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Integral Inequalities for The Strongly-Generalized Nonconvex Functions

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Abstract: In this paper, we establish some new inequalities of Hermite–Hadamard type for the strongly-generalized nonconvex function by using the generalized fractional integral operator. Some new results as special cases are provided as well. At the end, some applications to special mean are obtained.

Keywords: Hermite–Hadamard inequalities, strongly convex function, generalized fractional integral operator.

1 Introduction

Definition 1.[1] *A function $\mathscr{F}: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be convex on I if*

$$\mathscr{F}(\iota\ell_1 + (1-\iota)\ell_2) \leq \iota\mathscr{F}(\ell_1) + (1-\iota)\mathscr{F}(\ell_2)$$

holds for every $\ell_1, \ell_2 \in I$ and $\iota \in [0,1]$.

Definition 2.[2] *A function $\mathscr{F}: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is called strongly convex with modulus $\theta \in \mathbb{R}^+$, if*

$$\mathscr{F}(\iota\ell_1 + (1-\iota)\ell_2) \leq \iota\mathscr{F}(\ell_1) + (1-\iota)\mathscr{F}(\ell_2) - \theta\iota(1-\iota)(\ell_2-\ell_1)^2$$

holds for every $\ell_1, \ell_2 \in I$ and $\iota \in [0,1]$.

Strongly convex functions have been introduced by Polyak, see [2] and references therein. Since strong convexity is a strengthening feature of the notion of convexity, some properties of strongly convex functions are just stronger versions of known properties of convex functions. Strongly convex functions have been used for proving the convergence of a gradient-type algorithm for minimizing a function. They play an important role in optimization theory and mathematical economics.
The most significant inequality is the Hermite–Hadamard integral inequality, see [3]. This double inequality is expressed as:

$$\mathscr{F}\left(\frac{\ell_1+\ell_2}{2}\right) \leq \frac{1}{\ell_2-\ell_1}\int_{\ell_1}^{\ell_2} \mathscr{F}(\tau)d\tau \leq \frac{\mathscr{F}(\ell_1)+\mathscr{F}(\ell_2)}{2}. \quad (1)$$

The double inequality (1) has became a very important foundation within the field of mathematical analysis and optimization, several applications of these inequalities have been found in number of settings. Furthermore, several inequalities of special means can be discovered for the specific options of the function $\mathscr{F}$. Due to large applications of double inequality (1), literature is growing and giving its some new proofs, augmentations, improvements and generalizations, see [4]-[10],[11,12,13,14] and the references therein.

In [5], Raina R. K. introduced a class of functions defined formally by

$$\mathscr{F}^{\sigma}_{\rho,\lambda}(x) = \mathscr{F}^{\sigma(0),\sigma(1),\dots}_{\rho,\lambda}(x) = \sum_{k=0}^{+\infty} \frac{\sigma(k)}{\Gamma(\rho k+\lambda)}x^k, \quad (2)$$

where $\rho, \lambda > 0, |x| < R$ and $\sigma = (\sigma(0), \dots, \sigma(k), \dots)$ is a bounded sequence of positive real numbers. Note that, if we take in (2) $\rho = 1, \lambda = 0$ and $\sigma(k) = \frac{((\alpha)_k(\beta)_k)}{(\gamma)_k}$ for $k = 0,1,2,\dots$, where α, β and γ are parameters which can take arbitrary real or complex values (provided that $\gamma \neq 0,-1,-2,\dots$), and the symbol $(a)_k$ denotes the quantity

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\dots(a+k-1), \quad k = 0,1,2,\dots,$$

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and restrict its domain to $|x| \leq 1$ (with $x \in \mathbb{C}$), then we have the classical hypergeometric function, that is

$$\mathscr{F}^{\sigma}_{\rho,\lambda}(x) = F(\alpha,\beta;\gamma;x) = \sum_{k=0}^{+\infty} \frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k} x^k.$$

Also, if $\sigma = (1,1,\dots)$ with $\rho = \alpha, (Re(\alpha) > 0), \lambda = 1$ and restricting its domain to $x \in \mathbb{C}$ in (2) then we have the classical Mittag-Leffler function

$$E_{\alpha}(x) = \sum_{k=0}^{+\infty} \frac{1}{\Gamma(1+\alpha k)} x^k.$$

Now we are able to define a new class of function involving $\mathscr{F}^{\sigma}_{\rho,\lambda}(\cdot)$ (Raina function).

Definition 3. *If a function $\mathscr{G} : I \to \mathbb{R}$ satisfies the following inequality*

$$\mathscr{G}(\ell_1 + \iota \mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \leq (1-\iota)\mathscr{G}(\ell_1) + \iota \mathscr{G}(\ell_2) - \theta\iota(1-\iota)(\ell_2-\ell_1)^2,$$

for all $\iota \in [0,1]$ and $\ell_1, \ell_2 \in I$, where $\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1) > 0$, then $\mathscr{G}$ is called strongly-generalized nonconvex with modulus value $\theta \geq 0$. Taking $\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1) = \ell_2 - \ell_1 > 0$ in our definition, then we obtain definition 2.

Definition 4. *[13] The left and right side generalized fractional integrals for a function $\mathscr{G}$ are defined as*

$$_{\ell_1^+}I_{\phi}\mathscr{G}(\tau) = \int_{\ell_1}^{\tau} \frac{\phi(\tau-\iota)}{\tau-\iota}\mathscr{G}(\iota)d\iota, \ \tau > \ell_1,$$

$$_{\ell_2^-}I_{\phi}\mathscr{G}(\tau) = \int_{\tau}^{\ell_2} \frac{\phi(\iota-\tau)}{\iota-\tau}\mathscr{G}(\iota)d\iota, \ \tau < \ell_2.$$

Our main goal during this paper is to prove in Section 2 inequalities for the strongly generalized nonconvex functions by using the generalized fractional integral operator. We prove several corollaries as special cases of our main results. In Section 3, some applications to special mean are obtained. In Section 4, a briefly conclusion is given as well.

2 Main Results

Throughout this section the following notation is used:

$$O = \left[\ell_1, \ell_1 + \mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right] \quad \text{where} \quad \mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1) > 0.$$

Theorem 1. *Let $\mathscr{G} : O \to (0,+\infty)$ be strongly-generalized nonconvex function with modulus value $\theta \geq 0$. Then the following inequality for the generalized fractional integral holds:*

$$_{\ell_1^+}I_{\phi}\mathscr{G}(\ell_1 + \mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + {}_{(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}I_{\phi}\mathscr{G}(\ell_1) \leq (\mathscr{G}(\ell_1)+\mathscr{G}(\ell_2))P_1 - 2\theta(\ell_2-\ell_1)^2 P_2, \tag{3}$$

where

$$P_1 = \int_0^1 \frac{\phi(\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\iota)}{\iota} d\iota, \quad P_2 = \int_0^1 (1-\iota)\phi(\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\iota) d\iota.$$

Proof. Since $\mathscr{G}$ is strongly-generalized nonconvex function, then we have

$$\mathscr{G}(\ell_1 + \iota \mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \leq (1-\iota)\mathscr{G}(\ell_1) + \iota \mathscr{G}(\ell_2) - \theta\iota(1-\iota)(\ell_2-\ell_1)^2 \tag{4}$$

and

$$\mathscr{G}(\ell_1 + (1-\iota)\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \leq \iota\mathscr{G}(\ell_1) + (1-\iota)\mathscr{G}(\ell_2) - \theta\iota(1-\iota)(\ell_2-\ell_1)^2. \tag{5}$$

Adding (4) and (5), we obtain

$$\begin{aligned} &\mathscr{G}(\ell_1 + \iota \mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + \mathscr{G}(\ell_1 + (1-\iota)\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ &\leq \left((1-\iota)\mathscr{G}(\ell_1) + \iota\mathscr{G}(\ell_2) - \theta\iota(1-\iota)(\ell_2-\ell_1)^2\right) \\ &\quad + \left(\iota\mathscr{G}(\ell_1) + (1-\iota)\mathscr{G}(\ell_2) - \theta\iota(1-\iota)(\ell_2-\ell_1)^2\right) \\ &= (\mathscr{G}(\ell_1)+\mathscr{G}(\ell_2)) - 2\theta\iota(1-\iota)(\ell_2-\ell_1)^2. \end{aligned} \tag{6}$$

Multiplying (6) with $\frac{\phi(\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\iota)}{\iota}$ on both sides and integrating the resultant inequality with respect to ι over $[0,1]$, we have required result (3).

Corollary 2 *Let $\mathscr{G} : O \to (0,+\infty)$ be strongly generalized nonconvex function with modulus value $\theta \geq 0$. Then the following inequality for the Riemann integral holds:*

$$\frac{2}{\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \int_{\ell_1}^{\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \mathscr{G}(\tau)d\tau \leq (\mathscr{G}(\ell_1)+\mathscr{G}(\ell_2)) - \frac{\theta(\ell_2-\ell_1)^2}{6}. \tag{7}$$

Proof. We can easily prove this result by using $\phi(\iota) = \iota$ in Theorem 1.

Corollary 3 *Let $\mathscr{G} : O \to (0,+\infty)$ be strongly generalized nonconvex function with modulus value $\theta \geq 0$. Then the following inequality for the Riemann fractional integral holds:*

$$\begin{aligned} &\frac{\Gamma(\alpha)}{[\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^{\alpha}} \left[I^{\alpha}_{\ell_1^+}\mathscr{G}(\ell_1 + \mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + I^{\alpha}_{\ell_2^-}\mathscr{G}(\ell_1)\right] \\ &\leq \frac{(\mathscr{G}(\ell_1)+\mathscr{G}(\ell_2))}{\alpha} - \frac{\theta(\ell_2-\ell_1)^2}{\alpha^2+3\alpha+2}. \end{aligned} \tag{8}$$

Proof. We can easily prove this result by using $\phi(\iota) = \frac{\iota^{\alpha}}{\Gamma(\alpha)}$ in Theorem 1.

Corollary 4 *Let $\mathscr{G} : O \to (0,+\infty)$ be strongly generalized nonconvex function with modulus value $\theta \geq 0$. Then the following inequality for the k-Riemann fractional integral holds: mmm*

$$\begin{aligned} &\frac{\Gamma_k(\alpha)}{[\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^{\frac{\alpha}{k}}} \left[I^{\alpha}_{\ell_1^+,k}\mathscr{G}(\ell_1 + \mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + I^{\alpha}_{(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-,k}\mathscr{G}(\ell_1)\right] \\ &\leq \frac{(\mathscr{G}(\ell_1)+\mathscr{G}(\ell_2))}{\alpha} - \frac{\theta(\ell_2-\ell_1)^2 k}{(k+\alpha)(2k+\alpha)}. \end{aligned} \tag{9}$$

Proof. We can easily prove this result by using $\phi(\iota) = \frac{\iota^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 1.

Corollary 5*Let $\mathscr{G}:O\rightarrow(0,+\infty)$ be strongly generalized nonconvex function with modulus value $\theta\geq 0$. Then the following inequality for the conformable fractional integral holds:*

$$2I^{\alpha}_{\ell_1}\mathscr{G}(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\leq\frac{(\mathscr{G}(\ell_1)+\mathscr{G}(\ell_2))(\ell_2^{\alpha}-\ell_1^{\alpha})}{\alpha}-\theta\left\{\frac{\ell_1^{\alpha+2}-\ell_2^{\alpha+2}}{\alpha+2}+\frac{(\ell_1+\ell_2)(\ell_2^{\alpha+1}-\ell_1^{\alpha+1})}{\alpha+1}-\frac{\ell_1\ell_2(\ell_2^{\alpha}-\ell_1^{\alpha})}{\alpha}\right\}. \tag{10}$$

Proof. We can easily prove this result by using $\phi(\iota)=\iota(\ell_2-\iota)^{\alpha-1}$ in Theorem 1 and α in $(0,1)$.

Corollary 6. *Let $\mathscr{G}:O\rightarrow(0,+\infty)$ be strongly generalized nonconvex function with modulus value $\theta\geq 0$. Then the following inequality for the fractional integral with exponential kernel holds:*

$$\left[I^{\alpha}_{\ell_1^+}\mathscr{G}(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+I^{\alpha}_{(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathscr{G}(\ell_1)\right]\leq\frac{(\mathscr{G}(\ell_1)+\mathscr{G}(\ell_2))}{\alpha-1}\left\{e^{\frac{\alpha-1}{\alpha}\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}-1\right\}-\frac{\theta\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)(\ell_2-\ell_1)^2}{\alpha}Q, \tag{11}$$

where

$$Q=\int_0^1\iota(1-\iota)\exp\left(-\frac{1-\alpha}{\alpha}\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\,\iota\right)d\iota.$$

Proof. We can prove this result by using $\phi(\iota)=\frac{\iota}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}\iota\right)$ in Theorem 1 and α in $(0,1)$.

Theorem 7.*Let $\mathscr{G}_1,\mathscr{G}_2:O\rightarrow(0,+\infty)$ be strongly-generalized nonconvex functions with modulus value $\theta\geq 0$. Then the following inequality for the generalized fractional integral holds:*

$$\begin{aligned}&{}_{\ell_1^+}I_{\phi}\mathscr{G}_1(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathscr{G}_2(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+{}_{(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}I_{\phi}\mathscr{G}_1(\ell_1)\mathscr{G}_2(\ell_1)\\ \leq\;&(P_1-2P_2)M(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)-\theta(\ell_2-\ell_1)^2P_2L(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)\\ +\;&2P_2N(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)+2\theta^2(\ell_2-\ell_1)^4P_3,\end{aligned} \tag{12}$$

where

$$M(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)=\mathscr{G}_1(\ell_1)\mathscr{G}_2(\ell_1)+\mathscr{G}_1(\ell_2)\mathscr{G}_2(\ell_2),$$
$$N(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)=\mathscr{G}_1(\ell_1)\mathscr{G}_2(\ell_2)+\mathscr{G}_1(\ell_2)\mathscr{G}_2(\ell_1),$$
$$L(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)=\mathscr{G}_1(\ell_1)+\mathscr{G}_1(\ell_2)+\mathscr{G}_2(\ell_1)+\mathscr{G}_2(\ell_2)$$

and

$$P_3=\int_0^1\iota(1-\iota)^2\phi(\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\iota)d\iota.$$

Proof. Since $\mathscr{G}_1,\mathscr{G}_2$ are strongly-generalized nonconvex functions, we have

$$\begin{aligned}&\mathscr{G}_1(\ell_1+\iota\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathscr{G}_2(\ell_1+\iota\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\\ \leq\;&\left((1-\iota)\mathscr{G}_1(\ell_1)+\iota\mathscr{G}_1(\ell_2)-\theta\iota(1-\iota)(\ell_2-\ell_1)^2\right)\\ \times\;&\left((1-\iota)\mathscr{G}_2(\ell_1)+\iota\mathscr{G}_2(\ell_2)-\theta\iota(1-\iota)(\ell_2-\ell_1)^2\right)\\ =\;&(1-\iota)^2\mathscr{G}_1(\ell_1)\mathscr{G}_2(\ell_1)+\iota^2\mathscr{G}_1(\ell_2)\mathscr{G}_2(\ell_2)-\theta\iota(1-\iota)^2(\ell_2-\ell_1)^2(\mathscr{G}_1(\ell_1)+\mathscr{G}_2(\ell_1))\\ -\;&\theta\iota^2(1-\iota)(\ell_2-\ell_1)^2(\mathscr{G}_1(\ell_2)+\mathscr{G}_2(\ell_2))+\iota(1-\iota)(\mathscr{G}_1(\ell_1)\mathscr{G}_2(\ell_2)+\mathscr{G}_1(\ell_2)\mathscr{G}_2(\ell_1))\\ +\;&\left(\theta\iota(1-\iota)(\ell_2-\ell_1)^2\right)^2\end{aligned} \tag{13}$$

and

$$\begin{aligned}&\mathscr{G}_1(\ell_1+(1-\iota)\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathscr{G}_2(\ell_1+(1-\iota)\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\\ \leq\;&\left(\iota\mathscr{G}_1(\ell_1)+(1-\iota)\mathscr{G}_1(\ell_2)-\theta\iota(1-\iota)(\ell_2-\ell_1)^2\right)\\ \times\;&\left(\iota\mathscr{G}_2(\ell_1)+(1-\iota)\mathscr{G}_2(\ell_2)-\theta\iota(1-\iota)(\ell_2-\ell_1)^2\right)\\ =\;&\iota^2\mathscr{G}_1(\ell_1)\mathscr{G}_2(\ell_1)+(1-\iota)^2\mathscr{G}_1(\ell_2)\mathscr{G}_2(\ell_2)-\theta\iota^2(1-\iota)(\ell_2-\ell_1)^2(\mathscr{G}_1(\ell_1)+\mathscr{G}_2(\ell_1))\\ -\;&\theta\iota(1-\iota)^2(\ell_2-\ell_1)^2(\mathscr{G}_1(\ell_2)+\mathscr{G}_2(\ell_2))+\iota(1-\iota)(\mathscr{G}_1(\ell_1)\mathscr{G}_2(\ell_2)+\mathscr{G}_1(\ell_2)\mathscr{G}_2(\ell_1))\\ +\;&\left(\theta\iota(1-\iota)(\ell_2-\ell_1)^2\right)^2.\end{aligned} \tag{14}$$

Adding (13) and (14), we have

$$\begin{aligned}&\mathscr{G}_1(\ell_1+\iota\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathscr{G}_2(\ell_1+\iota\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\\ &+\mathscr{G}_1(\ell_1+(1-\iota)\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathscr{G}_2(\ell_1+(1-\iota)\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\\ \leq\;&(1-2\iota(1-\iota))M(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)-\theta\iota(1-\iota)(\ell_2-\ell_1)^2L(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)\\ +\;&2\iota(1-\iota)N(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)+2\left(\theta\iota(1-\iota)(\ell_2-\ell_1)^2\right)^2.\end{aligned} \tag{15}$$

Multiplying (15) with $\frac{\phi(\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\iota)}{\iota}$ on both sides and integrating the obtained inequality with respect to ι over $[0,1]$, then we have required inequality (12).

Corollary 8*Let $\mathscr{G}_1,\mathscr{G}_2:O\rightarrow(0,+\infty)$ be strongly-generalized nonconvex functions with modulus value $\theta\geq 0$. Then the following inequality for the Riemann integral holds:*

$$\frac{2}{[\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^2}\int_{\ell_1}^{\ell_2}\mathscr{G}_1(\tau)\mathscr{G}_2(\tau)d\tau\leq\frac{2(M(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)+N(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2))}{3}-\frac{\theta(\ell_2-\ell_1)^2}{6}L(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)+\frac{\theta^2(\ell_2-\ell_1)^4}{15}. \tag{16}$$

Proof. We can easily prove this result by using $\phi(\iota)=\iota$ in Theorem 7.

Corollary 9*Let $\mathscr{G}_1,\mathscr{G}_2:O\rightarrow(0,+\infty)$ be strongly-generalized nonconvex functions with modulus value $\theta\geq 0$. Then the following inequality for the Riemann fractional integral holds:*

$$\begin{aligned}&\frac{\Gamma(\alpha)}{[\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^{2\alpha}}\Big[I^{\alpha}_{\ell_1^+}\mathscr{G}_1(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathscr{G}_2(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\\ &+I^{\alpha}_{(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathscr{G}_1(\ell_1)\mathscr{G}_2(\ell_1)\Big]\\ \leq\;&\left(\frac{1}{\Gamma(\alpha+1)}-\frac{2\alpha}{\Gamma(\alpha+3)}\right)M(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)-\frac{\theta\alpha(\ell_2-\ell_1)^2}{\Gamma(\alpha+3)}L(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)\\ &+\frac{2\alpha}{\Gamma(\alpha+3)}N(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)+\frac{4\alpha(\alpha+1)\theta^2(\ell_2-\ell_1)^4}{\Gamma(\alpha+5)}.\end{aligned} \tag{17}$$

Proof. We can easily prove result by using $\phi(\iota)=\frac{\iota^{\alpha}}{\Gamma(\alpha)}$ in Theorem 7.

Corollary 10*Let $\mathscr{G}_1,\mathscr{G}_2:O\rightarrow(0,+\infty)$ be strongly-generalized nonconvex functions with modulus value $\theta\geq 0$. Then the following inequality for the k-Riemann fractional integral holds:*

$$\begin{aligned}&\frac{\Gamma_k(\alpha)}{[\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^{\frac{2\alpha}{k}}}\Big[I^{\alpha}_{\ell_1^+,k}\mathscr{G}_1(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathscr{G}_2(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\\ &+I^{\alpha}_{(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-,k}\mathscr{G}_1(\ell_1)\mathscr{G}_2(\ell_1)\Big]\\ \leq\;&\left(\frac{1}{\Gamma_k(\alpha+k)}-\frac{2\alpha}{k\Gamma_k(\alpha+2k+1)}\right)M(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)-\frac{\theta\alpha(\ell_2-\ell_1)^2}{k\Gamma_k(\alpha+2k+1)}L(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)\\ &+\frac{2\alpha}{k\Gamma_k(\alpha+2k+1)}N(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)+\frac{4\alpha(\alpha+k)\theta^2(\ell_2-\ell_1)^4}{k\Gamma_k(\alpha+4k+1)}.\end{aligned} \tag{18}$$

Proof. We can prove this result by using $\phi(\iota)=\frac{\iota^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 7.

Corollary 11*Let $\mathscr{G}_1,\mathscr{G}_2:O\rightarrow(0,+\infty)$ be strongly-generalized nonconvex functions with modulus* *value* $\theta \geq 0$. *Then the following inequality for the conformable fractional integral holds:*

$$\begin{aligned} & 2I^{\alpha}_{\ell_1}\mathscr{G}_1(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathscr{G}_2(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ & \leq (R_1-2R_2)M(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)-\theta(\ell_2-\ell_1)^2R_2L(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2) \\ & + 2R_2N(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)+2\theta^2(\ell_2-\ell_1)^4R_3, \end{aligned}$$

where

$$R_1 = \frac{\ell_2^{\alpha}-[\ell_2-\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^{\alpha}}{\alpha},$$

$$R_2 = \frac{1}{[\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^2}\int_{\ell_2-\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}^{\ell_2}(\imath+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\ell_2)(\ell_2-\imath)\imath^{\alpha-1}d\imath$$

and

$$R_3 = \frac{1}{[\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^2}\int_{\ell_2-\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}^{\ell_2}(\imath+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\ell_2)^2(\ell_2-\imath)^2\imath^{\alpha-1}d\imath.$$

*Proof.*We can prove this result by using $\phi(\imath)=\imath(\ell_2-\imath)^{\alpha-1}$ in Theorem 7 and α in $(0,1)$.

Corollary 12*Let* $\mathscr{G}_1,\mathscr{G}_2 : O \to (0,+\infty)$ *be strongly-generalized nonconvex functions with modulus value* $\theta \geq 0$. *Then the following inequality for the fractional integral with exponential kernel holds:*

$$\begin{aligned} & \frac{\alpha}{\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\Big[I^{\alpha}_{\ell_1^+}\mathscr{G}_1(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathscr{G}_2(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ & + I^{\alpha}_{(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathscr{G}_1(\ell_1)\mathscr{G}_2(\ell_1)\Big] \\ & \leq (U_1-2U_2)M(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)-\theta(\ell_2-\ell_1)^2U_2L(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2) \\ & + 2U_2N(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)+2\theta^2(\ell_2-\ell_1)^4U_3, \end{aligned}$$

where

$$U_1 = \frac{1-\exp(-A\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))}{1-\alpha},$$

$$U_2 = \frac{\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}{\alpha}\int_0^1 \imath(1-\imath)\exp(-A\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath)d\imath$$

and

$$U_3 = \frac{\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}{\alpha}\int_0^1 \imath^2(1-\imath)^2\exp(-A\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath)d\imath.$$

*Proof.*We can prove this result by using $\phi(\imath)=\frac{\imath}{\alpha}\exp(-A\imath)$, where $A=\frac{1-\alpha}{\alpha}$ in Theorem 7 and α in $(0,1)$.

Theorem 13.*Let* $\mathscr{G}_1,\mathscr{G}_2 : O \to (0,+\infty)$ *strongly-generalized nonconvex functions with modulus value* $\theta \geq 0$. *Then the following inequality for the generalized fractional integral holds:*

$$\begin{aligned} & \Big[{}_{\ell_1^+}I_{\phi}\mathscr{G}_1(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathscr{G}_2(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ & \qquad + {}_{(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}I_{\phi}\mathscr{G}_1(\ell_1)\mathscr{G}_2(\ell_1)\Big] \\ & \leq \frac{1}{2}\Big[(P_1-2P_2)F(\mathscr{G}_1^2,\mathscr{G}_2^2;\ell_1,\ell_2)+4P_2G(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2) \\ & -2\theta(\ell_2-\ell_1)^2P_2L(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)+2\theta^2(\ell_2-\ell_1)^4P_1\Big], \end{aligned} \tag{19}$$

where

$$F(\mathscr{G}_1^2,\mathscr{G}_2^2;\ell_1,\ell_2)=\mathscr{G}_1^2(\ell_1)+\mathscr{G}_1^2(\ell_2)+\mathscr{G}_2^2(\ell_1)+\mathscr{G}_2^2(\ell_2),$$

$$G(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)=\mathscr{G}_1(\ell_1)\mathscr{G}_1(\ell_2)+\mathscr{G}_2(\ell_1)\mathscr{G}_2(\ell_2).$$

*Proof.*Since $\mathscr{G}_1,\mathscr{G}_2$ are strongly-generalized nonconvex functions, then we have

$$\begin{aligned} & \mathscr{G}_1(\ell_1+\imath\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathscr{G}_2(\ell_1+\imath\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ & \leq \left((1-\imath)\mathscr{G}_1(\ell_1)+\imath\mathscr{G}_1(\ell_2)-\theta\imath(1-\imath)(\ell_2-\ell_1)^2\right) \\ & \quad \left((1-\imath)\mathscr{G}_2(\ell_1)+\imath\mathscr{G}_2(\ell_2)-\theta\imath(1-\imath)(\ell_2-\ell_1)^2\right) \\ & \leq \frac{1}{2}\Big\{\left((1-\imath)\mathscr{G}_1(\ell_1)+\imath\mathscr{G}_1(\ell_2)-\theta\imath(1-\imath)(\ell_2-\ell_1)^2\right)^2 \\ & + \left((1-\imath)\mathscr{G}_2(\ell_1)+\imath\mathscr{G}_2(\ell_2)-\theta\imath(1-\imath)(\ell_2-\ell_1)^2\right)^2\Big\} \\ & = \frac{1}{2}\Big[(1-\imath)^2\mathscr{G}_1^2(\ell_1)+\imath^2\mathscr{G}_1^2(\ell_2)+2\imath(1-\imath)\mathscr{G}_1(\ell_1)\mathscr{G}_1(\ell_2) \\ & \quad -2\theta(\ell_2-\ell_1)^2(\imath(1-\imath)^2\mathscr{G}_1(\ell_1)+\imath^2(1-\imath)\mathscr{G}_1(\ell_2)) \\ & + \left(\theta\imath(1-\imath)(\ell_2-\ell_1)^2\right)^2+(1-\imath)^2\mathscr{G}_2^2(\ell_1)+\imath^2\mathscr{G}_2^2(\ell_2)+2\imath(1-\imath)\mathscr{G}_2(\ell_1)\mathscr{G}_2(\ell_2) \\ & - 2\theta(\ell_2-\ell_1)^2[\imath(1-\imath)^2\mathscr{G}_2(\ell_1)+\imath^2(1-\imath)\mathscr{G}_2(\ell_2)]+\left(\theta\imath(1-\imath)(\ell_2-\ell_1)^2\right)^2\Big]. \end{aligned} \tag{20}$$

In a similar way, we have

$$\begin{aligned} & \mathscr{G}_1(\ell_1+(1-\imath)\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathscr{G}_2(\ell_1+(1-\imath)\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ & \leq \frac{1}{2}\Big[\imath^2\mathscr{G}_1^2(\ell_1)+(1-\imath)^2\mathscr{G}_1^2(\ell_2)+2\imath(1-\imath)\mathscr{G}_1(\ell_1)\mathscr{G}_1(\ell_2) \\ & \quad -2\theta(\ell_2-\ell_1)^2(\imath^2(1-\imath)\mathscr{G}_1(\ell_1)+\imath(1-\imath)^2\mathscr{G}_1(\ell_2)) \\ & + \left(\theta\imath(1-\imath)(\ell_2-\ell_1)^2\right)^2+\imath^2\mathscr{G}_2^2(\ell_1)+(1-\imath)^2\mathscr{G}_2^2(\ell_2)+2\imath(1-\imath)\mathscr{G}_2(\ell_1)\mathscr{G}_2(\ell_2) \\ & - 2\theta(\ell_2-\ell_1)^2[\imath^2(1-\imath)\mathscr{G}_2(\ell_1)+\imath(1-\imath)^2\mathscr{G}_2(\ell_2)]+\left(\theta\imath(1-\imath)(\ell_2-\ell_1)^2\right)^2\Big]. \end{aligned} \tag{21}$$

Adding (20) and (21), we have

$$\begin{aligned} & \mathscr{G}_1(\ell_1+\imath\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathscr{G}_2(\ell_1+\imath\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ & +\mathscr{G}_1(\ell_1+(1-\imath)\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathscr{G}_2(\ell_1+(1-\imath)\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ & \leq \frac{1}{2}\Big[(2\imath(\imath-1)+1)F(\mathscr{G}_1^2,\mathscr{G}_2^2;\ell_1,\ell_2)+4\imath(1-\imath)G(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2) \\ & - 2\imath(1-\imath)\theta(\ell_2-\ell_1)^2L(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)+2\left(\theta(\ell_2-\ell_1)^2\right)^2\Big]. \end{aligned} \tag{22}$$

Multiplying (22) with $\frac{\phi(\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath)}{\imath}$ on both sides and integrating the obtained inequality with respect to $\imath$ over $[0,1]$ we have our required inequality (19).

Corollary 14. *Let* $\mathscr{G}_1,\mathscr{G}_2 : O \to (0,+\infty)$ *be strongly-generalized nonconvex functions with modulus value* $\theta \geq 0$. *Then the following inequality for the Riemann integral holds:*

$$\begin{aligned} & \frac{2}{[\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^2}\int_{\ell_1}^{\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\mathscr{G}_1(\tau)\mathscr{G}_2(\tau)d\tau \leq \frac{1}{2}\Big[\frac{2(F(\mathscr{G}_1^2,\mathscr{G}_2^2;\ell_1,\ell_2)+G(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2))}{3} \\ & -\frac{\theta(\ell_2-\ell_1)^2}{3}L(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)+2\theta^2(\ell_2-\ell_1)^4\Big]. \end{aligned} \tag{23}$$

*Proof.*We can easily prove this result by using $\phi(\imath)=\imath$ in Theorem 13.

Corollary 15*Let* $\mathscr{G}_1,\mathscr{G}_2 : O \to (0,+\infty)$ *be strongly-generalized nonconvex functions with modulus value* $\theta \geq 0$. *Then the following inequality for the*

Riemann fractional integral holds:

$$\frac{\Gamma(\alpha)}{[\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^{2\alpha}}\Big[I^{\alpha}_{\ell_1^+}\mathscr{G}_1(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathscr{G}_2(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))$$
$$+I^{\alpha}_{(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathscr{G}_1(\ell_1)\mathscr{G}_2(\ell_1)\Big] \tag{24}$$
$$\leq \frac{1}{2}\Big[\Big(\frac{1}{\Gamma(\alpha+1)}-\frac{2\alpha}{\Gamma(\alpha+3)}\Big)F(\mathscr{G}_1^2,\mathscr{G}_2^2;\ell_1,\ell_2)+\frac{4\alpha}{\Gamma(\alpha+3)}G(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)$$
$$-\theta(\ell_2-\ell_1)^2\frac{2\alpha}{\Gamma(\alpha+3)}L(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)+\theta^2(\ell_2-\ell_1)^4\frac{2\alpha}{\Gamma(\alpha+1)}\Big]. \tag{25}$$

*Proof.*We can easily prove result by using $\phi(\iota)=\frac{\iota^{\alpha}}{\Gamma(\alpha)}$ in Theorem 13.

Corollary 16*Let $\mathscr{G}_1,\mathscr{G}_2 : O \to (0,+\infty)$ be strongly-generalized nonconvex functions with modulus value $\theta \geq 0$. Then the following inequality for the k-Riemann fractional integral holds:*

$$\frac{\Gamma_k(\alpha)}{[\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^{\frac{2\alpha}{k}}}\Big[I^{\alpha}_{\ell_1^+,k}\mathscr{G}_1(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathscr{G}_2(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))$$
$$+I^{\alpha}_{(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-,k}\mathscr{G}_1(\ell_1)\mathscr{G}_2(\ell_1)\Big]$$
$$\leq \frac{1}{2}\Big[\Big(\frac{1}{\Gamma_k(\alpha+k)}-\frac{2\alpha}{k\Gamma_k(\alpha+2k+1)}\Big)F(\mathscr{G}_1^2,\mathscr{G}_2^2;\ell_1,\ell_2)+\frac{4\alpha}{k\Gamma_k(\alpha+2k+1)}G(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)$$
$$-\theta(\ell_2-\ell_1)^2\frac{2\alpha}{k\Gamma_k(\alpha+2k+1)}L(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)+\theta^2(\ell_2-\ell_1)^4\frac{2\alpha}{k\Gamma_k(\alpha+k)}\Big]. \tag{26}$$

*Proof.*We can prove this result by using $\phi(\iota)=\frac{\iota^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 13.

Corollary 17*Let $\mathscr{G}_1,\mathscr{G}_2 : O \to (0,+\infty)$ be strongly-generalized nonconvex functions with modulus value $\theta \geq 0$. Then the following inequality for the conformable fractional integral holds:*

$$2I^{\alpha}_{\ell_1}\mathscr{G}_1(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathscr{G}_2(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))$$
$$\leq (R_1-2R_2)F(\mathscr{G}_1^2,\mathscr{G}_2^2;\ell_1,\ell_2)-\theta(\ell_2-\ell_1)^2R_2G(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)$$
$$+2R_2L(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)+2\theta^2(\ell_2-\ell_1)^4R_3,$$

where R_1,R_2 and R_3 are defined as in Corollary 11.

*Proof.*We can prove this result by using $\phi(\iota)=\iota(\ell_2-\iota)^{\alpha-1}$ in Theorem 13 and α in $(0,1)$.

Corollary 18*Let $\mathscr{G}_1,\mathscr{G}_2 : O \to (0,+\infty)$ be strongly-generalized nonconvex functions with modulus value $\theta \geq 0$. Then the following inequality for the fractional integral with exponential kernel holds:*

$$\frac{\alpha}{\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\Big[I^{\alpha}_{\ell_1^+}\mathscr{G}_1(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathscr{G}_2(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))$$
$$+I^{\alpha}_{(\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathscr{G}_1(\ell_1)\mathscr{G}_2(\ell_1)\Big]$$
$$\leq (U_1-2U_2)F(\mathscr{G}_1^2,\mathscr{G}_2^2;\ell_1,\ell_2)-\theta(\ell_2-\ell_1)^2U_2G(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)$$
$$+2U_2L(\mathscr{G}_1,\mathscr{G}_2;\ell_1,\ell_2)+2\theta^2(\ell_2-\ell_1)^4U_3, \tag{27}$$

where U_1,U_2 and U_3 are defined as in Corollary 12.

*Proof.*We can prove this result by using $\phi(\iota)=\frac{\iota}{\alpha}\exp(-A\iota)$, where $A=\frac{1-\alpha}{\alpha}$ in Theorem 13 and α in $(0,1)$.

*Remark.*If we tend to suppose $\theta=0$ in all proved results of this paper, then all results hold for the generalized nonconvex function.

*Remark.*Taking $\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)=\ell_2-\ell_1>0$ in Theorems 7 and 13, then all results hold for the strongly convex function.

*Remark.*For different positive values of ρ,λ, where $\sigma=(\sigma(0),\ldots,\sigma(k),\ldots)$ is a bounded sequence of positive real involving $\mathscr{F}^{\sigma}_{\rho,\lambda}(\cdot)$ (Raina function) in our theorems, we have different fascinating inequalities of Hermite-Hadamard type.

3 Applications

Consider the following special means for different positive real numbers ℓ_1,ℓ_2, as follows:

1.The arithmetic mean:

$$A(\ell_1,\ell_2)=\frac{\ell_1+\ell_2}{2},$$

2.The harmonic mean:

$$H(\ell_1,\ell_2)=\frac{2}{\frac{1}{\ell_1}+\frac{1}{\ell_2}},$$

3.The logarithmic mean:

$$L(\ell_1,\ell_2)=\frac{\ell_2-\ell_1}{\ln|\ell_2|-\ln|\ell_1|},$$

4.The generalized log-mean:

$$L_r(\ell_1,\ell_2)=\left[\frac{\ell_2^{r+1}-\ell_1^{r+1}}{(r+1)(\ell_2-\ell_1)}\right]^{\frac{1}{r}};\ r\in\mathbb{Z}\setminus\{-1,0\}.$$

Using the theory results in Section 2, we give some applications to special means.

Proposition 1.*Let $0<\ell_1<\ell_2$ and $\theta\geq 0$. Then for $r\in\mathbb{N}$ and $r\geq 2$, the following inequality hold:*

$$\left|L_r^r\left(\ell_1,\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right)-A(\ell_1^r,\ell_2^r)\right|\leq\frac{\theta(\ell_2-\ell_1)^2}{12}. \tag{28}$$

*Proof.*Taking $\mathscr{G}(\tau)=\tau^r$ and $\phi(\iota)=\iota$ in Theorem 1, one can obtain the result immediately.

Proposition 2.*Let $0<\ell_1<\ell_2$ and $\theta\geq 0$. Then the following inequality holds:*

$$\left|\frac{1}{L\left(\ell_1,\ell_1+\mathscr{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right)}-\frac{1}{H(\ell_1,\ell_2)}\right|\leq\frac{\theta(\ell_2-\ell_1)^2}{12}. \tag{29}$$

*Proof.*Taking $\mathscr{G}(\tau)=\frac{1}{\tau}$ and $\phi(\iota)=\iota$ in Theorem 1, one can obtain the result immediately.

*Remark.*Taking different strongly-generalized nonconvex functions we can derive many interesting inequalities using results given in this part and special means.

4 Perspective

In this paper, we have defined a new class of functions involving $\mathscr{F}^{\sigma}_{\rho,\lambda}(\cdot)$ (Raina function) and some Hermite–Hadamard type integral inequalities are provided as well. Interested reader can establish new inequalities via fractional operators or multiplicative integrals. Also, these results can be applied in convex analysis, optimization and different areas of pure and applied sciences.

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