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Reduction of Second-Order Ordinary Differential Equations into General Linear Equations via Point Transformations and its Application

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Abstract: We study the linearization of nonlinear second-order ordinary differential equations from the point transformation viewpoint. A new algorithm for finding linearizing point transformation is constructed. The transformation is used to map the underlying class of equations into a linear second-order ordinary differential equation which is in the general form. The general solution of this class of equations is obtained by solving the linearized equation and applying the point transformation. Moreover, we apply the obtained linearization criteria to the interesting problems of nonlinear ordinary differential equations, nonlinear partial differential equations and system of nonlinear ordinary differential equations.

Keywords: Linearization problem, point transformation, nonlinear ordinary differential equation.

1 Introduction

Most differential equation systems in engineering, economics and also mathematics are naturally nonlinear problems; see, examples [1,2,3,4,5]. This system is always difficult to analyze its solution directly. One of the powerful method to solve them is to transform the original systems into the system of linear differential equations which is much more easier to analyze, this method is simply called linearization.

The main tools used to solve the linearization problem are transformations such as point, contact, tangent, generalized Sundman transformations. In this paper, we focus on linearization of nonlinear second-order ordinary differential equations from the point transformation viewpoint.

The linearization problem via a point transformation for a second-order ordinary differential equation was first investigated by Lie [6]. He found criteria that a nonlinear equation can be mapped into a linear equation via a point transformation. Later, Liouville [7] and Tresse [8] attacked the equivalence problem for second-order ordinary differential equations in terms of relative invariants of the equivalence group of point transformations. Moreover, Cartan [9] approached the second-order ordinary differential equations by geometric structure of a certain form.

Since the Lie test is the main tool of the algorithm development in the paper, let us recall it in detail.

Lie obtained the result that any second-order linearizable ordinary differential equation $y''=f(x,y,y')$ which can be mapped into the Laguerre canonical form of linear equation $u''=0$ via a point transformation $t=\varphi(x,y), u=\psi(x,y)$ has to be of the form

$$y''+a(x,y)y'^3+b(x,y)y'^2+c(x,y)y'+d(x,y)=0, \quad (1)$$

where

$$\begin{aligned}
a &=\Delta^{-1}\left(\varphi_y\psi_{yy}-\varphi_{yy}\psi_y\right),\\
b &=\Delta^{-1}\left(\varphi_x\psi_{yy}-\varphi_{yy}\psi_x+2\left(\varphi_y\psi_{xy}-\varphi_{xy}\psi_y\right)\right),\\
c &=\Delta^{-1}\left(\varphi_y\psi_{xx}-\varphi_{xx}\psi_y+2\left(\varphi_x\psi_{xy}-\varphi_{xy}\psi_x\right)\right),\\
d &=\Delta^{-1}\left(\varphi_x\psi_{xx}-\varphi_{xx}\psi_x\right),
\end{aligned}$$

and $\Delta=\varphi_x\psi_y-\varphi_y\psi_x\neq 0$ is a Jacobian of change of variables. Moreover, he also found that a second-order

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ordinary differential equation is linearizable if and only if it has the form (1) with the coefficients satisfying the conditions

$$3a_{xx}-2b_{xy}+c_{yy}-3a_xc+3a_yd+2b_xb-3c_xa \\ -c_yb+6d_ya=0,$$

$$b_{xx}-2c_{xy}+3d_{yy}-6a_xd+b_xc+3b_yd-2c_yc \\ -3d_xa+3d_yb=0.$$

Linearizing transformations are found by solving involutive systems of partial differential equations. These systems depend on the coefficient a.

If $a=0$, then $\varphi=\varphi(x)$ and the involutive system is

$$\begin{aligned} &\psi_{yy}=\psi_y b,\ 2\psi_{xy}=(\varphi_x{}^{-1}\psi_y\varphi_{xx}+\psi_y c),\\ &\psi_{xx}=\varphi_x{}^{-1}\psi_x\varphi_{xx}+\psi_y d,\\ &2\varphi_x\varphi_{xxx}-3\varphi_{xx}^2-\varphi_x{}^2(4(d_y+bd)-(2c_x+c^2))=0. \end{aligned} \tag{2}$$

If $a\neq 0$, then $\varphi_y\neq 0$ and the functions $\varphi(x,y)$ and $\psi(x,y)$ satisfy the involutive system of partial differential equations

$$\begin{aligned} \varphi_y\psi_{yy}&=\varphi_{yy}\psi_y+a\Delta,\\ 2\varphi_y^2\psi_{xy}&=2\varphi_{xy}\varphi_y\psi_y-\varphi_{yy}\Delta-(a\varphi_x-b\varphi_y)\Delta,\\ \varphi_y^2\psi_{xx}&=2\varphi_{xy}\varphi_y\psi_x-\varphi_x\varphi_{yy}\psi_x-\varphi_x^2\psi_x a\\ &\quad+\varphi_x\varphi_y\psi_x b+\varphi_y^2(\psi_y d-\psi_x c),\\ \varphi_y^2\varphi_{xx}&=2\varphi_{xy}\varphi_x\varphi_y-\varphi_x^2\varphi_{yy}-\varphi_x^3a+\varphi_x^2\varphi_y b\\ &\quad-\varphi_x\varphi_y^2c+\varphi_y^3d,\\ 2\varphi_y\varphi_{yyy}&=3\left(\varphi_{yy}^2-2\varphi_{xy}\varphi_y a+2\varphi_x\varphi_{yy}a+\varphi_x^2a^2\right)\\ &\quad-2\varphi_x\varphi_y\left(a_y+ab\right)\\ &\quad+\varphi_y^2\left(2b_y-4a_x+4ac-b^2\right),\\ 6\varphi_y^2\varphi_{xyy}&=3(4\varphi_{xy}\varphi_{yy}\varphi_y-\varphi_x\varphi_{yy}^2+2\varphi_x\varphi_{yy}\varphi_y b\\ &\quad-2\varphi_{xy}\varphi_y^2b)+3\varphi_x^3a^2\\ &\quad+3\varphi_x\varphi_y^2(-2a_x+2ac-b^2)\\ &\quad+2\varphi_y^3(-b_x+2c_y+3ad). \end{aligned} \tag{3}$$

In this paper, we focus on finding the necessary and sufficient conditions which allow the nonlinear second-order ordinary differential equation to be transformed to the general linear equation $u''+\alpha(t)u'+\beta(t)u+\gamma(t)=0$. We have found a new algorithm for finding linearizing point transformation. It is worth mentioning that among the examples one can find such well-known equations as parachute equation, equation for the variable frequency oscillator, equation describing the geodesics on pseudosphere and equation for non-polynomial oscillator. We also make use of a travelling wave solution to obtain the solution of modified generalized Vakhnenko equation. Furthermore, the linearization criteria can be applied to Newtonian systems as well. Note also that after obtaining the linearizing transformation the general solution of the original equation is obtained in quadrature.

2 Formulation of the linearization theorems

2.1 Obtaining necessary condition of linearization

We begin with investigation the necessary conditions for linearization. Recall that a general linear second-order ordinary differential equation has the form

$$u''+\alpha(t)u'+\beta(t)u+\gamma(t)=0. \tag{4}$$

Here we consider the nonlinear second-order ordinary differential equations

$$y''=f\left(x,y,y'\right), \tag{5}$$

which can be transformed to the general linear equation (4) by the point transformation

$$\begin{aligned} t&=\varphi\left(x,y\right),\\ u&=\psi\left(x,y\right). \end{aligned} \tag{6}$$

Notice that if $\varphi_y=0$, a transformation (6) is called a fiber-preserving transformation. So, we arrive at the following theorem.

Theorem 1. *Any second-order ordinary differential equations linearizable by point transformation has to be the form*

$$y''+a(x,y)y'^3+b(x,y)y'^2+c(x,y)y'+d(x,y)=0, \tag{7}$$

where

$$\begin{aligned} a&=\Delta^{-1}(-\varphi_{yy}\psi_y+\varphi_y^3\beta\psi+\varphi_y^3\gamma+\varphi_y^2\psi_y\alpha\\ &\quad+\varphi_y\psi_{yy}),\\ b&=\Delta^{-1}(-2\varphi_{xy}\psi_y+3\varphi_x\varphi_y^2\beta\psi+3\varphi_x\varphi_y^2\gamma\\ &\quad+2\varphi_x\varphi_y\psi_y\alpha+\varphi_x\psi_{yy}-\varphi_{yy}\psi_x+\varphi_y^2\psi_x\alpha\\ &\quad+2\varphi_y\psi_{xy}),\\ c&=\Delta^{-1}(-2\varphi_{xy}\psi_x-\varphi_{xx}\psi_y+3\varphi_x^2\varphi_y\beta\psi+3\varphi_x^2\varphi_y\gamma\\ &\quad+\varphi_x^2\psi_y\alpha+2\varphi_x\varphi_y\psi_x\alpha+2\varphi_x\psi_{xy}+\varphi_y\psi_{xx}),\\ d&=\Delta^{-1}(-\varphi_{xx}\psi_x+\varphi_x^3\beta\psi+\varphi_x^3\gamma+\varphi_x^2\psi_x\alpha\\ &\quad+\varphi_x\psi_{xx}), \end{aligned} \tag{8}$$

and $\Delta=\varphi_x\psi_y-\varphi_y\psi_x\neq 0$ *is a Jacobian of change of variables.*

Proof. The derivatives are changed by the formulae

$$\begin{aligned} u'(t)&=\frac{D_x\psi}{D_x\varphi}\\ &=\frac{\psi_x+y'\psi_y}{\varphi_x+y'\varphi_y}\\ &=g(x,y,y'), \end{aligned}$$

$$u''(t)=\frac{D_xg}{D_x\varphi}=\frac{g_x+y'g_y+y''g_{y'}}{\varphi_x+y'\varphi_y}=P(x,y,y',y''),$$

where

$$g_x=\frac{(\varphi_x+y'\varphi_y)(\psi_{xx}+y'\psi_{xy})-(\psi_x+y'\psi_y)(\varphi_{xx}+y'\varphi_{xy})}{(\varphi_x+y'\varphi_y)^2},$$

$$g_y=\frac{(\varphi_x+y'\varphi_y)(\psi_{xy}+y'\psi_{yy})-(\psi_x+y'\psi_y)(\varphi_{xy}+y'\varphi_{yy})}{(\varphi_x+y'\varphi_y)^2},$$

$$g_{y'}=\frac{(\varphi_x+y'\varphi_y)(\psi_y)-(\psi_x+y'\psi_y)(\varphi_y)}{(\varphi_x+y'\varphi_y)^2},$$

and $D_x=\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}+y''\frac{\partial}{\partial y'}+...$ is a total derivative. Substituting the resulting expression into the linear equation (4) we arrive at the necessary form (7), where a,b,c and d are some functions of x and y as defined in system of equation (8).

2.2 Obtaining sufficient conditions of linearization, linearizing transformation and coefficients of linear equation

We have shown in the previous subsection that every linearizable second-order ordinary differential equation belongs to the class of equation (7). In this subsection, we formulate the main theorems containing sufficient conditions for linearization as well as the methods for constructing the linearizing transformations and the coefficients of linear equation.

To obtain sufficient conditions, one has to solve the compatibility problem. Consider the representations of the coefficients a,b,c and d through the unknown functions φ and ψ in system of equation (8). The compatibility analysis depends on the value of φ_y. A complete study of all cases is given here.

Case $\varphi_y=0$

From relations (8), one defines

$$a=0,\quad \psi_{yy}=\psi_yb,$$

$$\alpha=(\varphi_{xx}\psi_y-2\varphi_x\psi_{xy}+\varphi_x\psi_yc)/(\varphi_x^2\psi_y),\tag{9}$$

$$\beta=(-\varphi_x^2\psi_y\gamma+2\psi_{xy}\psi_x-\psi_{xx}\psi_y-\psi_x\psi_yc+\psi_y^2d)/(\varphi_x^2\psi_y\psi).\tag{10}$$

Since $\alpha_y=0$, differentiating equation (9) with respect to y, one arrives at the condition

$$c_y=2b_x.$$

Since $\beta_y=0$, differentiating equation (10) with respect to y, one arrives at the coefficient

$$\gamma=(-d_y\psi_y^2\psi-2\psi_{xy}^2\psi+2\psi_{xy}\psi_x\psi_y+\psi_{xy}\psi_yc\psi+\psi_{xxy}\psi_y\psi-\psi_{xx}\psi_y^2-\psi_x\psi_y^2c+\psi_y^3d-\psi_y^2bd\psi)/(\varphi_x^2\psi_y^2).\tag{11}$$

Since $\gamma_y=0$, differentiating equation (11) with respect to y, one arrives at the condition

$$b_{xx}=-b_xc+b_yd+d_{yy}+d_yb.$$

Case $\varphi_y\neq 0$

According to our notations, the following equations hold

$$\psi_x=(\varphi_x\psi_y-\Delta)/\varphi_y,\tag{12}$$

and

$$\alpha_x=(\varphi_x\alpha_y)/\varphi_y,\tag{13}$$

$$\beta_x=(\varphi_x\beta_y)/\varphi_y,\tag{14}$$

$$\gamma_x=(\varphi_x\gamma_y)/\varphi_y.\tag{15}$$

So, a system of equation (8) becomes

$$-\varphi_{yy}\psi_y+\varphi_y^3\beta\psi+\varphi_y^3\gamma+\varphi_y^2\psi_y\alpha+\varphi_y\psi_{yy}-a\Delta=0,\tag{16}$$

$$-2\Delta_y\varphi_y-3\varphi_x\varphi_{yy}\psi_y+3\varphi_x\varphi_y^3\beta\psi+3\varphi_x\varphi_y^3\gamma+3\varphi_x\varphi_y^2\psi_y\alpha+3\varphi_x\varphi_y\psi_{yy}+3\varphi_{yy}\Delta-\varphi_y^2\alpha\Delta-\varphi_yb\Delta=0,\tag{17}$$

$$-\Delta_x\varphi_y^2-3\Delta_y\varphi_x\varphi_y+3\varphi_{xy}\varphi_y\Delta-3\varphi_x^2\varphi_{yy}\psi_y+3\varphi_x^2\varphi_y^3\beta\psi+3\varphi_x^2\varphi_y^3\gamma+3\varphi_x^2\varphi_y^2\psi_y\alpha+3\varphi_x^2\varphi_y\psi_{yy}+3\varphi_x\varphi_{yy}\Delta-2\varphi_x\varphi_y^2\alpha\Delta-\varphi_y^2c\Delta=0,\tag{18}$$

$$-\Delta_x\varphi_x\varphi_y^2-\Delta_y\varphi_x^2\varphi_y+\varphi_{xy}\varphi_x\varphi_y\Delta+\varphi_{xx}\varphi_y^2\Delta-\varphi_x^3\varphi_{yy}\psi_y+\varphi_x^3\varphi_y^3\beta\psi+\varphi_x^3\varphi_y^3\gamma+\varphi_x^3\varphi_y^2\psi_y\alpha+\varphi_x^3\varphi_y\psi_{yy}+\varphi_x^2\varphi_{yy}\Delta-\varphi_x^2\varphi_y^2\alpha\Delta-\varphi_y^3\Delta d=0.\tag{19}$$

From equation (16), one obtains the coefficient

$$\gamma=(\varphi_{yy}\psi_y-\varphi_y^3\beta\psi-\varphi_y^2\psi_y\alpha-\varphi_y\psi_{yy}+a\Delta)/\varphi_y^3.\tag{20}$$

And from equation (17), one obtains the coefficient

$$\alpha=(-2\Delta_y\varphi_y+3\varphi_xa\Delta+3\varphi_{yy}\Delta-\varphi_yb\Delta)/(\varphi_y^2\Delta).\tag{21}$$

Substituting equation (20) into equation (15), one obtains the coefficient

$$\beta=(-a_x\varphi_y\Delta^2+a_y\varphi_x\Delta^2\Delta_x\varphi_ya\Delta-\Delta_{yy}\varphi_y\Delta+2\Delta_y^2\varphi_y-2\Delta_y\varphi_xa\Delta-2\Delta_y\varphi_{yy}\Delta+\Delta_y\varphi_yb\Delta+3\varphi_{xy}a\Delta^2+\varphi_{yyy}\Delta^2-\varphi_{yy}b\Delta^2)/(\varphi_y^3\Delta^2).\tag{22}$$

From equation (18), one gets the derivative

$$\Delta_x = (\Delta_y \varphi_x \varphi_y + 3\varphi_{xy}\varphi_y \Delta - 3\varphi_x^2 a\Delta - 3\varphi_x \varphi_{yy}\Delta + 2\varphi_x\varphi_y b\Delta - \varphi_y^2 c\Delta)/\varphi_y^2.$$

From equation (19), one gets the derivative

$$\varphi_{xx} = (2\varphi_{xy}\varphi_x\varphi_y - \varphi_x^3 a - \varphi_x^2\varphi_{yy} + \varphi_x^2\varphi_y b - \varphi_x\varphi_y^2 c + \varphi_y^3 d)/\varphi_y^2.$$

Substituting equation (21) into equation (13), one gets the derivative

$$\begin{aligned}\varphi_{xyy} = (&3a_x\varphi_x\varphi_y^2 + 3a_y\varphi_x^2\varphi_y - b_x\varphi_y^3 - 3b_y\varphi_x\varphi_y^2 \\ &+2c_y\varphi_y^3 + 9\varphi_{xy}\varphi_x\varphi_y a + 6\varphi_{xy}\varphi_{yy}\varphi_y - 3\varphi_{xy}\varphi_y^2 b \\ &-3\varphi_x^3 a^2 - 9\varphi_x^2\varphi_{yy}a + 3\varphi_x^2\varphi_y ab + 3\varphi_x\varphi_{yyy}\varphi_y \\ &-6\varphi_x\varphi_{yy}^2 + 3\varphi_x\varphi_{yy}\varphi_y b - 3\varphi_x\varphi_y^2 ac \\ &+3\varphi_y^3 ad)/(3\varphi_y^2).\end{aligned}$$

Substituting equation (22) into equation (14), one gets the condition

$$a_{xx} = (3a_x c - 3a_y d + 2b_{xy} - 2b_x + 3c_x a - c_{yy} + c_y b - 6d_y a)/3.$$

Comparing the mixed derivative $(\varphi_{xx})_{yy} = (\varphi_{xyy})_x$, one gets the condition

$$b_{xx} = 6a_x d - b_x c - b_y d + 2c_{xy} + 2c_y c + 3d_x a - 3d_{yy} - 3d_y b.$$

All obtained results can formulate the main theorems containing sufficient conditions for linearization, the methods for constructing the linearizing transformations and the coefficients of linear equation as follows.

Theorem 2.*Equation (7) is linearizable by point transformation (6) with the function $\varphi = \varphi(x)$ if and only if its coefficients satisfy the conditions*

$$a = 0,\ \ c_y = 2b_x,\ \ b_{xx} = -b_x c + b_y d + d_{yy} + d_y b. \quad (23)$$

Provided that the conditions (23) are satisfied, the linearizing transformation (6) is obtained by solving the compatible system of equations

$$\varphi_y = 0, \psi_{yy} = \psi_y b, \quad (24)$$

and the coefficients α, β and γ of the resulting linear equation (4) are given by equations

$$\begin{aligned}\alpha &= (\varphi_{xx}\psi_y - 2\varphi_x\psi_{xy} + \varphi_x\psi_y c)/(\varphi_x^2\psi_y), \\ \beta &= (-\varphi_x^2\psi_y\gamma + 2\psi_{xy}\psi_x - \psi_{xx}\psi_y - \psi_x\psi_y c \\ &\quad + \psi_y^2 d)/(\varphi_x^2\psi_y\psi), \\ \gamma &= (-d_y\psi_y^2\psi - 2\psi_{xy}^2\psi + 2\psi_{xy}\psi_x\psi_y + \psi_{xy}\psi_y c\psi \\ &\quad + \psi_{xxy}\psi_y\psi - \psi_{xx}\psi_y^2 - \psi_x\psi_y^2 c + \psi_y^3 d \\ &\quad - \psi_y^2 bd\psi)/(\varphi_x^2\psi_y^2).\end{aligned}$$

Theorem 3.*Equation (7) is linearizable by point transformation (6) if and only if its coefficients satisfy the conditions*

$$\begin{aligned}a_{xx} &= (3a_x c - 3a_y d + 2b_{xy} - 2b_x + 3c_x a - c_{yy} \\ &\quad + c_y b - 6d_y a)/3, \\ b_{xx} &= 6a_x d - b_x c - b_y d + 2c_{xy} + 2c_y c + 3d_x a \\ &\quad - 3d_{yy} - 3d_y b.\end{aligned} \quad (25)$$

Provided that the conditions (25) are satisfied, the linearizing transformation (6) is obtained by solving the compatible system of equations

$$\begin{aligned}\psi_x &= (\varphi_x\psi_y - \Delta)/\varphi_y, \\ \Delta_x &= \Delta_y\varphi_x\varphi_y + 3\varphi_{xy}\varphi_y\Delta - 3\varphi_x^2 a\Delta - 3\varphi_x\varphi_{yy}\Delta \\ &\quad + 2\varphi_x\varphi_y b\Delta - \varphi_y^2 c\Delta)/\varphi_y^2, \\ \varphi_{xx} &= (2\varphi_{xy}\varphi_x\varphi_y - \varphi_x^3 a - \varphi_x^2\varphi_{yy} + \varphi_x^2\varphi_y b \\ &\quad - \varphi_x\varphi_y^2 c + \varphi_y^3 d)/\varphi_y^2, \\ \varphi_{xyy} &= (3a_x\varphi_x\varphi_y^2 + 3a_y\varphi_x^2\varphi_y - b_x\varphi_y^3 - 3b_y\varphi_x\varphi_y^2 \\ &\quad + 2c_y\varphi_y^3 + 9\varphi_{xy}\varphi_x\varphi_y a + 6\varphi_{xy}\varphi_{yy}\varphi_y \\ &\quad - 3\varphi_{xy}\varphi_y^2 b - 3\varphi_x^3 a^2 - 9\varphi_x^2\varphi_{yy}a + 3\varphi_x^2\varphi_y ab \\ &\quad + 3\varphi_x\varphi_{yyy}\varphi_y - 6\varphi_x\varphi_{yy}^2 + 3\varphi_x\varphi_{yy}\varphi_y b \\ &\quad - 3\varphi_x\varphi_y^2 ac + 3\varphi_y^3 ad)/(3\varphi_y^2),\end{aligned} \quad (26)$$

and the coefficients α, β and γ of the resulting linear equation (4) are given by equations

$$\begin{aligned}\alpha &= (-2\Delta_y\varphi_y + 3\varphi_x a\Delta + 3\varphi_{yy}\Delta - \varphi_y b\Delta)/(\varphi_y^2\Delta), \\ \beta &= (-a_x\varphi_y\Delta^2 + a_y\varphi_x\Delta^2\Delta_x\varphi_y a\Delta - \Delta_{yy}\varphi_y\Delta + 2\Delta_y^2\varphi_y \\ &\quad - 2\Delta_y\varphi_x a\Delta - 2\Delta_y\varphi_{yy}\Delta + \Delta_y\varphi_y b\Delta + 3\varphi_{xy}a\Delta^2 \\ &\quad + \varphi_{yyy}\Delta^2 - \varphi_{yy}b\Delta^2)/(\varphi_y^3\Delta^2), \\ \gamma &= (\varphi_{yy}\psi_y - \varphi_y^3\beta\psi - \varphi_y^2\psi_y\alpha - \varphi_y\psi_{yy} + a\Delta)/\varphi_y^3.\end{aligned}$$

3 Some applications

In this section we focus on finding some applications which satisfy Theorem 1, Theorem 2 and Theorem 3. The obtained results are as follows.

3.1 Parachute equation

The idea of this application is based on a model for movement of a parachutist during the air using Newton's II law is $\sum F = ma$. The motion of skydiver when the coefficient of air resistance changes between free-fall and the final steady state descent with the parachute is dully deployed.

Consider the parachute equation [10] in the form

$$y'' + ky'^2 - g = 0, \quad (27)$$

with initial conditions $y(0) = 0$ and $y'(0) = 0$.
Here $k = \frac{\pi\rho C_d D^2}{8m}$ where

m is the mass of the body and parachute,
ρ is the density of the fluid in which the body moves,
C_d is the drag coefficient for the parachute (1.5 for parabolic profile and 0.75 for flat),
D is the effective diameter of the parachute.
The solution of equation (27) is

$$y=\frac{1}{k}(\log(\frac{e^{\sqrt{gk}x}+1}{2})-\sqrt{gk}x).$$

3.1.1 Applying the obtained theorems to the problem

By using the obtained theorems, we get the results as follow. Equation (27) is an equation of the form (7) in Theorem 1 with the coefficients

$$a=0,\ b=k,\ c=0,\ d=-g.$$

One can check that these coefficients obey the conditions in the Theorem 2. Hence, an equation (27) is linearizable by point transformation. The linearizing transformation is found by solving the following equations

$$\varphi_y=0,\ \psi_{yy}=k\psi_y. \tag{28}$$

One can find the particular solution for equations in (28) as

$$\varphi=x,\psi=\frac{e^{ky}}{k}.$$

Therefore, the linearizing point transformation of (27) takes the form

$$t=x,\ u=\frac{e^{ky}}{k}. \tag{29}$$

So, the coefficients of the resulting linear equation (4) are

$$\alpha=0,\ \beta=-gk,\ \gamma=0.$$

Hence, the nonlinear equation (27) can be mapped by transformation (29) into the linear equation

$$u''-gku=0. \tag{30}$$

The general solution of equation (30) is

$$u=C_1e^{\sqrt{gk}t}+C_2e^{-\sqrt{gk}t}, \tag{31}$$

where C_1 and C_2 are arbitrary constants. Substituting equation (29) into equation (31), we obtain the following general solution of (27) given by

$$e^{ky}=k(C_1e^{\sqrt{gk}x}+C_2e^{-\sqrt{gk}x}). \tag{32}$$

Substituting the initial condition $y(0)=0$ and $y'(0)=0$ into equation (32), one obtains the particular solution of equation (27) as

$$y=\frac{1}{2}(\log(\frac{e^{\sqrt{gk}x}+1}{2})-\sqrt{gk}x).$$

3.2 Equation for the variable frequency oscillator

A variable frequency oscillator (VFO) in electronics is an oscillator whose frequency can be tuned (i.e. varied) over some range. It is a necessary component in any tunable radio receiver or transmitter that works by the superheterodyne principle and controls the frequency to which the apparatus is tuned.

In 2013, Mastafa, Al-Dueik and Mara'beh [11] considered the ordinary differential for the variable frequency oscillator

$$y''+yy'^2=0. \tag{33}$$

They showed that this equation can be linearizable by generalized Sundman transformation

$$u(t)=\psi(x,y),\ dt=\phi(x,y)dx,\ \psi_y\neq 0.$$

By using their method, the solution of equation (33) is

$$\mathrm{erfi}(\frac{y}{\sqrt{2}})=C_1x+C_2,$$

where $\mathrm{erfi}(y)=\frac{2}{\sqrt{\Pi}}\int_0^y e^{t^2}dt$ is an imaginary error function and C_1, C_2 are arbitrary constants.

3.2.1 Applying the obtained theorems to the problem

By using the obtained theorems, we get the results as follow. Equation (33) is an equation of the form (7) in Theorem 1 with the coefficients

$$a=0,\ b=y,\ c=0,\ d=0.$$

One can check that these coefficients obey the conditions in the Theorem 2. Hence, an equation (33) is linearizable by point transformation. The linearizing transformation is found by solving the following equations

$$\varphi_y=0,\ \psi_{yy}=\psi_y y. \tag{34}$$

One can find the particular solution for equations (34) as

$$\varphi=x,\ \psi=\int e^{\frac{y^2}{2}}dy.$$

Therefore, the linearizing point transformation of (33) takes the form

$$t=x,\ u=\int e^{\frac{y^2}{2}}dy. \tag{35}$$

So, the coefficients of the resulting linear equation (4) are

$$\alpha=0,\ \beta=0,\ \gamma=0.$$

Hence, the nonlinear equation (33) can be mapped by transformation (35) into the linear equation

$$u''=0.$$

So that

$$u = C_1 t + C_2, \tag{36}$$

where C_1 and C_2 are arbitrary constants. Substituting equation (35) into equation (36), we get the general solution of (33) as

$$\int e^{\frac{y^2}{2}} dy = C_1 x + C_2,$$

or

$$\mathrm{erfi}\left(\frac{y}{\sqrt{2}}\right) = C_1 x + C_2.$$

3.3 Equation describe the geodesics on pseudosphere

In 2013, Mastafa, Al-Dueik and Mara'beh [11] considered the ordinary differential for describes the geodesics on pseudosphere.

$$y'' + 2y'^2 - e^{2y} = 0. \tag{37}$$

They showed that this equation can be linearizable by generalized Sundman transformation

$$u(t) = \psi(x,y),\ dt = \phi(x,y,y')dx,\ \psi_y \neq 0.$$

By using their method, the solution of equation (37) is

$$e^{-2y} + x^2 = C_1 x + C_2,$$

where C_1 and C_2 are arbitrary constants.

3.3.1 Applying the obtained theorems to the problem

By using the obtained theorems, we get the results as follow. Equation (37) is an equation of the form (7) in Theorem 1 with the coefficients

$$a = 0,\ b = -2,\ c = 0,\ d = -e^{2y}.$$

One can check that these coefficients obey the conditions in the Theorem 2. Hence, an equation (37) is linearizable by point transformation. The linearizing transformation is found by solving the following equations

$$\varphi_y = 0,\ \psi_{yy} = -2\psi_y. \tag{38}$$

One can find the particular solution for equations (38) as

$$\varphi = x, \psi = -\frac{1}{2}e^{-2y}.$$

Therefore, one obtains the linearizing transformation

$$t = x,\ u = -\frac{1}{2}e^{-2y}. \tag{39}$$

So, the coefficients of the resulting linear equation (4) are

$$\alpha = 0,\ \beta = 0,\ \gamma = -1.$$

Hence, the nonlinear equation (37) can be mapped by transformation (39) into the linear equation

$$u'' - 1 = 0. \tag{40}$$

The general solution of equation (40) is

$$u = \frac{t^2}{2} + C_1 t + C_2, \tag{41}$$

where C_1 and C_2 are arbitrary constant. Substituting equation (39) into equation (41), we get the general solution of (37) as

$$e^{-2y} + x^2 = C_1 x + C_2.$$

3.4 The one-dimensional non-polynomial oscillator

In the note [12], Mathew and Lakshmanan presented a remarkable nonlinear system that all its bounded periodic motions are simple harmonic. The system is a particle obeying the highly nonlinear equation of motion

$$(1+\lambda y^2)y'' + (\alpha - \lambda y'^2)y = 0, \tag{42}$$

where λ and α are arbitrary parameters.

3.4.1 Applying the obtained theorems to the problem

By using the obtained theorems, we get the results as follow. Equation (42) is an equation of the form (7) in Theorem 1 with the coefficients

$$a = 0,\ b = -\frac{\lambda y}{\lambda y^2 + 1},\ c = 0,\ d = \frac{\alpha y}{\lambda y^2 + 1}.$$

One can check that the first and second conditions of equation (23) in Theorem 2 are satisfied. Now, the last condition of equation (23) is satisfied when the following condition holds, that is,

$$(\lambda y^2 - 2)\alpha\lambda y = 0.$$

Two cases arise.

Case 1: $\alpha = 0$
In this case, the equation (42) takes the form

$$(1+\lambda y^2)y'' - \lambda y'^2 y = 0. \tag{43}$$

The linearizing transformation is found by solving the following equations

$$\varphi_y = 0,\ \psi_{yy} = -\frac{\psi_y \lambda y}{\lambda y^2 + 1}. \tag{44}$$

One can find the particular solution for equation (44) as

$$\varphi = x,\ \psi = \frac{1}{\sqrt{\lambda}}(\ln|\sqrt{\lambda y^2+1}+\sqrt{\lambda}y).$$

Therefore, one obtains the linearizing transformation

$$t = x,\ u = \frac{1}{\sqrt{\lambda}}(\ln|\sqrt{\lambda y^2+1}+\sqrt{\lambda}y). \tag{45}$$

So, the coefficients of the resulting linear equation (4) are

$$\tilde{\alpha} = 0,\ \tilde{\beta} = 0,\ \tilde{\gamma} = 0.$$

Hence, the nonlinear equation (43) can be mapped by transformation (45) into the linear equation

$$u'' = 0. \tag{46}$$

The general solution of (46) is

$$u = C_1 t + C_2, \tag{47}$$

where C_1 and C_2 are arbitrary constants. Substituting equation (45) into equation (47), we get the general solution of equation (43) as

$$\frac{1}{\sqrt{\lambda}}(\ln|\sqrt{\lambda y^2+1}+\sqrt{\lambda}y) = C_1 x + C_2.$$

Case 2: $\lambda = \frac{2}{y^2}$

In this case, the equation (42) takes the form

$$3y''y - 2y'^2 + \alpha y^2 = 0. \tag{48}$$

The linearizing transformation is found by solving the following equations

$$\varphi_y = 0,\ \psi_{yy} = -\frac{2\psi_y}{3y}. \tag{49}$$

One can find the particular solution for equation (49) as

$$\varphi = x,\ \psi = y^{\frac{1}{3}}.$$

Therefore, one obtains the linearizing transformation

$$t = x,\ u = y^{\frac{1}{3}}. \tag{50}$$

So, the coefficients of the resulting linear equation (4) are

$$\tilde{\alpha} = 0,\ \tilde{\beta} = \frac{\alpha}{9},\ \tilde{\gamma} = 0.$$

Hence, the nonlinear equation (48) can be mapped by transformation (50) into the linear equation

$$u'' + \frac{\alpha}{9}u = 0. \tag{51}$$

The general solution of equation (51) is

$$u = C_1 \cos\frac{\sqrt{\alpha}}{3}t + C_2 \sin\frac{\sqrt{\alpha}}{3}t, \tag{52}$$

where C_1 and C_2 are arbitrary constants. Substituting equation (50) into equation (52), we get the general solution of equation (48) as

$$y = (C_1 \cos\frac{\sqrt{\alpha}}{3}x + C_2 \sin\frac{\sqrt{\alpha}}{3}x)^3.$$

3.5 Modified generalized Vakhnenko equation

In 2009, Ma, Li and Wang [13] considered a modified generalized Vakhnenko equation (mGVE),

$$\frac{\partial}{\partial x}(\mathfrak{D}^2 u + \frac{1}{2}pu^2 + \beta u) + q\mathfrak{D}u = 0,\ \mathfrak{D} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}, \tag{53}$$

where p, q, β are arbitrary non-zero constants.

To construct the exact solutions for mGVE is all important. For examples, when $p = \beta = 0$ and $q = 1$, equation (53) is reduced to well-known Vakhnenko equation(VE), which governs the nonlinear propagation of high-frequency wave in a relaxing medium [14]-[16] and the VE has soliton solutions as in ref [16]. When $p = q = 1$ and β an arbitrary non-zero constant, equation (53) is reduced as the generalized VE (GVE), in [17] it was shown that GVE has N-soliton solution. When $p = 2q$ and β is an arbitrary non-zero constant, equation (53) has a loop-like, hump-like and cusp-like soliton solutions [18,19,20]. In [21], it was shown that equation (53) has travelling wave solution and single-soliton solution.

3.5.1 Applying the obtained theorems to the problem

Consider a modified generalized Vakhnenko equation (53), we can rewrite it in the form

$$\begin{aligned} &2u_t u_{tx} + 2[uu_x u_{tx} + u_t(uu_{xx} + u_x^2)] + 2u^2 u_{xx} \\ &+2u(u_x)^3 + puu_x + \beta u_x + q(u_t + uu_x) = 0. \end{aligned} \tag{54}$$

Of particular interest among solutions of equation (54) are travelling wave solutions:

$$u(x,t) = H(x - Dt),$$

where D is a constant phase velocity and the argument $x - Dt$ is a phase of the wave. Substituting the representation of a solution into equation (54), one finds

$$\begin{aligned} &2D^2 H'H'' - 2DH'(2HH'' + H'^2) + 2H^2 H'H'' \\ &+2HH'^3 + pHH' + \beta H' + q(-DH' + HH') = 0. \end{aligned} \tag{55}$$

By using the obtained theorems, we get the results as follow. Equation (55) is an equation of the form (7) in Theorem 1 with the coefficients

$$a = 0,\ b = -\frac{1}{D-H},\ c = 0,\ d = \frac{\beta - Dq + pH + qH}{2(D^2 - 2DH + H^2)}.$$

From Theorem 2, equation (55) is linearizable if and only if

$$\beta = -Dp.$$

The linearizing transformation is found by solving the following equations

$$\varphi_H = 0,\ \psi_{HH} = -\frac{\psi_H}{D-H}. \tag{56}$$

One can find the particular solution for equation (56) as

$$\varphi = x - Dt,\ \psi = 2DH - H^2.$$

Therefore, one obtains the linearizing transformation

$$\tilde{t} = x - Dt,\ \tilde{u} = 2DH - H^2. \tag{57}$$

So, the coefficients of the resulting linear equation (4) are

$$\tilde{\alpha} = 0,\ \tilde{\beta} = 0,\ \tilde{\gamma} = -(p+q).$$

Hence, the nonlinear equation (55) can be mapped by transformation (57) into the linear equation

$$\tilde{u}'' - (p+q) = 0. \tag{58}$$

The general solution of equation (58) is

$$\tilde{u} = (p+q)\frac{\tilde{t}^2}{2} + C_1\tilde{t} + C_2, \tag{59}$$

where C_1 and C_2 are arbitrary constants. Substituting equation (57) into equation (59), we get the general solution of ordinary differential (55) as

$$2DH - H^2 = (p+q)\frac{(x-Dt)^2}{2} + C_1(x-Dt) + C_2.$$

So, the general solution of partial differential equation is

$$2Du - u^2 = (p+q)\frac{(x-Dt)^2}{2} + C_1(x-Dt) + C_2.$$

3.6 Newtonian System

In 2001, Soh and Mahomed [22] considered the Newtonian system:

$$x'' = x'^2 + y'^2, \tag{60}$$

$$y'' = 2x'y'. \tag{61}$$

Note that $x' = \frac{dx}{dz}$ and $y' = \frac{dy}{dz}$. By using the Lie algorithm, they obtained all the symmetries of equations (60) and (61). They showed that these equations can be transformed to the system of equation $X'' = 0,\ Y'' = 0$.

3.6.1 Applying the obtained theorems to the problem

Consider nonlinear second-order ordinary differential system (60) and (61), combining equations (60) and (61), one gets

$$x'' + y'' = x'^2 + 2x'y' + y'^2. \tag{62}$$

Let $\omega = x + y$, so that equation (62) becomes

$$\omega'' - \omega'^2 = 0. \tag{63}$$

By using the obtained theorems, we get the results as follow. Equation (63) is an equation of the form (7) in Theorem 1 with the coefficients

$$a = 0,\ b = -1,\ c = 0,\ d = 0.$$

One can check that these coefficients obey the conditions in the Theorem 2. Hence, an equation (63) is linearizable by point transformation. The linearizing transformation is found by solving the following equations

$$\varphi_\omega = 0,\ \psi_{\omega\omega} = -\psi_\omega. \tag{64}$$

One can find the particular solution for equation (64) as

$$\varphi = z, \psi = e^{-\omega}.$$

Therefore, one obtains the linearizing transformation

$$t = z,\ u = e^{-\omega}. \tag{65}$$

So, the coefficients of the resulting linear equation (4) are

$$\alpha = 0,\ \beta = 0,\ \gamma = 0.$$

Hence, the nonlinear equation (63) can be mapped by transformation (65) into the linear equation

$$u'' = 0. \tag{66}$$

The general solution of equation (66) is

$$u = C_1 t + C_2, \tag{67}$$

where C_1 and C_2 are arbitrary constants. Substituting equation (65) into equation (67), we get the particular solution

$$\omega = -\ln(z).$$

Since $\omega = x + y$, we have

$$x = -\ln(z) - y \tag{68}$$

so that

$$x' = -\frac{1}{z} - y'. \tag{69}$$

Substituting equation (69) into equation (61), we have

$$y'' + 2y'^2 + \frac{2}{z}y' = 0. \tag{70}$$

By using the obtained theorems again, we get the results as follow. Equation (70) is an equation of the form (7) in Theorem 1 with the coefficients

$$a = 0,\ b = 2,\ c = \frac{2}{z},\ d = 0.$$

One can check that these coefficients obey the conditions in the Theorem 2. Hence, an equation (70) is linearizable by point transformation. The linearizing transformation is found by solving the following equations

$$\varphi_y = 0, \ \psi_{yy} = 2\psi_y. \tag{71}$$

One can solve the particular solution for equation in (71) as

$$\varphi = z, \ \psi = ze^{2y}.$$

Therefore, one obtains the linearizing transformation

$$t = z, \ u = ze^{2y}. \tag{72}$$

So, the coefficients of the resulting linear equation (4) are

$$\alpha = 0, \ \beta = 0, \ \gamma = 0.$$

Hence, the nonlinear equation (70) can be mapped by transformation (72) into the linear equation

$$u'' = 0. \tag{73}$$

The general solution of equation (73) is

$$u = C_3 t + C_4, \tag{74}$$

where C_3 and C_4 are arbitrary constants. Substituting equation (72) into equation (74), we get the solution

$$y = \frac{1}{2}\ln(\frac{C_3 z + C_4}{z}). \tag{75}$$

Substituting equation (75) into equation (68), we have

$$x = -\ln z - \frac{1}{2}\ln(\frac{C_3 z + C_4}{z}).$$

So, we get the solution of Newtonian system (60) and (61) as

$$x = -\ln(z) - \tfrac{1}{2}\ln(\tfrac{C_3 z + C_4}{z}),$$

$$y = \frac{1}{2}\ln(\frac{C_3 z + C_4}{z}).$$

3.7 Example for case $\varphi_y \neq 0$

Consider the nonlinear second-order differential equation [23]

$$y'' - 2y'^3 y = 0. \tag{76}$$

By setting $v = y'$. The formulas above lead to

$$v\frac{dv}{dy} - 2v^3 y = 0.$$

This a first-order separable differential equation. Its resolution gives

$$v = -\frac{1}{y^2 + C},$$

where C is an arbitrary constant. Since $y' = \frac{dy}{dx} = v$, we get

$$\frac{dy}{dx} = -\frac{1}{y^2 + C}.$$

Since this is a separable first-order differential equation, one gets, after resolution,

$$y^3 + Cy = -x + C^*,$$

where C^* is an arbitrary constant.

3.7.1 Applying the obtained theorems to the problem

By using the obtained theorems, we get the results as follow. Equation (76) is an equation of the form (7) in Theorem 1 with the coefficients

$$a = -2y, \ b = 0, \ c = 0, \ d = 0.$$

One can check that these coefficients obey the conditions in the Theorem 3. Hence, an equation (76) is linearizable by point transformation. The linearizing transformation is found by solving the following equations

$$\psi_x = (\varphi_x \psi_y - \Delta)/\varphi_y, \tag{77}$$

$$\varphi_{xx} = (\varphi_x(2\varphi_{xy}\varphi_y + 2\varphi_x^2 y - \varphi_x\varphi_{yy}))/\varphi_y^2, \tag{78}$$

$$\begin{aligned}\varphi_{xyy} = (&-6\varphi_{xy}\varphi_x\varphi_y y + 2\varphi_{xy}\varphi_{yy}\varphi_y - 4\varphi_x^3 y^2 \\ &+6\varphi_x^2\varphi_{yy}y - 2\varphi_x^2\varphi_y + \varphi_x\varphi_{yyy}\varphi_y \\ &-2\varphi_x\varphi_{yy}^2)/\varphi_y^2,\end{aligned} \tag{79}$$

$$\Delta_x = (3\Delta(\varphi_{xy}\varphi_y + 2\varphi_x^2 y - \varphi_x\varphi_{yy}))/\varphi_y^2. \tag{80}$$

One can find the particular solution for the equations (77)-(80) as

$$\varphi = y, \ \psi = -x.$$

Therefore, one obtains the linearizing transformation

$$t = y, \ u = -x. \tag{81}$$

So, the coefficients of the resulting linear equation (4) are

$$\alpha = 0, \ \beta = 0, \ \gamma = -2t.$$

Hence, the nonlinear equation (76) can be mapped by transformation (81) into the linear equation.

$$u'' - 2t = 0. \tag{82}$$

Therefore, the general solution of equation (82) is

$$u = C_1 + C_2 t + \frac{t^3}{3}, \tag{83}$$

where C_1 and C_2 are arbitrary constants. Substituting equation (81) into equation (83), we get the solution

$$-x = C_1 + C_2 y + \frac{y^3}{3}.$$

4 Conclusion

This paper is devoted to find the conditions which allow the second-order ordinary differential equation to be transformed to the general linear equation. Necessary conditions which guarantee that the second-order ordinary differential equation can be linearized are found in Theorem 1. Theorem 2 and Theorem 3 are sufficient conditions for the linearization problem, they are selected by the way of finding a linearizing transformation. We have found that a new algorithm for finding linearizing point transformation (24) and (26) which can be solved easier than (2) and (3). Finally, some applications are provided to demonstrate our procedure.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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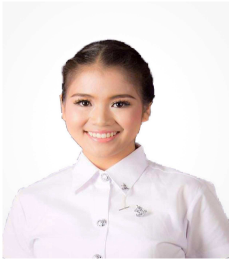

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