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Edge Subcategories of the Edge Ideal of a Graph Simple and Local Cohomology Modules

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Abstract: This paper is concerned with the relation between local cohomology modules of a module defined by an ideal and the edge subcategories of the category of modules, together with the edge ideal of a graph simple and finite, with no isolated vertices.

Keywords: local cohomology modules, edge subcategory, edge weakly Laskerian, edge ideal of a graph.

1 Introduction

Throughout this paper, R is a commutative Noetherian ring with non-zero identity.

Let $\mathfrak{a}$ be an ideal of R and let $I(G)$ be the edge ideal of a graph simple, finite and with no isolated vertices. By $\mathrm{H}^i_{\mathfrak{a}}(I(G))$ we mean

$$\varinjlim_{t\in\mathbb{N}} \mathrm{Ext}^i_R(R/\mathfrak{a}^t, I(G)),$$

the i-th local cohomology module of $I(G)$ with respect to the ideal $\mathfrak{a}$, for $i \geq 0$. For more details about local cohomology modules, see [1].

Local cohomology was introduced by Grothendieck and many people have worked about the understanding of their structure, (non)-vanishing and finiteness properties. For example, Grothendieck's non-vanishing theorem is one of the important theorems in local cohomology

We provide here results for local cohomology modules which involve the theory of graphs, together with the edge ideal of a graph. In this study, we investigate modules which involve the theory of graphs, together with the edge ideal of a graph simple and finite.

In the Section 2, we put some definitions and prerequisites for a better understanding of the theory and results. We introduce preliminaries of the theory of graphs which involving the edge ideal of a graph G; associated to the graph G is a monomial ideal

$$I(G) = (v_iv_j \mid v_iv_j \text{ is an edge of } G),$$

with $v_iv_j = v_jv_i$ and with $i \neq j$, in the polynomial ring $R = K[v_1, v_2, \ldots, v_s]$ over a field K, called the **edge ideal** of G. The preliminaries of the theory of graphs were introduced in this Section 2 together with the concepts suitable for the work.

In the Section 3, we prove some properties of modules and submodules with respect to theory in question, properties that involve the edge ideal of a graph G, which is a graph simple and finite, with no isolated vertices. Moreover, in this section we use some definitions as edge subcategory, edge weakly Laskerian module and the concept of local cohomology module J-cofinite, for J an ideal of the ring $R = K[v_1, \ldots, v_s]$.

Throughout the paper, we mean by a graph G, a finite simple graph with the vertex set $V(G)$ and with no isolated vertices.

Here we use properties of commutative algebra and homological algebra for the development of the results (see [3] and [5]).

Moreover, we observe that this theory of the edge ideal together with local cohomology can be found in [8] and [9].

2 Prerequisites of the graphs theory

Let us present in this section the concepts of the graphs theory that we are using in the course of this work.

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2.1 Edge ideal of a graph

This section is in accordance with [2] and [6].

Let $R = K[v_1,\ldots,v_s]$ be a polynomial ring over a field K, and let $Z = \{z_1,\ldots,z_q\}$ be a finite set of monomials in R. The **monomial subring** spanned by Z is the K-subalgebra,

$$K[Z] = K[z_1,\ldots,z_q] \subset R.$$

In general, it is very difficult to certify whether $K[Z]$ has a given algebraic property - e.g., Cohen-Macaulay, normal - or to obtain a measure of its numerical invariants - e.g., Hilbert function. This arises because the number q of monomials is usually large.

Thus, consider any graph G, simple and finite without isolated vertices, with vertex set $V(G) = \{v_1,\ldots,v_s\}$.

Let Z be the set of all monomials $v_iv_j = v_jv_i$, with $i \neq j$, in $R = K[v_1,\ldots,v_s]$, such that $\{v_iv_j\}$ is an edge of G, i.e., the graph finite and simple G, with no isolated vertices, is such that the squarefree monomials of degree two are defining the edges of the graph G.

Definition 21A **walk** of length s in G is an alternating sequence of vertices and edges $w = \{v_1, z_1, v_2,\ldots,v_{s-1}, z_h, v_s\}$, where $z_i = \{v_{i-1}v_i\}$ is the edge joining v_{i-1} and v_i.

Definition 22A walk is **closed** if $v_1 = v_s$. A walk may also be denoted by $\{v_1,\ldots,v_s\}$, where the edges are evident by context. A **cycle** of length s is a closed walk, in which the points $v_1,\ldots,v_s$ are distinct.

A **path** is a walk with all the points being distinct. A **tree** is a connected graph without cycles and a graph is **bipartite** if all its cycles are even. A vertex of degree one is called an **end point**.

Definition 23A subgraph $G' \subseteq G$ is called **induced** if $v_iv_j = v_jv_i$, with $i \neq j$, is an edge of G' whenever v_i and v_j are vertices of G' and v_iv_j is an edge of G.

The **complement** of a graph G, for which we write G^c, is the graph on the same vertex set in which $v_iv_j = v_jv_i$, with $j \neq i$, is an edge of G^c if and only if it is not an edge of G. Finally, let C_k denote the cycle on k vertices; a **chord** is an edge which is not in the edge set of C_k. A cycle is called **minimal** if it has no chord.

If G is a graph without isolated vertices, simple and finite, then let R denote the polynomial ring on the vertices of G over some fixed field K.

Definition 24([2]) According to the previous context, the **edge ideal** of a finite simple graph G, with no isolated vertices, is defined by

$$I(G) = (v_iv_j \mid v_iv_j \text{ is an edge of } G),$$

with $v_iv_j = v_jv_i$, and with $i \neq j$.

3 Results

In this section, present some results about the modules and submodules which involve the theory of graphs together with the edge ideal of a graph G, which is simple and finite and with no isolated vertices.

Here, we take K as a fixed field and we consider $K[v_1,v_2\ldots,v_s]$ as the ring polynomial over the field K. Since K is a field, we have that K is a Noetherian ring and then $K[v_1,\ldots,v_s]$ is also a Noetherian ring (Theorem of the Hilbert Basis).

Remark 31 By the previous context, $R = K[v_1,v_2\ldots,v_s]$ is a Noetherian ring. Thus, the edge ideal $I(G)$ is an R-module, and thus we can get characterizations for this module under certain hypothesis.

We denote by $\mathfrak{m} = (v_1,\ldots,v_s)$ the homogeneous maximal ideal of $R = K[v_1,\ldots,v_s]$, where we have $I(G)$ as a monomial ideal of R which is finitely generated.

Definition 32 Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ is the edge ideal in R of a finite simple graph G, with no isolated vertices. An edge subcategory $\mathfrak{S}$ of the category of R-modules with respect to the ideal $\mathfrak{m}$ is a class of R-modules such that if the R-module $I(G)$ is $\mathfrak{m}$-torsion and $(0:_{I(G)}\mathfrak{m})$ is in $\mathfrak{S}$ then we have that $I(G)$ is in $\mathfrak{S}$.

We have now the following proposition.

Proposition 33*Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ be the edge ideal in R of a finite simple graph G, with no isolated vertices. Let $\mathfrak{S}$ be an edge subcategory with respect to the ideal $\mathfrak{m}$ and let t be an integer. Suppose that T is an R-module such that $\mathrm{Ext}^i_R(R/\mathfrak{m},T)$ is in $\mathfrak{S}$ for all $i<t$. Then $\mathrm{H}^i_{\mathfrak{m}}(T)$ is in $\mathfrak{S}$ for all $i<t$.*

*Proof.*We prove the result by induction on i. It is straightforward to see that the result is true when $i = 0$. Suppose that $i > 0$ and that the result has been proved for $i-1$. It easily follows from the exact sequence

$$0 \to \Gamma_{\mathfrak{m}}(T) \to T \to T/\Gamma_{\mathfrak{m}}(T) \to 0,$$

that $\mathrm{Ext}^i_R(R/\mathfrak{m},T)$ is in $\mathfrak{S}$ if and only if $\mathrm{Ext}^i_R(R/\mathfrak{m},T/\Gamma_{\mathfrak{m}}(T))$ is in $\mathfrak{S}$. Also, by [1, Corollary 2.1.7], we have that

$$\mathrm{H}^i_{\mathfrak{m}}(T) \cong \mathrm{H}^i_{\mathfrak{m}}(T/\Gamma_{\mathfrak{m}}(T)),$$

for all $i > 0$. Therefore, we assume that $\Gamma_{\mathfrak{m}}(T) = 0$. Now, let E be an injective envelope of T. Then,

$$\Gamma_{\mathfrak{m}}(E) = \mathrm{Hom}_R(R/\mathfrak{m},E) = 0.$$

We put $L = E/T$ and we consider the exact sequence

$$0 \to T \to E \to L \to 0.$$

We obtain isomorphisms:

$$\mathrm{H}^i_{\mathfrak{m}}(T) \cong \mathrm{H}^{i-1}_{\mathfrak{m}}(L),$$

for all $i > 0$, and

$$\mathrm{Ext}^i_R(R/\mathfrak{m}, T) \cong \mathrm{Ext}^{i-1}_R(R/\mathfrak{m}, L),$$

for all $i > 0$.

We use the induction hypothesis applied to L, and we conclude that the R-module $\mathrm{H}^i_{\mathfrak{m}}(T)$ is in $\mathfrak{S}$ for all $i < t$. We conclude so the proof.

In the same context, it follows the following result which involve the definition of edge subcategory.

Proposition 34*Let $R = K[v_1,\dots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let $\mathfrak{S}$ be an edge subcategory with respect to the ideal $\mathfrak{m}$. Let t be an integer such that we have the index $j \leq t$. Suppose that*

$$\dim(I(G)/\mathfrak{m}I(G)) \leq 1.$$

Then, $\mathrm{Ext}^i_R(R/\mathfrak{m}, \mathrm{H}^j_{\mathfrak{m}}(I(G)))$ *is in* $\mathfrak{S}$*, for all* $i \geq 0$.

*Proof.*Consider the following spectral sequence

$$\mathrm{E}^{p,q}_2 := \mathrm{Ext}^p_R(R/\mathfrak{m}, \mathrm{H}^q_{\mathfrak{m}}(I(G))) \xrightarrow{p} \mathrm{Ext}^{p+q}_R(R/\mathfrak{m}, I(G)) = \mathrm{H}^{p+q}.$$

In view of [7, Theorem 4.3], we have $\mathrm{E}^{p,q}_2 = 0$ unless $q = 0, 1$. It follows that the exact sequence

$$\mathrm{H}^{p+1} \to \mathrm{E}^{p+1,0}_2 \to \mathrm{E}^{p-1,1}_2 \to \mathrm{H}^p \to \mathrm{E}^{p,0}_2 \to \mathrm{E}^{p-2,1}_2 \to \mathrm{H}^{p-1},$$

which in turn yields the exact sequence

$$\mathrm{Ext}^{p+1}_R(R/\mathfrak{m}, I(G)) \to \mathrm{Ext}^{p+1}_R(R/\mathfrak{m}, \Gamma_{\mathfrak{m}}(I(G))) \to \mathrm{Ext}^{p-1}_R(R/\mathfrak{m}, \mathrm{H}^1_{\mathfrak{m}}(I(G))) \to$$
$$\mathrm{Ext}^{p}_R(R/\mathfrak{m}, I(G)) \to \mathrm{Ext}^{p}_R(R/\mathfrak{m}, \Gamma_{\mathfrak{m}}(I(G))) \to \mathrm{Ext}^{p-2}_R(R/\mathfrak{m}, \mathrm{H}^1_{\mathfrak{m}}(I(G))).$$

Since, by our assumption, the R-modules

$$\mathrm{Ext}^i_R(R/\mathfrak{m}, \Gamma_{\mathfrak{m}}(I(G))) \text{ and } \mathrm{Ext}^i_R(R/\mathfrak{m}, I(G)),$$

are in $\mathfrak{S}$ for all $i \geq 0$, we have that the result it follows.

Definition 35 Let $R = K[v_1,\dots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. The R-module $I(G)$ is said to be edge weakly Laskerian if any quotient of $I(G)$ has finitely many of associated prime ideals, where the prime ideals are in $R = K[v_1,\dots,v_s]$.

4 Some results of applications

Recall that the *spectrum* of R, denoted by $\mathrm{Spec}(R)$, is the set of prime ideals of R with the Zariski topology, which is the topology where the closed sets are

$$\mathrm{V}(J) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid J \subseteq \mathfrak{p}\},$$

for ideals $J \subseteq R$.

We presented now the following result.

Theorem 41*Let $R = K[v_1,\dots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let $\mathfrak{S}$ be an edge subcategory with respect to the ideal $\mathfrak{m}$. Let t be an integer. Suppose that $\mathfrak{m} \in \mathrm{V}(J)$. Moreover, suppose that $I(G)$ is edge weakly Laskerian module. Then $\mathrm{H}^i_{\mathfrak{m}}(I(G)) \in \mathfrak{S}$ for all $i < t$.*

*Proof.*By using the induction on t, the theorem is proved. It is straightforward to see that the result is true when $t = 1$. Suppose that $t > 1$, and the result holds for the case $t - 1$. Since,

$$\mathrm{H}^i_J(I(G)) \cong \mathrm{H}^i_J(I(G)/\Gamma_J(I(G))),$$

for all $i > 0$, we may replace $I(G)$ by $I(G)/\Gamma_J(I(G))$ and hence to assume that there exists an element $v \in \mathfrak{m}$, such that v is a non-zero divisor on $I(G)$. The exact sequence

$$0 \to I(G) \xrightarrow{v} I(G) \to I(G)/vI(G) \to 0,$$

induces two exact sequences

$$\to \mathrm{H}^{i-1}_J(I(G)/vI(G)) \to \mathrm{H}^i_J(I(G)) \xrightarrow{v} \mathrm{H}^i_J(I(G)) \to \mathrm{H}^i_J(I(G)/vI(G)),$$

and

$$\to \mathrm{H}^{i-1}_{\mathfrak{m}}(I(G)/vI(G)) \to \mathrm{H}^i_{\mathfrak{m}}(I(G)) \xrightarrow{v} \mathrm{H}^i_{\mathfrak{m}}(I(G)) \to \mathrm{H}^i_{\mathfrak{m}}(I(G)/vI(G)) \quad (*)$$

of local cohomology modules which involve the edge ideal of G. The induction hypothesis and the above sequences yield that the R-modules $\mathrm{H}^i_{\mathfrak{m}}(I(G))$ and $\mathrm{H}^i_{\mathfrak{m}}(I(G)/vI(G))$ are in $\mathfrak{S}$ for all $i < t-1$. It suffices to show that $\mathrm{H}^{t-1}_{\mathfrak{m}}(I(G))$ is in $\mathfrak{S}$. Now, the exactness of $(*)$, in conjunction with the fact that

$$\left(0 :_{\mathrm{H}^{t-1}_{\mathfrak{m}}(I(G))} \mathfrak{m}\right) \subseteq \left(0 :_{\mathrm{H}^{t-1}_{\mathfrak{m}}(I(G))} v\right),$$

and our hypotheses, show that $\mathrm{H}^{t-1}_{\mathfrak{m}}(I(G))$ is in $\mathfrak{S}$, and this proves our claim.

Now, we have the following theorem which involves the definition of edge weakly Laskerian module.

Theorem 42*Let $R = K[v_1,\dots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let $\mathfrak{S}$ be an edge subcategory with respect to the ideal $\mathfrak{m}$. Suppose that $I(G)$ is edge weakly Laskerian module and let r be a non-negative integer such that*

$$\mathrm{H}^r_{\mathfrak{m}}(R/\mathfrak{p}) \in \mathfrak{S} \text{ for all } \mathfrak{p} \in \mathrm{Supp}(I(G)).$$

Then, $\mathrm{H}^r_{\mathfrak{m}}(I(G)) \in \mathfrak{S}$.

*Proof.*Note that, there exists a filtration of the submodules of $I(G)$:

$$0 \subseteq M_0 \subset M_1 \subset \ldots \subset M_l = M,$$

such that for each $1 \leq j \leq l$, we have then

$$M_j/M_{j-1} \cong R/\mathfrak{p}_j,$$

where $\mathfrak{p}_j \in \mathrm{Supp}(I(G))$. We use induction on l. When $l = 1$, we have

$$\mathrm{H}^r_{\mathfrak{m}}(R/\mathfrak{p}) = \mathrm{H}^r_{\mathfrak{m}}(I(G)),$$

is in $\mathfrak{S}$, where we put $\mathfrak{p} = \mathfrak{p}_j$. Now, suppose that $l > 1$, and the result has been proved for $l-1$. The exact sequence

$$0 \to M_{l-1} \to M_l \to M_l/M_{l-1} \to 0,$$

induces the long exact sequence

$$\mathrm{H}^r_{\mathfrak{m}}(M_{l-1}) \to \mathrm{H}^r_{\mathfrak{m}}(M_l) \to \mathrm{H}^r_{\mathfrak{m}}(M_l/M_{l-1}).$$

It follows that we have

$$\mathrm{H}^r_{\mathfrak{m}}(M_l) \in \mathfrak{S},$$

and this completes the proof.

Definition 43 Let $R = K[v_1,\dots,v_s]$ be the ring polynomial, and $I(G)$ be the edge ideal in R of a finite simple graph G, with no isolated vertices. The R-module $\mathrm{H}^i_{\mathfrak{m}}(M)$, for an R-module any M and $i \geq 0$, is called J-cofinite if we have $J \neq \mathfrak{m}$, and J is not any maximal ideal of R, with

$$\mathrm{Supp}(I(G)) \subseteq \mathrm{V}(J),$$

and $\mathrm{Ext}^i_R(R/\mathfrak{m}, I(G))$ is a finitely generated R-module, for every $i \geq 0$.

In the following result, we use the Theorem 42.

Proposition 44*Let $R = K[v_1,\dots,v_s]$ be the ring polynomial, and $I(G)$ be the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $\dim(I(G)) = d$, and let $\mathfrak{S}$ be an edge subcategory with respect to the ideal $\mathfrak{m}$ such that $\mathfrak{S}$ is contained in the class of J-cofinite modules, according to the Definition 43, and Artinian modules. Moreover, suppose that $I(G)$ is edge weakly Laskerian module. Then, $\mathrm{H}^d_{\mathfrak{m}}(I(G))$ is an R-module Artinian and J-cofinite, yet of according to the Definition 43.*

*Proof.*It is enough, in view of the Theorem 42, to show that the R-module $\mathrm{H}^d_{\mathfrak{m}}(R/\mathfrak{p})$ is Artinian and J-cofinite, according to the Definition 43, for all $\mathfrak{p} \in \mathrm{Supp}(I(G))$. If $J \subseteq \mathfrak{p}$, then $R/\mathfrak{p}$ is J-torsion and then

$$\mathrm{H}^d_{\mathfrak{m}}(R/\mathfrak{p}) \cong \mathrm{H}^d_{\mathfrak{m}}(I(G)).$$

Since $\dim(R/\mathfrak{p}) \leq d$ then, in view of [4, Proposition 5.1], we have that $\mathrm{H}^d_{\mathfrak{m}}(R/\mathfrak{p}) \cong \mathrm{H}^d_{\mathfrak{m}}(I(G))$ is Artinian and J-cofinite. If J is not in contained in $\mathfrak{p}$, then we have that

$$\dim[(R/\mathfrak{p})/J(R/\mathfrak{p})] < \dim(R/\mathfrak{p}) \leq d,$$

and so

$$\mathrm{H}^d_{\mathfrak{m}}(R/\mathfrak{p}) \cong \mathrm{H}^d_{\mathfrak{m}}(I(G)) = 0,$$

by [7, Theorem 4.3].

Thus, the proof is completed.

We conclude the paper with a theorem that involves the previously presented concepts.

Theorem 45*Let $R = K[v_1,\dots,v_s]$ be the ring polynomial, and $I(G)$ be the edge ideal in R of a finite simple graph G, with no isolated vertices. Let $\mathfrak{S}$ be an edge subcategory with respect to the ideal $\mathfrak{m}$. Suppose that $I(G)$ is edge weakly Laskerian module and let t be a non-negative integer. Then, we have that the R-module $\mathrm{H}^i_{\mathfrak{m}}(R/\mathfrak{p})$ is in $\mathfrak{S}$ for all $i > t$ and $\mathfrak{p} \in \mathrm{Supp}(I(G))$.*

*Proof.*We use descending induction on i. Now, assume that $i > t$ and that the claim holds for $i+1$. We want to show that $\mathrm{H}^i_{\mathfrak{m}}(R/\mathfrak{p})$ is in $\mathfrak{S}$ for all $\mathfrak{p} \in \mathrm{Supp}(I(G))$. Suppose the contrary. We set:

$$\mathrm{A} := \left\{\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}(I(G)),\ \mathrm{H}^i_{\mathfrak{m}}(R/\mathfrak{p}) \text{ is not in } \mathfrak{S}\right\}.$$

Thus $\mathrm{A} \neq \emptyset$; it follows that the set A has a maximal element, and let $\mathfrak{p}$ be one such element. Since $\mathfrak{p} \in \mathrm{Supp}(I(G))$, there exists a non-zero map $f : I(G) \to R/\mathfrak{p}$. The exact sequence

$$0 \to \mathrm{Ker}(f) \to I(G) \to \mathrm{Im}(f) \to 0,$$

yields the exact sequence

$$\mathrm{H}^i_{\mathfrak{m}}(I(G)) \to \mathrm{H}^i_{\mathfrak{m}}(\mathrm{Im}(f)) \to \mathrm{H}^{i+1}_{\mathfrak{m}}(\mathrm{Ker}(f)).$$

Since $\mathrm{Supp}(\mathrm{Ker}(f)) \subset \mathrm{Supp}(I(G))$, it follows from the inductive hypothesis that the R-module $\mathrm{H}^{i+1}_{\mathfrak{m}}(R/\mathfrak{p})$ is in $\mathfrak{S}$ for all $\mathfrak{p} \in \mathrm{Supp}(\mathrm{Ker}(f))$, so that, in view of the Theorem 42, and the above exact sequence, the R-module $\mathrm{H}^i_{\mathfrak{m}}(\mathrm{Im}(f))$ is in $\mathfrak{S}$. There exists a filtration

$$0 = N_s \subset N_{s-1} \subset N_{s-2} \subset \ldots \subset N_0 = \mathrm{Coker}(f),$$

of submodules of $\mathrm{Coker}(f)$, such that for each $0 \leq i \leq s$, we have that

$$N_{i-1}/N_i \cong R/\mathfrak{q}_i,$$

where $\mathfrak{q}_i \in \mathrm{Supp}(\mathrm{Coker}(f))$. Then by the maximality of $\mathfrak{p}$, we have $\mathrm{H}^i_{\mathfrak{m}}(R/\mathfrak{q}_i)$ is in $\mathfrak{S}$. Next, the exact sequence

$$0 \to \mathrm{Im}(f) \to R/\mathfrak{p} \to \mathrm{Coker}(f) \to 0,$$

yields the exact sequence

$$\mathrm{H}^i_{\mathfrak{m}}(\mathrm{Im}(f)) \to \mathrm{H}^i_{\mathfrak{m}}(R/\mathfrak{p}) \to \mathrm{H}^i_{\mathfrak{m}}(\mathrm{Coker}(f)).$$

It follows that $\mathrm{H}^i_{\mathfrak{m}}(R/\mathfrak{p})$ is in $\mathfrak{S}$, which is a contradiction. We finalize then the proof.

We finished the article with the following conclusion.

5 Conclusion

In this article, we can relate the theory of graphs, with respect to the edge ideal of a simple graph, to the theory of local cohomology modules. With the results of the article, we show the importance of local cohomology theory as a study tool within the commutative algebra theory.

Moreover, by making this relationship, we get applications for the edge ideal in a general theory of modules.

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