac{(b-x)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)b)\right|dt.$$

Since $|f'|$ is strongly h-convex on $[a,b]$ with modulus $c > 0$ and $|f'| \le M$, we have

$$\int_0^1 t\left|f'(tx+(1-t)a)\right|dt$$
$$\le \int_0^1 \left[th(t)\left|f'(x)\right| + th(1-t)\left|f'(a)\right|\right.$$
$$\left.-ct^2(1-t)(x-a)^2\right]dt$$
$$\le M\int_0^1 t\,(h(t)+h(1-t))\,dt - \frac{c(x-a)^2}{12}$$
$$= M\int_0^1 h(t)\,dt - \frac{c(x-a)^2}{12}$$

and

$$\int_0^1 t\left|f'(tx+(1-t)b)\right|dt$$
$$\le \int_0^1 \left[th(t)\left|f'(x)\right| + th(1-t)\left|f'(b)\right|\right.$$
$$\left.-ct^2(1-t)(b-x)^2\right]dt$$
$$\le M\int_0^1 h(t)\,dt - \frac{c(b-x)^2}{12}.$$

This implies that

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\le \frac{(x-a)^2}{b-a}\left[M\int_0^1 h(t)\,dt - \frac{c(x-a)^2}{12}\right]$$
$$+ \frac{(b-x)^2}{b-a}\left[M\int_0^1 h(t)\,dt - \frac{c(b-x)^2}{12}\right].$$

Remark 2.1.

1. When $h(t) = t$ for each $t \in [0,1]$, Inequality(4) reduces to the inequality

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\le \frac{(x-a)^2}{2(b-a)}\left[M - \frac{c(x-a)^2}{6}\right]$$
$$+ \frac{(b-x)^2}{2(b-a)}\left[M - \frac{c(b-x)^2}{6}\right],$$

which is obtained in [7].

2. If $c \longrightarrow 0^+$ Inequality(4) reduces to the inequality

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \quad (5)$$
$$\le \frac{(x-a)^2+(b-x)^2}{b-a}\left[M\int_0^1 h(t)\,dt\right].$$

3. In [8], Tunc has obtained the inequality

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \quad (6)$$
$$\le \frac{M\left[(x-a)^2+(b-x)^2\right]}{b-a}\left[\int_0^1 h(t^2) + h(t-t^2)\,dt\right],$$

where h is super multiplicative, that is, $h(xy) \ge h(x)h(y)$ for each $x, y \in [0,1]$, $h(t) \ge t$ for each $t \in [0,1]$, and f satisfies the conditions of Theorem 2.1 except that strong h-convexity of $|f'|$ is replaced by $h-$convexity. Clearly, Inequality (5) is better than Inequality (6) because of the fact that

$$\begin{aligned}\int_0^1 h(t)\,dt &= \int_0^1 h(t)(t+(1-t))\,dt \\ &\le \int_0^1 h(t)(h(t)+h(1-t))\,dt \\ &= \int_0^1 h\left(t^2\right)+h\left(t-t^2\right)dt.\end{aligned}$$

Corollary 2.1. In Inequality(4), if we choose $x=\frac{a+b}{2}$ then we obtain the midpoint inequality

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \quad (7)\\ &\le \frac{M(b-a)}{2}\left[\int_0^1 h(t)\,dt\right]-\frac{c(b-a)3}{96}.\end{aligned}$$

Corollary 2.2. Suppose that $h(t)=t^s$ for each $t\in[0,1]$, where $s\in[0,1)$. Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, and $a,b\in I$ with $a<b$. Let $f'\in L^1[a,b]$. If $\left|f'\right|$ is strongly h-convex on $[a,b]$ with modulus $c>0$ and $\left|f'\right|\le M$, where $M>0$, then

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \quad (8)\\ &\le \frac{(x-a)^2}{b-a}\left[\frac{M}{s+1}-\frac{c(x-a)^2}{12}\right] \\ &+\frac{(b-x)^2}{b-a}\left[\frac{M}{s+1}-\frac{c(b-x)^2}{12}\right].\end{aligned}$$

for each $x\in[a,b]$.

Proof. Note that

$$\begin{aligned}\int_0^1 h(t)\,dt &= \int_0^1 t^s dt \\ &= \frac{1}{s+1}.\end{aligned}$$

Theorem 2.2. Suppose that $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, and $a,b\in I$ with $a<b$. If $f'\in L^1[a,b]$ and $\left|f'\right|^q$ is strongly h-convex on $[a,b]$ with modulus $c>0$ and $\left|f'\right|\le M$, where $M>0$, then

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \quad (9)\\ &\le\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \\ &\times\left[\frac{(x-a)^2}{b-a}\left[2M^q\int_0^1 h(t)\,dt-\frac{c(x-a)^2}{6}\right]^{\frac{1}{q}}\right. \\ &\left.+\frac{(b-x)^2}{b-a}\left[2M^q\int_0^1 h(t)\,dt-\frac{c(b-x)^2}{6}\right]^{\frac{1}{q}}\right].\end{aligned}$$

for each $x\in[a,b]$.

Proof. Using Identity (3) and the triangle inequality, we get that

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\ &\le\frac{(x-a)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)a)\right|dt \\ &+\frac{(b-x)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)b)\right|dt.\end{aligned}$$

By Holder's inequality,

$$\begin{aligned}&\int_0^1 t\left|f'(tx+(1-t)a)\right|dt \\ &\le\left(\int_a^b t^p dy\right)^{\frac{1}{p}}\left(\int_a^b\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\ &=\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_a^b\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}},\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 t\left|f'(tx+(1-t)b)\right|dt \\ &\le\left(\int_a^b t^p dy\right)^{\frac{1}{p}}\left(\int_a^b\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\ &=\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_a^b\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

Since $\left|f'\right|^q$ is strongly h - convex with modulus c and $\left|f'\right|\le M$,

$$\begin{aligned}&\int_a^b\left|f'(tx+(1-t)a)\right|^q dt \\ &\le\int_0^1\left[h(t)\left|f'(x)\right|^q+h(1-t)\left|f'(a)\right|^q\right. \\ &\left.-ct(1-t)(x-a)^2\right]dt \\ &\le 2M^q\int_0^1 h(t)\,dt-\frac{c(x-a)^2}{6},\end{aligned}$$

and

$$\int_a^b \left|f'(tx+(1-t)b)\right|^q dt$$
$$\leq \int_0^1 \left[h(t)\left|f'(x)\right|^q + h(1-t)\left|f'(b)\right|^q\right.$$
$$\left.-ct(1-t)(b-x)^2\right] dt$$
$$\leq 2M^q \int_0^1 h(t)\,dt - \frac{c(b-x)^2}{6}.$$

Therefore,

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$
$$\times \left[\frac{(x-a)^2}{b-a}\left[2M^q\int_0^1 h(t)\,dt - \frac{c(x-a)^2}{6}\right]^{\frac{1}{q}}\right.$$
$$\left.+\frac{(b-x)^2}{b-a}\left[2M^q\int_0^1 h(t)\,dt - \frac{c(b-x)^2}{6}\right]^{\frac{1}{q}}\right].$$

Remark 2.2.

1. When $h(t)=t$ for each $t\in[0,1]$, Inequality(9) reduces to the inequality

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \tag{10}$$
$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$
$$\times \left[\frac{(x-a)^2}{b-a}\left[M^q - \frac{c(x-a)^2}{6}\right]^{\frac{1}{q}}\right.$$
$$\left.+\frac{(b-x)^2}{b-a}\left[M^q - \frac{c(b-x)^2}{6}\right]^{\frac{1}{q}}\right],$$

which is obtained in [7].

2. If $c \longrightarrow 0^+$, Inequality(4) reduces to the inequality

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \tag{11}$$
$$\leq \frac{(x-a)^2+(b-x)^2}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[2M^q\int_0^1 h(t)\,dt\right]^{\frac{1}{q}}.$$

3. In [8], Tunc has obtained the inequality

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \tag{12}$$
$$\leq \frac{M(h(1))^{\frac{1}{q}}\left[(x-a)^2+(b-x)^2\right]}{b-a}\left[\int_0^1 h(t^p)\,dt\right]^{\frac{1}{p}},$$

where h is super additive, that is, $h(x+y)\geq h(x)+h(y)$ for each $x,y\in[0,1]$, $h(t)\geq t$ for each $t\in[0,1]$, and f satisfies the conditions of Theorem 2 except that strong h-convexity of $\left|f'\right|$ is replaced by $h-$convexity. Clearly, Inequality (11) is better than Inequality (12) because of the facts that

$$\frac{1}{p+1} = \int_0^1 t^p\,dt$$
$$\leq \int_0^1 h(t^p)\,dt,$$

and

$$2\int_0^1 h(t)\,dt = \int_0^1 h(t)+h(1-t)\,dt$$
$$\leq \int_0^1 h(1)\,dt = h(1).$$

Corollary 2.3. In Inequality (9), if we choose $x=\frac{a+b}{2}$ then we obtain the midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[2M^q\int_0^1 h(t)\,dt - \frac{c(b-a)^2}{24}\right]^{\frac{1}{q}}. \tag{13}$$

The following Theorem gives an improvement of Inequality (9).

Theorem 2.3. Suppose that $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, and $a,b\in I$ with $a<b$. If $f'\in L^1[a,b]$ and $\left|f'\right|^q$ is strongly h-convex on $[a,b]$ with modulus $c>0$ and $\left|f'\right|\leq M$, where $M>0$, then

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \tag{14}$$
$$\leq \frac{(x-a)^2}{2(b-a)}\left[2M^q\int_0^1 h(t)\,dt - \frac{c(x-a)^2}{6}\right]^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{2(b-a)}\left[2M^q\int_0^1 h(t)\,dt - \frac{c(b-x)^2}{6}\right]^{\frac{1}{q}}.$$

for each $x\in[a,b]$.

Proof. Using Identity (3) and the triangle inequality, we get that

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{(x-a)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)a)\right| dt$$
$$+\frac{(b-x)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)b)\right| dt.$$

Using Holder's inequality and the fact that $t = t^{\frac{1}{p}} t^{\frac{1}{q}}$, we get that

$$\begin{aligned}
&\int_0^1 t\left|f'(tx+(1-t)a)\right| dt \\
&\leq \left(\int_0^1 t dt\right)^{\frac{1}{p}} \left(\int_0^1 t\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{2}\right)^{\frac{1}{p}} \left(\int_0^1 t\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 t\left|f'(tx+(1-t)a)\right| dt \\
&\leq \left(\int_0^1 t dt\right)^{\frac{1}{p}} \left(\int_0^1 t\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{2}\right)^{\frac{1}{p}} \left(\int_0^1 t\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|f'\right|^q$ is strongly h - convex with modulus c and $\left|f'\right| \leq M$,

$$\begin{aligned}
&\int_0^1 t\left|f'(tx+(1-t)a)\right|^q dt \\
&\leq \int_0^1 t\left[h(t)\left|f'(x)\right|^q + h(1-t)\left|f'(a)\right|^q \right.\\
&\left. -ct(1-t)(x-a)^2\right] dt \\
&\leq M^q \int_0^1 h(t)dt - \frac{c(x-a)^2}{12},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 t\left|f'(tx+(1-t)b)\right|^q dt \\
&\leq \int_0^1 t\left[h(t)\left|f'(x)\right|^q + h(1-t)\left|f'(b)\right|^q \right.\\
&\left. -ct(1-t)(b-x)^2\right] dt \\
&\leq M^q \int_0^1 h(t)dt - \frac{c(b-x)^2}{12}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \\
&\leq \frac{(x-a)^2}{2^{\frac{1}{p}}(b-a)}\left[M^q \int_0^1 h(t)dt - \frac{c(x-a)^2}{12}\right]^{\frac{1}{q}} \\
&+ \frac{(b-x)^2}{2^{\frac{1}{p}}(b-a)}\left[M^q \int_0^1 h(t)dt - \frac{c(b-x)^2}{12}\right]^{\frac{1}{q}} \\
&= \frac{(x-a)^2}{2(b-a)}\left[2M^q \int_0^1 h(t)dt - \frac{c(x-a)^2}{6}\right]^{\frac{1}{q}} \\
&+ \frac{(b-x)^2}{2(b-a)}\left[2M^q \int_0^1 h(t)dt - \frac{c(b-x)^2}{6}\right]^{\frac{1}{q}}.
\end{aligned}$$

Corollary 2.4. In Inequality (9), if we choose $x = \frac{a+b}{2}$ then we obtain the midpoint inequality

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)du\right| \\
&\leq \frac{(b-a)}{4}\left[2M^q \int_0^1 h(t)dt - \frac{c(b-a)^2}{24}\right]^{\frac{1}{q}}. \qquad (15)
\end{aligned}$$

Remark 2.3. If $c \longrightarrow 0^+$, Inequality (15) reduces to the inequality

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{M(b-a)}{2^{2-\frac{1}{q}}}\left[\int_0^1 h(t)dt\right]^{\frac{1}{q}}. \qquad (16)$$

3 Some Applications

We start this section with an application to cumulative distribution functions. Let X be a random variable taking its values in the finite interval $[a,b]$, where $a < b$,with a probability density function $f : [a,b] \to [0,1]$. The cumulative distribution function is defined as:

$$F(x) = \Pr(X \leq x) = \int_a^x f(t)dt.$$

The expectation of X is defined as:

$$E(X) = \int_a^b tf(t)dt.$$

Proposition 3.1. Let X be a random variable taking its values in the finite interval $[a,b]$, where $a < b$, with a probability density function $f : [a,b] \to [0,1]$. If

$f \in L^1[a,b]$ such that $|f|$ is strongly h-convex on $[a,b]$ with modulus $c>0$ and $|f| \le M$, where $M>0$, then

$$\left|F(x)-\frac{b-E(X)}{b-a}\right| \le \frac{(x-a)^2}{b-a}\left[M\int_0^1 h(t)\,dt-\frac{c(x-a)^2}{12}\right]+\frac{(b-x)^2}{b-a}\left[M\int_0^1 h(t)\,dt-\frac{c(b-x)^2}{12}\right]. \tag{17}$$

for each $x \in [a,b]$.

Proof. Using integration by parts and the facts that $F' = f$ and $F(b)=1$, we get that

$$E(X)=b-\int_a^b F(t)\,dt.$$

Using Inequality (4) with f replaced by F, we have

$$\left|F(x)-\frac{1}{b-a}\int_a^b F(u)\,du\right| \le \frac{(x-a)^2}{b-a}\left[M\int_0^1 h(t)\,dt-\frac{c(x-a)^2}{12}\right]+\frac{(b-x)^2}{b-a}\left[M\int_0^1 h(t)\,dt-\frac{c(b-x)^2}{12}\right].$$

But

$$\int_a^b F(t)\,dt=b-E(X),$$

which implies that

$$\left|F(x)-\frac{b-E(X)}{b-a}\right| \le \frac{(x-a)^2}{b-a}\left[M\int_0^1 h(t)\,dt-\frac{c(x-a)^2}{12}\right]+\frac{(b-x)^2}{b-a}\left[M\int_0^1 h(t)\,dt-\frac{c(b-x)^2}{12}\right].$$

The second application of the results obtained is devoted to arithmetic and generalized log means. Recall that for any two positive numbers a,b with $a \ne b$, the arithmetic mean is defined as:

$$A(a,b)=\frac{a+b}{2},$$

and the generalized log-mean is defined as:

$$L_r(a,b)=\left(\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\right)^{\frac{1}{r}}$$

for $r \in \mathbb{R}-\{-1,0\}$.

Proposition 3.2. Let $0<a<b$, $\alpha \in (0,1]$, and $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Then for each $s \in [2,\infty)$, we have

$$|A^s(a,b)-L_s^s(a,b)| \le \frac{M(b-a)}{2^{2-\frac{1}{q}}}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}. \tag{18}$$

Proof. Let $s \in [2,\infty)$ and $f(x)=x^s$ for $x \in [a,b]$. Let $h(t)=t^{\alpha}$ for $t \in [0,1]$. Note that $f' \in L^1[a,b]$, $\left|f'\right|^q$ is h-convex on $[a,b]$, and $\left|f'\right| \le M = sb^{s-1}$. Using Inequality (16), we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \le \frac{M(b-a)}{2^{2-\frac{1}{q}}}\left[\int_0^1 h(t)\,dt\right]^{\frac{1}{q}} = \frac{M(b-a)}{2^{2-\frac{1}{q}}}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}.$$

But

$$f\left(\frac{a+b}{2}\right)=A^s(a,b),$$

and

$$\frac{1}{b-a}\int_a^b f(u)\,du=L_s^s(a,b),$$

which implies that

$$|A^s(a,b)-L_s^s(a,b)| \le \frac{M(b-a)}{2^{2-\frac{1}{q}}}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}.$$

For our next application, recall that a tagged partition P of a finite interval $[a,b]$ is a finite sequence of numbers $a=x_0<x_1<\cdots<x_n=b$, with corresponding tags $t_i \in [x_{i-1},x_i]$, for $i=1,\ldots,n$. In addition P is said to be uniform if $(x_i-x_{i-1})=\frac{(b-a)}{n}$ for each $i=1,\ldots,n$. The Riemann sum of a Riemann integrable function f over the interval $[a,b]$ with respect to the tagged partition P is given as:

$$S(f,P)=\sum_{k=1}^{n} f(t_i)(x_i-x_{i-1}).$$

Proposition 3.3. Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on *on* (a,b), where $a<b$. Let $f' \in L^1[a,b]$ such that $\left|f'\right|$ is strongly h-convex on $[a,b]$ with modulus $c>0$ and $\left|f'\right| \le M$, where $M>0$. If $P: a=x_0<x_1<\cdots<x_n=b$ is a uniform tagged

partition with corresponding tags $t_i \in [x_{i-1}, x_i]$, then

$$\left| S(f,P) - \int_a^b f(u)\,du \right| \leq \frac{M(b-a)^2}{n}\left[\int_0^1 h(t)\,dt\right] - \frac{c(b-a)^4}{96n^3}. \tag{19}$$

Proof. Let $P: a = x_0 < x_1 < \cdots < x_n = b$ be a uniform tagged partition with corresponding tags $t_i \in [x_{i-1}, x_i]$. For each $1 \leq i \leq n$, applying inequality (4) over the interval $[x_{i-1}, x_i]$, we have

$$\left| f(t_i)(x_i - x_{i-1}) - \int_{x_{i-1}}^{x_i} f(u)\,du \right| \leq M\left[(t_i - x_{i-1})^2 + (x_i - t_i)^2\right]\left[\int_0^1 h(t)\,dt\right] - \frac{c}{12}\left[(t_i - x_{i-1})^4 + (x_i - t_i)^4\right].$$

Using the triangle inequality and properties of Riemann integral, we get that

$$\begin{aligned} \left| S(f,P) - \int_a^b f(u)\,du \right| &= \left| \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) - \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(u)\,du \right| \\ &\leq \sum_{i=1}^n \left| f(t_i)(x_i - x_{i-1}) - \int_{x_{i-1}}^{x_i} f(u)\,du \right| \\ &\leq M\left[\int_0^1 h(t)\,dt\right]\sum_{i=1}^n \left[(t_i - x_{i-1})^2 + (x_i - t_i)^2\right] \\ &\quad - \frac{c}{12}\sum_{i=1}^n \left[(t_i - x_{i-1})^4 + (x_i - t_i)^4\right] \end{aligned}$$

Note that

$$\sum_{i=1}^n \left[(t_i - x_{i-1})^2 + (x_i - t_i)^2\right] \leq \sum_{i=1}^n (x_i - x_{i-1})^2 \leq \frac{(b-a)^2}{n},$$

and for each $i = 1, 2, \ldots, n$, $x \in [x_{i-1}, x_i]$,

$$\begin{aligned} \left[(x - x_{i-1})^4 + (x_i - x)^4\right] &\geq \left[\left(\frac{x_{i-1} + x_i}{2} - x_{i-1}\right)^4 + \left(x_i - \frac{x_{i-1} + x_i}{2}\right)^4\right] \\ &= \frac{1}{8}(x_i - x_{i-1})^4 = \frac{(b-a)^4}{8n^4}. \end{aligned}$$

Therefore,

$$\left| S(f,P) - \int_a^b f(u)\,du \right| \leq \frac{M(b-a)^2}{n}\left[\int_0^1 h(t)\,dt\right] - \frac{c(b-a)^4}{96n^3}.$$

Acknowledgement

The author is grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

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