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Bayesian Inference for the Randomly-Censored Three-Parameter Burr XII Distribution

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Abstract: In this paper, we obtain the point and interval estimations for a three-parameter Burr-XII distribution (TPBXIID) based on randomly-censored data. The maximum likelihood (ML) and Bayes estimation method are used to estimate the unknown parameters of the TPBXIID. Furthermore, approximate confidence intervals (ACIs) for the unknown parameters are constructed. Markov chain Monte Carlo (MCMC) method is applied to find the Bayes estimation. Also, highest posterior density (HPD) credible intervals (CRIs) are obtained for the parameters. Gibbs within Metropolis-Hasting samplers are used to generate samples from the posterior density functions. A couple of real data sets are discussed to illustrate the proposed methods. Finally, to compare different estimates proposed in this paper, a Monte Carlo simulation study has been performed.

Keywords: Three-parameter Burr-XII distribution; Randomly censored data; Maximum likelihood estimators; Bayesian estimation; MCMC method.

1 Introduction

The Burr-XII distribution originally introduced by Burr [1] has been used for the lifetime modeling in reliability analysis, life-testing problems and acceptance sampling plans see Abbasi et al. [2]. Also, it has been fitted a wide range of observational information in different areas such as meteorology, finance and hydrology see Chen et al. [3]. For more details about applications of Burr-XII, see Ali and Jaheen [4] and Burr [1]. Shao [5] expanded the TPBXIID and studied the maximum likelihood estimation and Shao et al.[6] studied the models for extremes for the TPBXIID with application to flood frequency analysis. Wu et al. [7] studied the estimation problems by using Burr-XII based on progressive type-II censoring with random removals. Silva et al. [8] suggested a location-scale regression model based on Burr-XII distribution. Paranaiba et al. [9] suggested the beta Burr-XII. Paranaiba et al. [10] discussed Kumaraswamy Burr-XII. Mead [11] presented the beta exponentiated Burr-XII and Al-Saiarie et al. [12] studied the Marshall-Olkin extended Burr-XII.
Recently, Gomes et al. [13] discussed theory and practice for two extended Burr models (McDonald Burr-XII) and Mead and Affy [14] studied the properties and applications for five parameters Burr-XII distribution which are called the Kumaraswamy exponentiated Burr-XII distribution. The cumulative distribution function (CDF) of the TPBXIID is given by

$$F(x;\alpha,\theta,\gamma)=1-\left[1+\left(\frac{x}{\alpha}\right)^{\theta}\right]^{-\gamma},\quad x>0,\quad \alpha,\theta,\gamma>0, \tag{1}$$

and the probability density function (PDF) is

$$f(x;\alpha,\theta,\gamma)=\theta\,\gamma\,\alpha^{-\theta}\,x^{\theta-1}\left[1+\left(\frac{x}{\alpha}\right)^{\theta}\right]^{-(\gamma+1)},\quad x>0,\quad \alpha,\theta,\gamma>0. \tag{2}$$

Here γ and θ are the shape parameters and α is a scale parameter. Random censoring is a situation when the units in the experiment are lost or removed from the experiment randomly before its failure in a medical study, for example some patients may leave the course of treatment or die before its completion. Similarly, in reliability engineering, an item may have to be removed from the test before its complete failure because of its breakage or for saving time and money. When items are removed from the test at different random time points, it is called randomly-censored. Gilbert [15] was the first who introduced the random censoring. Many authors

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studied this type of censoring such as Breslow and Crowley [16], Koziol and Green [17] and Csorgo and Horvath [18]. Kim [19] implemented chi-square goodness of fit tests for randomly-censored data. Ghitany [20] analyzed Rayleigh survival model and its application to randomly-censored data. Ghitany and Al-Awadhi [21] studied maximum likelihood estimation of Burr-XII distribution parameters under random censoring. Friesl and Hurt [22] exponential distribution under random censorship. Saleem and Aslam [23] discussed the Bayesian analysis of the Rayleigh survival time assuming the random censor time. Saleem and Raza [24] studied the Bayesian analysis of the exponential survival time assuming the exponential censor time. Danish and Aslam [25,26] discussed the Bayesian estimation for generalized exponential and Weibull distributions under randomly-censored, respectively. Krishna and Vivekanand [27] studied the estimation in Maxwell distribution with randomly-censored data and Garg et al. [28] discussed the generalized inverted exponential distribution with randomly-censored data.
Recently, Krishna and Goel [29] dealt with maximum likelihood and Bayes estimation in randomly-censored geometric distribution. Danish et al. [30] dealt with Bayesian inference for the Burr-XII distribution under randomly-censored data and Krishna and Goel [31] studied the classical and Bayesian inference in two parameters exponential distribution with randomly-censored data.
The aim of this paper is organized as follows: Section 2, a mathematical formulation is discussed for TPBXIID based on randomly-censored data. The maximum likelihood estimators of the unknown parameters of TPBXIID are presented in Section 3. Section 4 deals with approximate confidence interval (ACI) based on MLEs. In Section 5, we cover Bayes estimates and construction of CRIs using the MCMC techniques under two different loss functions for the TPBXIID. In Section 6, we analyze two examples of real-data sets to illustrate the estimation methods developed in this paper and also deals with a Monte Carlo simulation results. Finally, conclusions appear in Section 7.

2 The Model and Assumptions

Assume n units are put on experiment with their lifetimes as $X_1,\ldots,X_n$ which are independent and identically distributed (iid) random variables with CDF $F_X(x)$ and PDF $f_X(x)$. Assume that another sequence $T_1,\ldots,T_n$ is the (iid) random censoring times of these units with CDF $F_T(t)$ and PDF $f_T(t)$. Assume that X_i's and T_i 's are mutually independent, so that one observes iid random pairs $(Y_1,D_1),\ldots,(Y_n,D_n)$, where $Y_i = \min(X_i,T_i)$, $i=1,\ldots,n$ and also define D_i as

$$D_i = \begin{cases} 1 & \text{if } X_i \le T_i \,, \\ 0 & \text{if } X_i \ge T_i \,. \end{cases}$$

Now, it is simple to show that the joint PDF of Y_i and D_i is

$$f_{Y,D}(y,d) = \big(f_X(y)(1-F_T(y))\big)^d \big(f_T(y)(1-F_X(y))\big)^{1-d}. \quad y \ge 0,\ d=0,1. \tag{3}$$

The random variables X and T satisfy the proportional hazards model with proportionality constant $\phi > 0$, if

$$\big(1-F_T(t)\big) = \big(1-F_X(t)\big)^{\phi}, \tag{4}$$

for ϕ equal zero Equation (4) describes the case of no censoring. From Equations (3) and (4) we get the joint PDF of Y_i and D_i

$$f_{Y,D}(y,d) = f_X(y)\big(1-F_X(y)\big)^{\phi}\phi^{1-d}, \qquad y \ge 0,\ d=0,1. \tag{5}$$

From Equations (1) and (2), the joint PDF in Equation (5) takes the following form:

$$f_{Y,D}(y,d;\alpha,\theta,\gamma,\phi) = \theta\,\gamma\,\alpha^{-\theta}\,y^{\theta-1}\left[1+\left(\frac{y}{\alpha}\right)^{\theta}\right]^{-\gamma(\phi+1)-1} \times \phi^{1-d}. \tag{6}$$

3 Maximum Likelihood Estimation

The likelihood function for TPBXIID is based on randomly-censored sample data $(y,d)=(y_1,d_1),\ \ldots,\ (y_n,d_n)$ of size n as displayed in Section 2, from Equation (6), the likelihood function is given by

$$L(y,d;\alpha,\theta,\gamma,\phi) = \theta^n\,\gamma^n\,\alpha^{-n\theta}\left[\prod_{i=1}^{n} y_i^{\theta-1}\right] \prod_{i=1}^{n}\left[1+\left(\frac{y_i}{\alpha}\right)^{\theta}\right]^{-\gamma(\phi+1)-1}\phi^{n-\sum_{i=1}^{n} d_i},$$

or, equivalently,

$$L(y,d;\alpha,\theta,\gamma,\phi) = \theta^n\,\gamma^n\,\alpha^{-n\theta}\,e^{(\theta-1)\sum_{i=1}^{n}\ln y_i} \times e^{-[\gamma(\phi+1)+1]\sum_{i=1}^{n}\ln\left[1+\left(\frac{y_i}{\alpha}\right)^{\theta}\right]} \times \phi^{n-\sum_{i=1}^{n} d_i}. \tag{7}$$

The log-likelihood function for the TPBXIID, corresponding to Equation (7) is

$$\ell(\alpha,\theta,\gamma,\phi) = n\ln\gamma + n\ln\theta - n\theta\ln\alpha + (\theta-1)\sum_{i=1}^{n}\ln y_i - [\gamma(\phi+1)+1]\sum_{i=1}^{n}\ln\left[1+\left(\frac{y_i}{\alpha}\right)^{\theta}\right] + \Big(n-\sum_{i=1}^{n} d_i\Big)\ln\phi. \tag{8}$$

Taking the first derivative of Equation (8) with respect to α, θ, γ and ϕ and setting each of these derivatives equal to zero, we obtain the likelihood equations for the parameters α, θ, γ and ϕ as follows:

$$\frac{-n\hat{\theta}}{\hat{\alpha}}+\left[\hat{\gamma}(\hat{\phi}+1)+1\right]\sum_{i=1}^{n}\frac{\hat{\theta}y_i\left(\frac{y_i}{\hat{\alpha}}\right)^{\hat{\theta}-1}}{\alpha^2\left[1+\left(\frac{y_i}{\hat{\alpha}}\right)^{\hat{\theta}}\right]}=0, \tag{9}$$

$$\frac{n}{\hat{\theta}}-n\ln\hat{\alpha}+\sum_{i=1}^{n}\ln y_i-\left[\hat{\gamma}(\hat{\phi}+1)+1\right]\sum_{i=1}^{n}\frac{\left(\frac{y_i}{\hat{\alpha}}\right)^{\hat{\theta}}\ln\left(\frac{y_i}{\hat{\alpha}}\right)}{1+\left(\frac{y_i}{\hat{\alpha}}\right)^{\hat{\theta}}}=0, \tag{10}$$

$$\frac{n}{\hat{\gamma}}-(\hat{\phi}+1)\sum_{i=1}^{n}\ln\left[1+\left(\frac{y_i}{\hat{\alpha}}\right)^{\hat{\theta}}\right]=0, \tag{11}$$

$$-\hat{\gamma}\sum_{i=1}^{n}\ln\left[1+\left(\frac{y_i}{\hat{\alpha}}\right)^{\hat{\theta}}\right]+\frac{1}{\hat{\phi}}(n-\sum_{i=1}^{n}d_i)=0. \tag{12}$$

From (11) we obtain the MLE of $\hat{\gamma}$ as

$$\hat{\gamma}=n\left[(\hat{\phi}+1)\sum_{i=1}^{n}\ln\left[1+\left(\frac{y_i}{\hat{\alpha}}\right)^{\hat{\theta}}\right]\right]^{-1}, \tag{13}$$

and from (12) we obtain the MLE of $\hat{\phi}$ as

$$\hat{\phi}=\frac{(n-\sum_{i=1}^{n}d_i)}{\hat{\gamma}\sum_{i=1}^{n}\ln\left[1+\left(\frac{y_i}{\hat{\alpha}}\right)^{\hat{\theta}}\right]}. \tag{14}$$

The maximum likelihood estimators of α and θ can be found by solving the system of Equations (9) and (10), but it is clear that is impossible to solve (9) and (10) analytically because it is very difficult to get closed forms for each parameter, we can use a suitable numerical technique such as Newton-Raphson iteration method to obtain the estimates, for more details see, EL-Sagheer[32].

4 Approximate Confidence Interval

The asymptotic variances-covariances of the MLEs for parameters α, θ, γ and ϕ are given by elements of the inverse of the Fisher information matrix are defined as

$$I_{ij}=-E\left(\frac{\partial^2\ell}{\partial\delta_i\partial\delta_j}\right),$$

where $i,j=1,2,3,4$ and $(\delta_1,\delta_2,\delta_3,\delta_4)=(\alpha,\theta,\gamma,\phi)$. Unfortunately, the exact mathematical expressions for the above expectations are very difficult to obtain. Therefore, we give the approximate asymptotic varaince-covariance matrix for the MLE, which is obtained by dropping the expectation operator E

$$I^{-1}(\alpha,\theta,\gamma,\phi)=\begin{pmatrix}-\frac{\partial^2\ell}{\partial\alpha^2} & -\frac{\partial^2\ell}{\partial\alpha\partial\theta} & -\frac{\partial^2\ell}{\partial\alpha\partial\gamma} & -\frac{\partial^2\ell}{\partial\alpha\partial\phi}\\ -\frac{\partial^2\ell}{\partial\theta\partial\alpha} & -\frac{\partial^2\ell}{\partial\theta^2} & -\frac{\partial^2\ell}{\partial\theta\partial\gamma} & -\frac{\partial^2\ell}{\partial\theta\partial\phi}\\ -\frac{\partial^2\ell}{\partial\gamma\partial\alpha} & -\frac{\partial^2\ell}{\partial\gamma\partial\theta} & -\frac{\partial^2\ell}{\partial\gamma^2} & -\frac{\partial^2\ell}{\partial\gamma\partial\phi}\\ -\frac{\partial^2\ell}{\partial\phi\partial\alpha} & -\frac{\partial^2\ell}{\partial\phi\partial\theta} & -\frac{\partial^2\ell}{\partial\phi\partial\gamma} & -\frac{\partial^2\ell}{\partial\phi^2}\end{pmatrix}^{-1}$$
$$=\begin{pmatrix}\widehat{var}(\hat{\alpha}) & cov(\hat{\alpha},\hat{\theta}) & cov(\hat{\alpha},\hat{\gamma}) & cov(\hat{\alpha},\hat{\phi})\\ cov(\hat{\theta},\hat{\alpha}) & \widehat{var}(\hat{\theta}) & cov(\hat{\theta},\hat{\gamma}) & cov(\hat{\theta},\hat{\phi})\\ cov(\hat{\gamma},\hat{\alpha}) & cov(\hat{\gamma},\hat{\theta}) & \widehat{var}(\hat{\gamma}) & cov(\hat{\gamma},\hat{\phi})\\ cov(\hat{\phi},\hat{\alpha}) & cov(\hat{\gamma},\hat{\theta}) & cov(\hat{\phi},\hat{\gamma}) & \widehat{var}(\hat{\phi})\end{pmatrix}. \tag{15}$$

With

$$\frac{\partial^2\ell}{\partial\alpha^2}=\frac{n\theta}{\alpha^2}+(\gamma(\phi+1)+1)\times\sum_{i=1}^{n}\left[\frac{\theta\left(\frac{y_i}{\alpha}\right)^{\theta}\left[-\alpha^2(\theta-1)\left[1+\left(\frac{y_i}{\alpha}\right)^{\theta}\right]\right.}{\left[\alpha^2\left[1+\left(\frac{y_i}{\alpha}\right)^{\theta}\right]\right]^2}-\frac{2\alpha y_i(\frac{y_i}{\alpha})^{-1}\left[1+\left(\frac{y_i}{\alpha}\right)^{\theta}\right]+\theta y_i^2(\frac{y_i}{\alpha})^{\theta-2}\left.\right]}{\left[\alpha^2\left[1+\left(\frac{y_i}{\alpha}\right)^{\theta}\right]\right]^2}\right], \tag{16}$$

$$\frac{\partial^2\ell}{\partial\theta^2}=\frac{-n}{\theta^2}-(\gamma(\phi+1)+1)\sum_{i=1}^{n}\frac{\left(\frac{y_i}{\alpha}\right)^{\theta}\left(\ln\left[\frac{y_i}{\alpha}\right]\right)^2}{\left[1+\left(\frac{y_i}{\alpha}\right)^{\theta}\right]^2}, \tag{17}$$

$$\frac{\partial^2\ell}{\partial\alpha\partial\theta}=\frac{\partial^2\ell}{\partial\theta\partial\alpha}=\frac{-R}{\alpha}+(\gamma(\phi+1)+1)\times\sum_{i=1}^{R}\frac{\frac{1}{\alpha}\left(\frac{y_i}{\alpha}\right)^{\theta}\left[\ln\left[\frac{y_i}{\alpha}\right]+\left(\frac{y_i}{\alpha}\right)^{\theta}+1\right]}{\left[1+\left(\frac{y_i}{\alpha}\right)^{\theta}\right]^2}, \tag{18}$$

$$\frac{\partial^2\ell}{\partial\gamma^2}=\frac{-n}{\gamma^2}, \tag{19}$$

$$\frac{\partial^2\ell}{\partial\phi^2}=\frac{-(n-\sum_{i=1}^{n}d_i)}{\phi^2}, \tag{20}$$

$$\frac{\partial^2\ell}{\partial\alpha\partial\gamma}=\frac{\partial^2\ell}{\partial\gamma\partial\alpha}=(\hat{\phi}+1)\sum_{i=1}^{n}\frac{\hat{\theta}y_i\left(\frac{y_i}{\hat{\alpha}}\right)^{\hat{\theta}-1}}{\alpha^2\left[1+\left(\frac{y_i}{\hat{\alpha}}\right)^{\hat{\theta}}\right]}, \tag{21}$$

$$\frac{\partial^2\ell}{\partial\theta\partial\gamma}=\frac{\partial^2\ell}{\partial\gamma\partial\theta}=-(\phi+1)\sum_{i=1}^{n}\frac{\left(\frac{y_i}{\hat{\alpha}}\right)^{\hat{\theta}}\ln\left(\frac{y_i}{\hat{\alpha}}\right)}{1+\left(\frac{y_i}{\hat{\alpha}}\right)^{\hat{\theta}}}, \tag{22}$$

$$\frac{\partial^2 \ell}{\partial\phi\partial\gamma} = \frac{\partial^2 \ell}{\partial\gamma\partial\phi} = -\sum_{i=1}^{n} \ln\left[1+\left(\frac{y_i}{\hat{\alpha}}\right)^{\hat{\theta}}\right], \tag{23}$$

$$\frac{\partial^2 \ell}{\partial\theta\partial\phi} = \frac{\partial^2 \ell}{\partial\phi\partial\theta} = -\gamma\sum_{i=1}^{n} \frac{\left(\frac{y_i}{\hat{\alpha}}\right)^{\hat{\theta}} \ln\left(\frac{y_i}{\hat{\alpha}}\right)}{1+\left(\frac{y_i}{\hat{\alpha}}\right)^{\hat{\theta}}}, \tag{24}$$

$$\frac{\partial^2 \ell}{\partial\alpha\partial\phi} = \frac{\partial^2 \ell}{\partial\phi\partial\alpha} = \gamma\sum_{i=1}^{n} \frac{\hat{\theta} y_i \left(\frac{y_i}{\hat{\alpha}}\right)^{\hat{\theta}-1}}{\alpha^2\left[1+\left(\frac{y_i}{\hat{\alpha}}\right)^{\hat{\theta}}\right]}. \tag{25}$$

The asymptotic normality of the MLEs can be used to compute the approximate confidence intervals for parameters α, θ, γ and ϕ. Therefore, $(1-\eta)100\%$ confidence intervals (CIs) for parameters α, θ, γ and ϕ become

$$\left(\hat{\alpha} \pm Z_{\eta/2}\sqrt{\widehat{var}(\hat{\alpha})}\right), \quad \left(\hat{\theta} \pm Z_{\eta/2}\sqrt{\widehat{var}(\hat{\theta})}\right), \quad \left(\hat{\gamma} \pm Z_{\eta/2}\sqrt{\widehat{var}(\hat{\gamma})}\right) \text{ and } \left(\hat{\phi} \pm Z_{\eta/2}\sqrt{\widehat{var}(\hat{\phi})}\right). \tag{26}$$

Where $Z_{\eta/2}$ is a standard normal value.

5 Bayes Estimation

In this section, we obtain the Bayesian inference of the unknown parameters TPBXIID based on randomly-censored data under squared error (SE) loss and general entropy (GE) loss functions. It is assumed here that the parameters α, θ, γ and ϕ are independent and follow the gamma prior distributions.

$$\pi_1(\alpha) \propto \alpha^{a_1-1}e^{-b_1\alpha}, \qquad \alpha > 0, \tag{27}$$

$$\pi_2(\theta) \propto \theta^{a_2-1}e^{-b_2\theta}, \qquad \theta > 0, \tag{28}$$

$$\pi_3(\gamma) \propto \gamma^{a_3-1}e^{-b_3\gamma}, \qquad \gamma > 0, \tag{29}$$

$$\pi_4(\phi) \propto \phi^{a_4-1}e^{-b_4\phi}, \qquad \phi > 0. \tag{30}$$

Here all the hyper parameters $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 are assumed to be known and non-negative.
The joint prior distribution for α, θ, γ and ϕ is

$$\pi(\alpha,\theta,\gamma,\phi) \propto \alpha^{a_1-1}\theta^{a_2-1}\gamma^{a_3-1}\phi^{a_4-1} \times e^{-b_1\alpha-b_2\theta-b_3\gamma-b_4\phi}. \tag{31}$$

From (7) and (31) we obtain the joint posterior density function as follows

$$\begin{aligned}\pi^*(\alpha,\theta,\gamma,\phi|\underline{y},\underline{d}) \propto\ & \alpha^{a_1-n\theta-1}\theta^{a_2+n-1}\gamma^{a_3+n-1} \\ & \times e^{-b_1\alpha-b_4\phi}\phi^{a_4+n-\Sigma_{i=1}^n d_i-1} \\ & \times e^{-\gamma\left[b_3+[(\phi+1)+1]\Sigma_{i=1}^n \ln\left[1+\left(\frac{y_i}{\alpha}\right)^{\theta}\right]\right]} \\ & \times e^{-\theta(b_2-sum_{i=1}^n \ln y_i)-\Sigma_{i=1}^n \ln y_i}.\end{aligned} \tag{32}$$

Therefore, the Bayes estimator of a function $U(\alpha,\theta,\gamma,\phi)$ under the SE loss function is as follows:

$$\begin{aligned}\hat{U}_{BS}(\alpha,\theta,\gamma,\phi) =& E\left[U(\alpha,\theta,\gamma,\phi)|\underline{y},\underline{d}\right] \\ =& \int_\alpha\int_\theta\int_\gamma\int_\phi U(\alpha,\theta,\gamma,\phi) \\ & \times \pi^*(\alpha,\theta,\gamma,\phi|y,d)d\alpha d\theta d\gamma d\phi.\end{aligned} \tag{33}$$

The Bayes estimator of a function $U(\alpha,\theta,\gamma,\phi)$ under the GE loss function is as follows

$$\begin{aligned}\hat{U}_{BG}(\alpha,\theta,\gamma,\phi) =& \left[E[U(\alpha,\theta,\gamma,\phi)]^{-a}|\underline{y},\underline{d}\right]^{-\frac{1}{a}} \\ =& \left(\int_\alpha\int_\theta\int_\gamma\int_\phi [U(\alpha,\theta,\gamma,\phi)]^{-a}\right. \\ & \left.\times \pi^*(\alpha,\theta,\gamma,\phi|\underline{y},\underline{d})d\alpha d\theta d\gamma d\phi\right)^{-\frac{1}{a}}.\end{aligned} \tag{34}$$

It is evident that is not possible to compute (33) and (34) analytically because it is very difficult to get explicit forms for the marginal posterior distributions for each parameter. Then, we propose using the MCMC method to approximate (33) and (34) under SE loss and GE loss functions.

5.1 MCMC method

In this section, we propose using the MCMC to generate samples from (32), we compute the Bayes estimates of α, θ, γ and ϕ and also, construct the corresponding HPD CRIs. A lot of papers dealt with MCMC technique such as, Chen and Shao [33], EL-Sagheer[32] and Ghazal and Hasaballah [34,35,36]. The conditional posterior densities distributions of α, θ, γ and ϕ can be obtained from Equation (32) up to proportionality as the following, to simply we use $\pi_1^*(\alpha)$, $\pi_2^*(\theta)$, $\pi_3^*(\gamma)$ and $\pi_4^*(\phi)$ instead of $\pi_1^*(\alpha|\theta,\gamma,\phi,(\underline{y},\underline{d}))$, $\pi_2^*(\theta|\alpha,\gamma,\phi,(\underline{y},\underline{d}))$, $\pi_3^*(\gamma|\alpha,\theta,\phi,(\underline{y},\underline{d}))$ and $\pi_4^*(\phi|\alpha,\theta,\gamma,(\underline{y},\underline{d}))$, respectively:

$$\pi_1^*(\alpha) \propto \alpha^{a_1-n\theta-1}e^{-b_1\alpha} \times e^{-[\gamma(\phi+1)+1]\Sigma_{i=1}^n \ln\left[1+\left(\frac{y_i}{\alpha}\right)^{\theta}\right]}, \tag{35}$$

$$\pi_2^*(\theta) \propto \theta^{a_2+n-1}\alpha^{a_1-n\theta-1}e^{-\theta(b_2-\sum_{i=1}^n \ln y_i)} \times e^{-[\gamma(\phi+1)+1]\sum_{i=1}^n \ln\left[1+\left(\frac{y_i}{\alpha}\right)^\theta\right]}, \tag{36}$$

$$\pi_3^*(\gamma) \propto \gamma^{a_3+n-1} e^{-\gamma\left[b_3+(\phi+1)\sum_{i=1}^n \ln\left[1+\left(\frac{y_i}{\alpha}\right)^\theta\right]\right]}, \tag{37}$$

$$\pi_4^*(\phi) \propto \phi^{a_4+n-\sum_{i=1}^n d_i-1} \times e^{-\phi\left[b_4+\gamma\sum_{i=1}^n \ln\left[1+\left(\frac{y_i}{\alpha}\right)^\theta\right]\right]}. \tag{38}$$

It can be seen that (37) is a gamma density with shape parameter $(a_3 + n)$ and scale parameter $\left(b_3+(\phi+1)\sum_{i=1}^n \ln\left[1+\left(\frac{y_i}{\alpha}\right)^\theta\right]\right)$ and (38) is a gamma density with shape parameter (a_4+n) and scale parameter $\left(b_4 + \gamma\sum_{i=1}^n \ln\left[1+\left(\frac{y_i}{\alpha}\right)^\theta\right]\right)$. Therefore, samples of γ and ϕ can be easily generated using any gamma-generating routine. However, the posterior densities in Equations (35) and (36) cannot be reduced analytically to well-known distributions and therefore, it is impossible to sample directly by standard methods, however, the plots of posterior distribution of α and θ show that they are similar to normal distribution, as seen in Figure (1) and in Figure (2) respectively. Furthermore, the conditional posterior densities in Equations (35) and (36) are log-concave. So, to generate random numbers from thes two distributions, we propose using the Metropolis-Hastings algorithm with normal proposal distribution, see Metropolis et al. [37] Now, we are applying the next MCMC algorithm to draw samples from the posterior density (32) and in turn to obtain the Bayes estimates of the parameters $(\alpha,\theta,\gamma,\phi)$ and the corresponding CRIs.

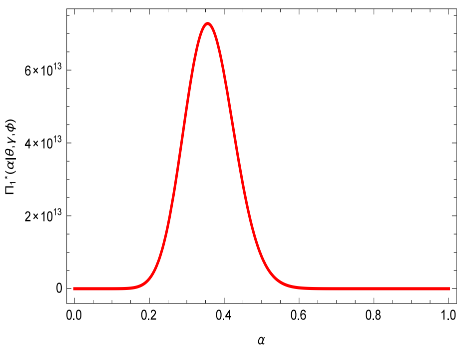

Fig. 1: Posterior density functions $\pi_1^*(\alpha|\theta,\gamma,\phi,(y,d))$ of α.

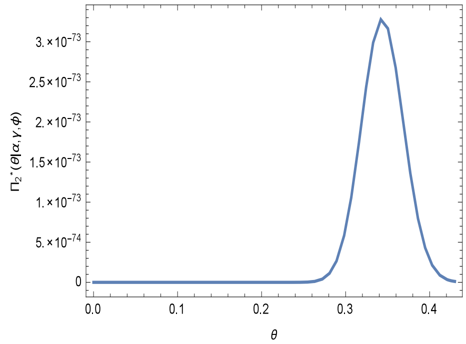

Fig. 2: Posterior density functions $\pi_2^*(\theta|\alpha,\gamma,\phi,(y,d))$ of θ.

$$\frac{\partial^2\pi_1^*(\alpha)}{\partial\alpha^2} = \frac{-(a_1-n\theta-1)}{\alpha^2} - [1+\gamma(\phi+1)] \times \sum_{i=1}^n \left[\frac{\theta\left(\frac{y_i}{\alpha}\right)^\theta\left[-\alpha^2(\theta-1)\left[1+\left(\frac{y_i}{\alpha}\right)^\theta\right]\right]}{\left[\alpha^2\left[1+\left(\frac{y_i}{\alpha}\right)^\theta\right]\right]^2} - \frac{2\alpha y_i\left(\frac{y_i}{\alpha}\right)^{-1}\left[1+\left(\frac{y_i}{\alpha}\right)^\theta\right]+\theta y_i^2\left(\frac{y_i}{\alpha}\right)^{\theta-2}]}{\left[\alpha^2\left[1+\left(\frac{y_i}{\alpha}\right)^\theta\right]\right]^2}\right] < 0, \tag{39}$$

$$\frac{\partial^2\pi_2^*(\theta)}{\partial\theta^2} = \frac{-(a_2+n-1)}{\theta^2} - \left[1+\gamma(\phi+1)\right]\sum_{i=1}^n \frac{\left(\frac{y_i}{\alpha}\right)^\theta \ln\left(\frac{y_i}{\alpha}\right)^2}{\left[1+\left(\frac{y_i}{\alpha}\right)^\theta\right]^2} < 0. \tag{40}$$

Metropolis-Hastings algorithm :

1.Take some initial guess of α, θ, γ and ϕ, say $\alpha^{(0)}$, $\theta^{(0)}$, $\gamma^{(0)}$ and $\phi^{(0)}$ respectively, M = burn-in.
2.Set $j = 1$.
3.Generate $\gamma^{(i)}$ from gamma $\left(n+a_3,\ b_3+(\phi+1)\sum_{i=1}^n \ln\left[1+\left(\frac{y_i}{\alpha}\right)^\theta\right]\right)$.
4.Generate $\phi^{(i)}$ from gamma $\left(n+a_4,\ b_4+\gamma\sum_{i=1}^n \ln\left[1+\left(\frac{y_i}{\alpha}\right)^\theta\right]\right)$.
5.Using Metropolis-Hastings, generate $\alpha^{(j)}$ and $\theta^{(j)}$ from $\pi_1^*(\alpha|\theta,\gamma,\phi,(\underline{y},\underline{d}))$ and $\pi_2^*(\theta|\alpha,\gamma,\phi,(\underline{y},\underline{d}))$ with normal proposal distribution $N(\alpha^{(j-1)},var(\alpha))$ and $N(\theta^{(j-1)},var(\theta))$ where $var(\alpha)$ and $var(\theta)$ can be obtained from the main diagonal in the inverse of the Fisher information matrix (15).
(i) Calculate the acceptance probability

$$r_1 = \min\left[1, \frac{\pi_1^*(\alpha^*|\theta^{j-1},\gamma^j,\phi^j,(\underline{y},\underline{d}))}{\pi_1^*(\alpha^{j-1}|\theta^{j-1},\gamma^j,\phi^j,(\underline{y},\underline{d}))}\right], \tag{41}$$

$$r_2 = \min\left[1, \frac{\pi_2^*(\theta^*|\alpha^j,\gamma^j,\phi^j,(\underline{y},\underline{d}))}{\pi_2^*(\theta^{j-1}|\alpha^j,\gamma^j,\phi^j,(\underline{y},\underline{d}))}\right]. \tag{42}$$

(ii) Generate u_1 and u_2 from a Uniform $(0,1)$ distribution.
(iii) If $u_1 \leq r_1$, accept the proposal and set $\alpha^i = \alpha^*$, else set $\alpha^i = \alpha^{i-1}$.
(iv) If $u_2 \leq r_2$, accept the proposal and set $\theta^i = \theta^*$, else set $\theta^i = \theta^{i-1}$.

6. Set $j = j+1$.
7. Repeat Steps $3-6$ N times to obtain $(\alpha^{(j)}, \theta^{(j)}, \gamma^{(j)}, \phi^{(j)})$, $j = M+1, ..., N$.
8. To compute the CRIs of α, θ, γ and ϕ, order $\alpha^{(j)}, \theta^{(j)}, \gamma^{(j)}$ and $\phi^{(j)}$, $j = M+1, ..., N$, as $(\alpha^{(1)} < \ldots, < \alpha^{(N-M)})$, $(\theta^{(1)} < \ldots, < \theta^{(N-M)})$, $(\gamma^{(1)} < \ldots, < \gamma^{(N-M)})$ and $(\phi^{(1)} < \ldots, < \phi^{(N-M)})$. Then the $100(1-\eta)\%$ CRIs of α, θ, γ and ϕ is $(\zeta_{(N-M)_{\eta/2}}, \zeta_{(N-M)_{(1-\eta/2)}})$. Then, the Bayes estimates of $\zeta = (\alpha,\ \theta,\ \gamma,\ \phi)$, under SE loss function are given by

$$\hat{\zeta}_{BS} = \frac{1}{N-M} \sum_{i=M+1}^{N} \zeta^{(i)}, \tag{43}$$

and the Bayes estimates of $\zeta = (\alpha,\ \theta,\ \gamma,\ \phi)$, under GE loss function are given by

$$\hat{\zeta}_{BG} = \left[\frac{1}{N-M} \sum_{i=M+1}^{N} [\zeta^{(i)}]^{-a} \right]^{\frac{-1}{a}}. \tag{44}$$

6 Data Analysis and Simulation Study

6.1 Data Analysis

we analyze two examples of real-data sets to illustrate the estimation methods developed in this paper and also deals with.

Example 1 In this example, a real data set is used to illustrate the proposed methods. The data set is obtained from Lawless [38]. The data given here arose in tests on endurance of deep groove ball bearings. The data are the numbers of million revolution before failure for each of the 23 ball bearings in the life test and they are:

17.88*	28.92	33.00	41.52*	42.12	45.60*
48.40*	51.84*	51.96*	54.12	55.56*	67.90
68.64*	68.64*	68.88	84.12*	93.12	98.64*
105.12	105.84*	127.92	127.04	173.40.	

The observations with asterisks indicate censored times. We have used Kolmogorov-Smirnov (K-S) test to fit whether the data distribution as TPBXIID or not. The calculated value of the K-S test is 0.111643 for the TPBXIID and this value is smaller than their corresponding values expected at 5% significance level, which is 0.274905 at $n = 23$ and P-value equal 0.9064. So, it can be seen that the TPBXIID fits the data very well and also we have just plotted the empirical $S(t)$ and the fitted $S(t)$ in Fig 3. Observe that the TPBXIID can be a good model fitting this data.

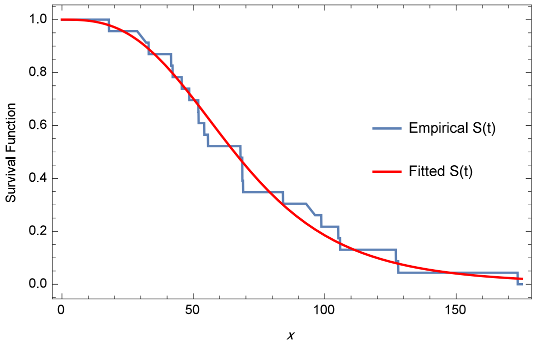

Fig. 3: Empirical and fitted survival functions.

Table 1: Different point estimates of $(\alpha, \theta, \gamma, \phi)$ for Example 1.

Parameters	MLE	MCMC		
		SE	GE	
			$a=-1$	$a=1$
α	86.1800	86.1898	86.1898	86.1898
θ	2.82503	2.81088	2.81088	2.81074
γ	1.03204	1.31504	1.31504	1.24072
ϕ	0.769231	0.406602	0.406602	0.346482

Table 2: 95% confidence intervals of $(\alpha, \theta, \gamma, \phi)$ for Example 1.

Parameters	ACIs		
	Lower	Upper	Length
α	-39.8294	212.189	252.0184
θ	0.783224	4.86683	4.08360
γ	-1.836190	3.90026	5.73645
ϕ	0.135073	1.40339	1.268317

Parameters	CRIs		
	Lower	Upper	Length
α	86.18500	86.1951	0.01010
θ	2.785160	2.83809	0.05293
γ	0.778654	1.99176	1.21311
ϕ	0.168291	0.796284	0.62799

Example 2 In this example, we analyze a real data set obtained from Egyptain Meteorological Authority. This data represents the wind speed measured by knots of 84 days. We have taken the maximum wind speed per day in the period from December, 21, 2014 to March, 14, 2015 for Alexandria city as follows:

11	18*	15	8.0*	9.0*	6.0*	11*	16*
11*	13*	9.0*	17*	12*	15*	11*	20*
32	28	20*	23*	21*	15*	17*	14*
16	11*	14	13	8.0	9.0	9.0*	8.0*
6.0*	9.0	11*	9.0*	8.0*	9.0*	14	14
18*	14	8.0	15*	12	6.0	8.0	8.0
14*	12	13	32*	28*	20	14	8.0*
8.0	10	9.0*	14	22	22	15	8.0
8.0	14*	10	13	16	10	9.0*	4.0
14*	14*	10	10	7.0	11*	12	10
12	16	7.0	13*.				

The observations with asterisks indicate censored times. Also, we have used (K-S). The calculated value of the K-S test is 0.09266 for the TPBXIID and this value is smaller than their corresponding values expected at 5% significance level, which is 0.14605 at $n = 84$ and P-value equal 0.44. So, it can be seen that the TPBXIID fits the data very well and also we have just plotted the empirical $S(t)$ and the fitted $S(t)$ in Fig 4. Observe that the TPBXIID can be a good model fitting this data.

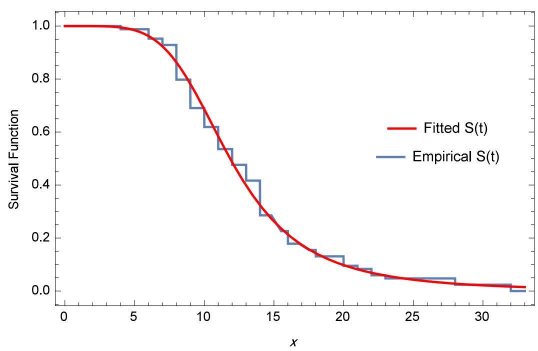

Fig. 4: Empirical and fitted survival functions.

Table 3: Different point estimates of $(\alpha,\theta,\gamma,\phi)$ for Example 2.

Parameters	MLE	MCMC		
		SE	GE	
			$a=-1$	$a=1$
α	10.71440	8.15489	8.09984	7.99008
θ	5.03672	4.37489	4.34005	4.26800
γ	0.36435	0.21598	0.21257	0.20356
ϕ	1.00000	2.44937	1.40025	1.21819

6.2 Simulation study

In this section, we present some simulation results to compare the performances of the different methods

Table 4: 95% confidence intervals of $(\alpha,\theta,\gamma,\phi)$ for Example 2.

Parameters	ACIs		
	Lower	Upper	Length
α	7.76024	13.6685	5.90826
θ	3.00934	7.06409	4.05466
γ	0.04461	0.68409	0.63948
ϕ	0.57230	1.42770	0.85539
Parameters	CRIs		
	Lower	Upper	Length
α	5.46735	10.5960	5.12865
θ	2.92674	5.90872	2.98198
γ	0.12609	0.31897	0.19288
ϕ	0.69873	3.34892	2.65019

proposed in this paper. We mainly compare the performances of the average values and absolute relative bias (ARB) of ML estimates and Bayes estimates of the unknown parameters of TPBXIID under two different loss functions, in terms of average interval length (AIL) of ACIs and CIs, and their coverage percentages.

1. For given hyperparameters $a_1,b_1,a_2,b_2,a_3,b_3,a_4$ and b_4 generate random values of α, θ, γ and ϕ from (35), (36), (37) and (38).
2. For given values of n with the initial values of α, θ, γ and ϕ given in Step (1), we generate random samples from the inverse CDF of TPBXIID and then ordered them.
3. The ML estimates of α, θ, γ and ϕ are then obtained by solving the four non-linear Equations (9), (10), (13) and (14) numerically and also computed the 95% confidence intervals using the observed Fisher information matrix.
4. The Bayes estimates of α, θ, γ and ϕ are computed and also computed the 95% HPD CRIs by applying the MCMC method with using 10000 MCMC samples and discard the first 1000 values as 'burn-in' under SE loss function, given by (43) and GE loss function, given by (44).
5. Steps 1-5 are repeated 1000 times and generated a sample from a TPBXIID with $\alpha = 7.8921$, $\theta = 22.2303$, $\gamma = 0.0683$, $\phi = 0.8074$ and $n = 20,30,40,50,60,70,80,90$. The ARB of the estimates are estimated by

$$\text{ARB}(\hat{\psi}) = \frac{\left|\frac{\sum_{i=1}^{1000}\hat{\psi}_i}{1000} - \psi\right|}{\psi}. \tag{45}$$

The main results of the simulation study are displayed in Tables 5,6, 7, 8, 9 and 10.

7 Conclusion

In this paper, based on randomly-censored data the MLEs and Bayesian estimates of the unknown parameters for

Table 5: The average and ARB (in parentheses) of estimates for the parameter α.

n	MLE	MCMC		
		SE	GE	
			$a=-1$	$a=1$
20	8.1603	7.5859	7.5859	7.2938
	(0.0398)	(0.0388)	(0.0389)	(0.0758)
30	8.0881	8.0421	8.0421	8.0227
	(0.0248)	(0.019)	(0.019)	(0.0165)
40	8.0316	7.5154	7.5154	7.5031
	(0.0577)	(0.0477)	(0.0477)	(0.0463)
50	7.9711	7.9388	7.9388	7.9295
	(0.060)	(0.0059)	(0.0059)	(0.0047)
60	8.0595	8.0361	8.0361	8.0273
	(0.0212)	(0.0182)	(0.0182)	(0.0171)
70	8.0220	7.9977	7.9977	7.9914
	(0.0565)	(0.0434)	(0.043)	(0.0126)
80	7.9558	7.938	7.938	7.9326
	(0.0081)	(0.0058)	(0.0058)	(0.0051)
90	7.9145	7.8979	7.8979	7.8942
	(0.0028)	(0.0007)	(0.0007)	(0.0003)

Table 6: The average and ARB (in parentheses) of estimates for the parameter θ.

n	MLE	MCMC		
		SE	GE	
			$a=-1$	$a=1$
20	20.9229	21.0678	21.0678	21.0343
	(0.0588)	(0.0523)	(0.0523)	(0.0518)
30	21.4172	21.4156	21.4156	21.4156
	(0.0376)	(0.0365)	(0.0366)	(0.0364)
40	22.2584	22.3427	22.3427	22.3342
	(0.0063)	(0.0051)	(0.0052)	(0.0047)
50	23.1579	23.1557	23.1557	23.1557
	(0.0417)	(0.0416)	(0.0416)	(0.0415)
60	20.9806	20.9804	20.9804	20.9804
	(0.0564)	(0.0562)	(0.0562)	(0.0560)
70	21.2733	21.2736	21.2736	21.2736
	(0.044)	(0.043)	(0.043)	(0.041)
80	22.3809	22.3812	22.3812	22.3812
	(0.0069)	(0.0066)	(0.0068)	(0.0065)
90	23.2819	23.282	23.282	23.282
	(0.0483)	(0.0473)	(0.0475)	(0.0472)

Table 7: The average and ARB (in parentheses) of estimates for the parameter γ.

n	MLE	MCMC		
		SE	GE	
			$a=-1$	$a=1$
20	0.1152	0.1111	0.1111	0.0757
	(0.6872)	(0.6268)	(0.6268)	(0.1077)
30	0.1048	0.0945	0.0945	0.0799
	(0.5348)	(0.3834)	(0.3834)	(0.170)
40	0.1001	0.0899	0.0899	0.0793
	(0.4654)	(0.3166)	(0.3166)	(0.1614)
50	0.0857	0.0799	0.0799	0.0735
	(0.2544)	(0.1693)	(0.1693)	(0.0768)
60	0.0973	0.0913	0.0913	0.0834
	(0.4251)	(0.3364)	(0.3364)	(0.2205)
70	0.0899	0.0843	0.0843	0.0805
	(0.3163)	(0.2345)	(0.2345)	(0.1787)
80	0.0799	0.0755	0.0755	0.0721
	(0.1703)	(0.1057)	(0.1057)	(0.055)
90	0.076	0.0733	0.0733	0.0716
	(0.1132)	(0.0737)	(0.0737)	(0.0476)

Table 8: The average and ARB (in parentheses) of estimates for the parameter ϕ.

n	MLE	MCMC		
		SE	GE	
			$a=-1$	$a=1$
20	0.7974	2.9028	2.9028	0.8068
	(2.7124)	(2.5952)	(2.5952)	(0.0008)
30	0.7935	2.1166	2.1166	0.8258
	(1.8172)	(1.6215)	(1.6215)	(0.0228)
40	0.8078	1.9583	1.9583	0.8567
	(1.5213)	(1.4254)	(1.4254)	(0.061)
50	0.8311	1.4569	1.4569	0.8565
	(0.9294)	(0.8044)	(0.8044)	(0.0608)
60	0.8449	1.6674	1.6674	0.8902
	(1.1465)	(1.0651)	(1.0651)	(0.1025)
70	0.8485	1.2471	1.2471	0.9031
	(0.6509)	(0.5446)	(0.5446)	(0.1186)
80	0.8843	1.5335	1.5335	0.9284
	(0.9953)	(0.8993)	(0.8993)	(0.1498)
90	0.8721	0.9483	0.9483	0.9033
	(0.1802)	(0.1745)	(0.1745)	(0.1188)

the TPBXIID has been obtained. It is observed that the Bayes estimators cannot be obtained in explicit forms. So, we have used MCMC technique to compute the Bayes estimates under SE and GE loss functions. We have applied the developed techniques on two real data sets. A simulation study is conducted to examine and compare the performance of the proposed methods. We observe the following from the tables.

1. It can be seen that, from Tables 5, 6, 7 and 8, the performance of the Bayes estimates for the parameters α, θ, γ and ϕ are better than the MLEs in the sense of having smaller ARB
2. It is clear from Tables 5, 6, 7 and 8 that the Bayes estimates under SE loss function is equal the Bayes estimates under GE loss function when $a=-1$.
3. It is clear from Table 6 that the Bayes estimates under SE loss function is equal the Bayes estimates under GE loss function and both are relatively close to the estimates under MLE.
4. It is clear from Table 5 that the Bayes estimates under SE loss function is equal the Bayes estimates under GE loss function and it also has the same ARB when $a=-1$ and both are relatively close to the estimates under MLE.

Table 9: 95% confidence / credible intervals and the corresponding coverage percentages for α and θ.

n	α		α	
	AIL (ACIs)	Coverage	AIL (CRIs)	Coverage
20	35.6631	0.911	2.3409	0.94
30	1.9561	0.874	1.4457	0.922
40	21.6145	0.855	1.1437	0.911
50	1.3362	0.865	0.9758	0.923
60	1.2476	0.853	0.9382	0.895
70	1.1139	0.854	0.8499	0.923
80	1.0198	0.832	0.7657	0.944
90	0.8990	0.857	0.6628	0.925

n	θ		θ	
	AIL (ACIs)	Coverage	AIL (CRIs)	Coverage
20	35.6500	0.802	0.6353	0.932
30	41.7900	0.881	0.0330	0.901
40	40.3230	0.834	0.3094	0.887
50	41.7858	0.872	0.0290	0.934
60	30.2230	0.842	0.0191	0.911
70	28.7684	0.862	0.0179	0.932
80	29.5663	0.821	0.0195	0.942
90	29.4496	0.871	0.0193	0.895

Table 10: 95% confidence / credible intervals and the corresponding coverage percentages for γ and ϕ.

n	γ		γ	
	AIL (ACIs)	Coverage	AIL (CRIs)	Coverage
20	3.5012	0.842	0.2352	0.933
30	0.2779	0.840	0.1189	0.914
40	2.0355	0.821	0.1007	0.858
50	0.1791	0.862	0.0752	0.936
60	0.1742	0.844	0.0838	0.895
70	0.1461	0.857	0.0603	0.887
80	0.1276	0.843	0.0512	0.895
90	0.1134	0.823	0.0448	0.887

n	ϕ		ϕ	
	AIL (ACIs)	Coverage	AIL (CRIs)	Coverage
20	2.6213	0.845	8.9495	0.895
30	1.1471	0.865	6.5543	0.921
40	2.7420	0.874	6.4066	0.925
50	0.9278	0.872	3.4532	0.942
60	0.8600	0.874	4.0255	0.942
70	0.7990	0.835	1.6889	0.929
80	0.7773	0.827	3.7569	0.941
90	0.7232	0.814	0.8250	0.936

5.It is evident that from Tables 9 and 10 the performance of the Bayes estimates for parameters α, θ, γ and ϕ is better than the ML estimates in terms of AIL and CP.

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