Applied Mathematics & Information Sciences
*An International Journal*

http://dx.doi.org/10.18576/amis/140206

# Pseudo Almost Periodic Solutions of the Third Order Differential Equation with Continuous Delay

*Manel Amdouni*[1,*] *and Farouk Chérif*[2]

[1] ISSAT, Université of Sousse, Sousse, Tunisia
[2] Department of Computer Science, ISSAT and MaPSFA , University of Sousse, Sousse, Tunisia.

Received: 5 Jul. 2019, Revised: 10 Oct. 2019, Accepted: 14 Oct. 2019
Published online: 1 Mar. 2020

**Abstract:** In this paper the pseudo almost periodic solutions of a class of nonautonomous third-order differential equation with multiple finite delay is studied by various fixed-point theorems. Moreover, by using new and sufficient conditions, we study the uniformly-bounded and global attractivity of the pseudo almost periodic solutions. Further, an example is given to illustrate the validity of the obtained results.

**Keywords:** pseudo almost periodic,nonautonomous third order differential equation, attractivity global

## 1 Introduction

As we all known in applied science some practical and complex problems are associated with higher order nonlinear differential equations, such as non-linear oscillations ( see [1]-[4]). Furthermore, the most attractive topics are dedicated to the stability, instability, boundedness, oscillations, non-oscillations of the solutions ( see [5]-[13]).

On the other hand, the study of real phenomena often requires notions that go beyond the concept of periodicity, which take into account the fact that these phenomena are not entirely periodic. The central tool in this work is the concept of pseudo almost periodic functions which is a naturale generalization of bohr almost periodic. This notion was introduced by Zhang in 1992 ( see [14]). Consequently, the existence of almost periodic, pseudo almost periodic solutions are one of the most attracting topics in the qualitative theory of differential equations thanks to their significance and applications in various fields ( see [15]-[19]).

Furthermore, systems of differential equations with delay argument occupy more than a place of central importance in all areas of science and particularly in the biological sciences. These equations are used as models to describe many physical and biological systems ( see [6]-[10]). Therefore, it is very important to determine the qualitative behavior of solutions when there is a delay.

Motivated by the above discussion, the main aim is to study the existence and uniqueness of the pseudo almost periodic of the nonautonomous third-order differential equation with multiple and finite delay described by the the following equation

$$x^{(3)}(t)+a(t)x^{(2)}(t) + b(t)x^{(1)}(t)+\sum_{i=1}^{n} g_i(t,x(t-r_i(t))) = p(t).$$

where $a(.)$, $b(.)$ and $p(.)$ and for $i=1,2,\ldots,n$, $g_i(.x(.-r_i(.)))$ are real valued and continuous, bounded functions, with $g_i(0,0)=0$ for all $i=1,\ldots,n$ and $r_i(.)$ is real-valued positive and continuous function.

By new and sufficiently condition we prove the existence of the pseudo almost periodic solution by using various fixed point. First, by Banach's fixed point theorem and some operator we prove the existence and uniqueness of the solution. In addition, by Schauder and Leray-Schauder fixed-point theorems, we establish the existence of the third-order differential equation as above, finally we prove the existence of the pseudo almost periodic solutions by Krasnoselskii fixed-point theorem.

Therefore, by using new technics and new sufficiently condition we prove the uniformly-bounded solution, and the attractivity global of the pseudo almost periodic solutions. Finally, an example is given to demonstrate our result.

* Corresponding author e-mail: manel.amdouni@hotmail.fr

This article is organized as follows. In section 2 we recall some basic definitions of the pseudo almost periodic functions. In section 3, we introduce some necessary notations, lemma which are used later, and we study the uniformly-bounded of the solutions. Section 4 is dedicated to prove the existence and uniqueness of the pseudo almost periodic solution from three theorems of fixed point. In the next section we study the attractivity global by new approach. Finally, an example is given to demonstrate the effectiveness of our results.

## 2 Pseudo almost periodic functions

This paragraph recalls some interesting properties of the pseudo almost periodic functions which is necessary for the study of the existence and uniqueness of the pseudo almost periodic solutions.

Let $BC(\mathbb{R},\mathbb{R})$ be the space of bounded continuous functions $f:\mathbb{R}\to\mathbb{R}$. Obviously, the space $BC(\mathbb{R},\mathbb{R})$ is equipped with the super norm

$$\|f\|_\infty = \sup_{t\in\mathbb{R}}|f(t)|,$$

is a Banach space.

**Definition 1.**[13] *A function $f\in BC(\mathbb{R},\mathbb{R})$ is called (Bohr) almost periodic if for each $\varepsilon>0$, there exists $l_\varepsilon>0$ such that every interval of length $l_\varepsilon$ contains at least a number $\tau$ with the following property:*

$$\sup_{t\in\mathbb{R}}|f(t+\tau)-f(t)|<\varepsilon.$$

The collection of all almost periodic functions $f:\mathbb{R}\to\mathbb{R}$ is denoted by $AP(\mathbb{R},\mathbb{R})$.

Besides, we define the class of functions $PAP_0(\mathbb{R},\mathbb{R})$ as follows

$$PAP_0(\mathbb{R},\mathbb{R}) = \left\{g\in BC(\mathbb{R},\mathbb{R}) : M[g]=0\right\},$$

where $M[g]=\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}|g(t)|dt$.

**Definition 2.**[14] *A function $f\in BC(\mathbb{R},\mathbb{R})$ is called pseudo almost periodic if it can be expressed as*

$$f=g+h$$

*where $g\in AP(\mathbb{R},\mathbb{R})$ and $h\in PAP_0(\mathbb{R},\mathbb{R})$.*

*The collection of such functions are denoted by $PAP(\mathbb{R},\mathbb{R})$.*

*Remark.* The functions $g$ and $h$ in the above definitions are respectively called the almost periodic and the ergodic perturbation components of the pseudo almost periodic function $f$. The decomposition given in definition above is unique. We observe that $(PAP(\mathbb{R},\mathbb{R}),\|.\|_\infty)$ is a Banach space.

*Example 1.* Consider the function defined by

$$f(t)=\sin t+\sin(\sqrt{2}t)+(1+t^2)^{-2},$$

for all $t\in\mathbb{R}$. It can be easily checked that the function $f$ is pseudo almost periodic. Indeed, the function $t\to\sin(t)+\sin(\sqrt{2}t)$ belongs to $AP(\mathbb{R},\mathbb{R})$ and the function $t\to(1+t^2)^{-2}$ is in $PAP_0(\mathbb{R},\mathbb{R})$.

**Lemma 1.**[16] *Suppose that $x_1(.)$, $r(.)\in AP(\mathbb{R},\mathbb{R}_+)$, $\dot r(.)\in BC(\mathbb{R},\mathbb{R}_+)$ and $x_2(.)\in PAP_0(\mathbb{R},\mathbb{R}_+)$, then*

*(1) $x_1(t-r(t))\in AP(\mathbb{R},\mathbb{R}_+)$.*
*(2) $x_2(t-r(t))\in PAP_0(\mathbb{R},\mathbb{R}_+)$, if $\inf_{t\in\mathbb{R}}(1-\dot r(t))>0$.*

## 3 Main assumptions and preliminary results

We consider the nonautonomous differential equation of the third order with multiple deviating arguments:

$$x^{(3)}(t)+a(t)x^{(2)}(t)+b(t)x^{(1)}(t)+\sum_{i=1}^{n}g_i(t,x(t-r_i(t))) = p(t), \tag{1}$$

where $a(.),r_i(.)$ are almost periodic functions and $p(.)$ is pseudo almost periodic function.

Throughout this paper, given a bounded continuous function $f$ defined on $\mathbb{R}$, let $\bar f$ and $\underline{f}$ be defined as

$$\bar f=\sup_t|f(t)|,\underline{f}=\inf_t|f(t)|.$$

Let us pose

$$x'(t)=y(t)-\alpha x(t),y'(t)=z(t)-\beta y(t)$$

where $\alpha,\beta$ are constants.

Then the equation (1) can be written as follows:

$$\begin{aligned}x'(t)&=y(t)-\alpha x(t)\\ y'(t)&=z(t)-\beta y(t)\\ z'(t)&=-(a(t)-\alpha-\beta)z(t)+[(\alpha+\beta)(a(t)-\alpha)-b(t)\\ &\quad-\beta^2]y(t)-[\alpha^2(a(t)-\alpha)-\alpha b(t)]x(t)\\ &\quad-\sum_{i=1}^{n}g_i(t,x(t-r_i(t)))+p(t).\end{aligned}\tag{2}$$

Let $\Lambda=(PAP(\mathbb{R},\mathbb{R}))^3$. Then, $\Lambda$ is a Banach space with the norm defined by

$$\|\omega\|_\Lambda=\sup_{t\in\mathbb{R}}\|\omega(t)\|=\sup_{t\in\mathbb{R}}\max_{1\le i\le 3}|\omega_i(t)|.$$

In order to establish our results let us consider the following conditions

$(H1)$ For all $1\le i\le n$, $r_i(t)$ is continuous differentiable on $t\in\mathbb{R}$, $\dot r_i(t)$ that is uniformly continuous on $\mathbb{R}$ with $\inf_{t\in\mathbb{R}}(1-\dot r_i(t))>0$.

$(H2)$ The function $g$ is global Lipschitz continuous, that is, there exists $L_g > 0$ such that for all $u, v \in \mathbb{R}$

$$|g_i(t,u) - g_i(t,v)| \leq L_g|u - v|.$$

$(H3)$ There exist constants $\alpha > 1, \beta > 1$ such that

$$0 < \sup_t \Big( |(\alpha+\beta)(a(t)-\alpha) - \beta^2 + \alpha^2(a(t)-\alpha) - b(t)(1+\alpha) - nL_g| \Big) < \inf_t\{a(t)-\alpha-\beta\},$$

which is denote by
$\mu = \frac{\sup_t(|(\alpha+\beta)(a(t)-\alpha)-\beta^2+\alpha^2(a(t)-\alpha)-b(t)(1+\alpha)-nL_g|)}{\inf_t\{a(t)-\alpha-\beta\}}$.

Now we establish some Lemmas in order to obtain our main results.

**Lemma 2.** *If $f(.) \in PAP(\mathbb{R},\mathbb{R})$, then the function $G : t \mapsto \int_{-\infty}^{t} e^{-(t-s)\alpha} f(s)ds$ is pseudo almost periodic function.*

*Proof.*First, the function $G$ satisfies

$$|G(t)| \leq \frac{1}{\alpha}\|f\|_\infty$$

which proves that the integral is well defined. Since $f$ is pseudo almost periodic function, one can write

$$f = f_1 + f_2,$$

where $f_1 \in AP(\mathbb{R},\mathbb{R})$ and $f_2 \in PAP_0(\mathbb{R},\mathbb{R})$. Hence

$$G(t) = \int_{-\infty}^{t} e^{-(t-s)\alpha} f_1(s)ds + \int_{-\infty}^{t} e^{-(t-s)\alpha} f_2(s)ds$$
$$= G_1(t) + G_2(t).$$

Let us prove the almost periodicity of $t \to G_1(t)$. $f_1 \in AP(\mathbb{R},\mathbb{R})$, then for all $\varepsilon > 0$ there exists a number $l_\varepsilon$ such that in any interval $[\rho, \rho + l_\varepsilon]$ $(\rho \in \mathbb{R})$ one finds a number $\tau$, with property

$$\sup_{t\in\mathbb{R}} |f_1(t+\tau) - f_1(t)| < \varepsilon.$$

We can write

$$\begin{aligned}
&| \ G_1(t+\tau) - G_1(t)| \\
&= |\int_{-\infty}^{t} f_1(s+\tau)e^{-(t-s)\alpha}ds - \int_{-\infty}^{t} f_1(s)e^{-(t-s)\alpha}ds| \\
&\leq \int_{\infty}^{t} |f_1(s+\tau) - f_1(s)|e^{-(t-s)\alpha}ds \\
&\leq \sup_{\xi} |f_1(\xi+\tau) - f_1(\xi)| \int_{\infty}^{t} e^{-(t-s)\alpha}ds < \frac{\varepsilon}{\alpha}.
\end{aligned}$$

Consequently, $G_1(.) \in AP(\mathbb{R},\mathbb{R})$. Next, we have to prove that $G_2 \in PAP_0(\mathbb{R},\mathbb{R})$, i.e.

$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} | \int_{\infty}^{t} e^{-(t-s)\alpha} f_2(s)ds|dt = 0.$

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} | \int_{\infty}^{t} e^{-(t-s)\alpha} f_2(s)ds|dt
&\leq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{\infty}^{t} e^{-(t-s)\alpha}|f_2(s)|dsdt \\
&\leq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{\infty}^{-T} e^{-(t-s)\alpha}|f_2(s)|dsdt \\
&\quad + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{t} e^{-(t-s)\alpha}|f_2(s)|dsdt \\
&= I_1 + I_2,
\end{aligned}$$

where

$$I_1 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{\infty}^{-T} e^{-(t-s)\alpha}|f_2(s)|dsdt,$$

$$I_2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{t} e^{-(t-s)\alpha}|f_2(s)|dsdt.$$

Now, we have to prove that $I_1 = I_2 = 0$.

$$\begin{aligned}
I_1 &\leq \|f_2\|_\infty \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{\infty}^{-T} e^{-(t-s)\alpha}dsdt \\
&\leq \|f_2\|_\infty \lim_{T\to\infty} \frac{1}{2T} \frac{e^{-T\alpha}}{\alpha} \int_{-T}^{T} e^{-t\alpha}dt \\
&= \|f_2\|_\infty \lim_{T\to\infty} \frac{1}{2T\alpha^2}[1 - e^{-2T\alpha}] = 0.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\tfrac{1}{2T} \int_{-T}^{T} \int_{-T}^{t} e^{-(t-s)\alpha}|f_2(s)|dsdt \\
&\leq \frac{1}{2T} \int_{-T}^{T} \int_{0}^{T+u} e^{-\alpha u}|f_2(t-u)|dudt \\
&\leq \frac{1}{2T} \int_{0}^{\infty} e^{-\alpha u}(\int_{-T-u}^{T+u} |f_2(s)|ds)du \\
&\leq \int_{0}^{\infty} e^{-\alpha u}(\frac{1}{2T} \int_{-T-u}^{T+u} |f_2(s)|ds)du.
\end{aligned}$$

Since $f_2 \in PAP_0(\mathbb{R},\mathbb{R})$, then

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T-u}^{T+u} |f_2(s)|ds = 0,$$

uniformly with respect to $u$. Finally, by the Lebesgue's dominated convergence Theorem, we obtain $I_2 = 0$. Consequently, $G$ belongs to $PAP(\mathbb{R},\mathbb{R})$.

**Lemma 3.** *Let $a(.) \in AP(\mathbb{R},\mathbb{R})$ and $f(.) \in PAP(\mathbb{R},\mathbb{R})$, then the function $F : t \mapsto \int_{-\infty}^{t} f(s)e^{-\int_s^t (a(u)-\alpha-\beta)du}ds$ is pseudo almost periodic function.*

*Proof.*Let us pose that $\psi(t) = a(t) - \alpha - \beta$. First, the function $F$ satisfies

$$|F(t)| \leq \frac{1}{\underline{\psi}}\|f\|_\infty < \infty.$$

Hence, $F$ is well defined. Since $f \in PAP(\mathbb{R},\mathbb{R})$, then one can write $f$ as follows $f = f_1 + f_2$, where $f_1 \in AP(\mathbb{R},\mathbb{R})$, and $f_2 \in PAP_0(\mathbb{R},\mathbb{R})$. Hence,

$$\begin{aligned} F(t) &= \int_{-\infty}^{t} f_1(s) e^{-\int_s^t (a(u)-\alpha-\beta)du} ds \\ &\quad + \int_{-\infty}^{t} f_2(s) e^{-\int_s^t (a(u)-\alpha-\beta)du} ds \\ &= F_1(t) + F_2(t). \end{aligned}$$

Let us prove that $F_1$ belongs to $AP(\mathbb{R},\mathbb{R})$. Since $f_1(.), a(.) \in AP(\mathbb{R},\mathbb{R})$, then for all $\varepsilon > 0$ there exists a number $l_\varepsilon$ such that in any interval $[\rho, \rho + l_\varepsilon]$ $(\rho \in \mathbb{R})$ one finds a number $\tau$, with property

$$\sup_{t\in\mathbb{R}} |f_1(t+\tau) - f_1(t)| < \frac{\varepsilon}{\underline{\psi}}, \quad \sup_{t\in\mathbb{R}} |a(t+\tau) - a(t)| < \varepsilon.$$

We can write

$$\begin{aligned} &| \ F_1(t+\tau) - F_1(t)| \\ &\leq \int_{-\infty}^{t} |f_1(s+\tau)||e^{-\int_s^t \psi(u+\tau)du} - e^{-\int_s^t \psi(u)du}| \\ &\quad + \int_{-\infty}^{t} |f_1(s+\tau) - f_1(s)| e^{-\int_s^t \psi(u)du} ds \\ &\leq \frac{\varepsilon}{\underline{\psi}} \int_{-\infty}^{t} e^{-(t-s)\underline{\psi}} ds \\ &\quad + \|f_1\|_\infty \int_{-\infty}^{t} e^{-\int_s^t \psi(u)du} |e^{-\int_s^t (\psi(u+\tau)-\psi(u))du} - 1| \\ &\leq \varepsilon + \|f_1\|_\infty \int_{-\infty}^{t} e^{-(t-s)\underline{\psi}} |e^{-\int_s^t (\psi(u+\tau)-\psi(u))du} - 1| \\ &< \varepsilon. \end{aligned}$$

Consequently, $F_1(.) \in AP(\mathbb{R},\mathbb{R})$. Let us study the ergodicity of $F_2(.)$.

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\int_{\infty}^{t} e^{-\int_s^t \psi(u)du} f_2(s) ds| dt \\ &\leq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{\infty}^{t} e^{-(t-s)\underline{\psi}} |f_2(s)| ds dt \\ &\leq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{\infty}^{-T} e^{-(t-s)\underline{\psi}} |f_2(s)| ds dt \\ &\quad + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{t} e^{-(t-s)\underline{\psi}} |f_2(s)| ds dt \leq I_1 + I_2, \end{aligned}$$

where

$$I_1 = \lim_{T\to\infty} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{\infty}^{-T} e^{-(t-s)\underline{\psi}} |f_2(s)| ds dt,$$

$$I_2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{t} e^{-(t-s)\underline{\psi}} |f_2(s)| ds dt.$$

Now, we shall prove that $I_1 = I_2 = 0$.

$$\begin{aligned} I_1 &\leq \|f_2\|_\infty \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{\infty}^{-T} e^{-(t-s)\underline{\psi}} ds dt \\ &\leq \|f_2\|_\infty \lim_{T\to\infty} \frac{1}{2T} \frac{e^{-T\underline{\psi}}}{\underline{\psi}} \int_{-T}^{T} e^{-t\underline{\psi}} dt \\ &= \|f_2\|_\infty \lim_{T\to\infty} \frac{1}{2T\underline{\psi}^2} [1 - e^{-2T\underline{\psi}}] = 0. \end{aligned}$$

On the other hand

$$\begin{aligned} &\tfrac{1}{2T} \int_{-T}^{T} \int_{0}^{T+u} e^{-u\underline{\psi}} |f_2(t-u)| du dt \\ &\leq \frac{1}{2T} \int_{0}^{\infty} e^{-\underline{\psi}u} \Big( \int_{-T-u}^{T+u} |f_2(s)| ds \Big) du \\ &\leq \int_{0}^{\infty} e^{-\underline{\psi}u} \Big( \frac{1}{2T} \int_{-T-u}^{T+u} |f_2(s)| ds \Big) du. \end{aligned}$$

Since $f_2 \in PAP_0(\mathbb{R},\mathbb{R})$, then

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T-u}^{T+u} |f_2(s)| ds = 0,$$

uniformly with respect to $u$. Finally, by the Lebesgue's dominated convergence Theorem, we obtain $I_2 = 0$. Consequently, $F$ belongs to $PAP(\mathbb{R},\mathbb{R})$.

### *3.1 Uniformly-bounded*

**Theorem 1.** *Assume that $(H1) - (H3)$ are satisfied, then all the solutions of equation (2) are uniformly bounded. In other words, there exists $M > 0$ such that*

$$\|X(t)\|_\Lambda \leq M.$$

*Proof.*In order to prove the inequalities as above, let us prove that for every $h > 1$, and for each $t \geq 1$

$$\|X(t)\|_\Lambda < hM.$$

Let us prove that by contradiction. Suppose there exists $t' > 0$, such that

$$\begin{cases} \|X(t')\|_\Lambda = hM, \\ \|X(t)\|_\Lambda < hM, 0 \leq t < t'. \end{cases}$$

We have to prove this result in three cases.

Case1.$\|X(t')\|_\Lambda = |x(t')|$. Then

$$\begin{aligned} hM = |x(t')| &= \left| [x(0) + \int_0^{t'} y(s) e^{\alpha s} ds] e^{-\alpha t'} \right| \\ &\leq hM[1 + \frac{1}{\alpha}(e^{\alpha t} - 1)] e^{-\alpha t'} < hM, \end{aligned}$$

which gives a contradiction. Consequently, for all $t \geq 0$, $\|X(t)\|_\Lambda \leq hM$. Let us take $h \to 1$, then $\|X(t)\|_\Lambda \leq M$.

Case2.$\|X(t')\|_\Lambda = |y(t')|$.

$y(t') = [y(0) + \int_0^{t'} z(s) e^{\beta s} ds] e^{-\beta t'}$, similarly of case 1, we obtain $\|X(t)\|_\Lambda \leq M$.

Case3.$\|X(t')\|_\Lambda = |z(t')|$. Then

$$\begin{aligned} hM &= |z(t')| \\ &= \Big|[z(0) + \int_0^{t'} e^{\int_0^s \psi(u)du}\Big[((\alpha+\beta)(a(s) \\ &\quad -\alpha) - \beta^2 - b(s))y(s) \\ &\quad -(\alpha^2(a(s)-\alpha) - \alpha b(s))x(s) \\ &\quad -\sum_{i=1}^n g_i(s, x(s-r_i(s))) + p(s)\Big]ds\Big| e^{-\int_0^{t'} \psi(u)du} \\ &\leq hMe^{-t'\underline{\psi}}[1 + \sup_t\Big|(\alpha+\beta)(a(t)-\alpha) \\ &\quad -\beta^2 - b(t) + (\alpha^2(a(t)-\alpha) - \alpha b(t)) \\ &\quad -nL_g\Big|\int_0^{t'} e^{s\underline{\psi}}ds] \\ &\leq hMe^{-t'\underline{\psi}}[1 + \mu(e^{t'\underline{\psi}} - 1)] \\ &< hM, \end{aligned}$$

which gives a contradiction. Consequently, for all $t \geq 0$, $\|X(t)\|_\Lambda \leq M$.

## 4 Existence of the pseudo almost periodic solutions

In this section, under new sufficient conditions we prove the existence of the pseudo almost periodic solution of (2). In our study we will use different fixed point theorems.

### *4.1 Banach's Fixed point Theorem*

**Theorem 2.** *Assume that $(H1)-(H3)$ hold, then the system (2) has a unique pseudo almost periodic solution $X^*(t)$.*

*Proof.*Let us consider the operator $T$ defined by, for all $(x,y,z) \in \Lambda$,

$$T(x,y,z)(t) = \Big(\int_{-\infty}^t y(s)e^{-(t-s)\alpha}ds, \int_{-\infty}^t z(s)e^{-(t-s)\beta}ds, \int_{-\infty}^t \varphi(s)e^{-\int_s^t \psi(u)du}ds\Big),$$

where

$$\varphi(t) = p(t) + [(\alpha+\beta)(a(t)-\alpha) - \beta^2 - b(t)]y(t) -[\alpha^2(a(t)-\alpha) - \alpha b(t)]x(t) - \sum_{i=1}^n g_i(t, x(t-r_i(t)))$$

In virtue of Lemma (2) and (3) the operator $T$ is a mapping of $\Lambda$ into itself. Now, we have to prove that the operator $T$ is a contraction. For $X = (x,y,z), V = (u,v,w) \in \Lambda$, we have

$$TV(t) = \Big(\int_{-\infty}^t v(s)e^{-(t-s)\alpha}ds, \int_{-\infty}^t w(s)e^{-(t-s)\beta}ds, \int_{-\infty}^t \phi(s)e^{-\int_s^t \psi(u)du}ds\Big)$$

where

$$\begin{aligned} \phi(t) &= p(t) + [(\alpha+\beta)(a(t)-\alpha) - \beta^2 - b(t)]v(t) \\ &\quad -[\alpha^2(a(t)-\alpha) - \alpha b(t)]u(t) \\ &\quad -\sum_{i=1}^n g_i(t, u(t-r_i(t))), \end{aligned}$$

and

$$\begin{aligned} \|TX - TV\|_\Lambda &= \sup_t \max\Big|\Big(\int_{-\infty}^t (y(s)-v(s))e^{-(t-s)\alpha}ds\Big|, \\ &\quad \int_{-\infty}^t (z(s)-w(s))e^{-(t-s)\beta}ds, \\ &\quad \int_{-\infty}^t (\varphi(s)-\phi(s))e^{-\int_s^t \psi(u)du}ds\Big)\Big|. \end{aligned}$$

Case 1.

$$\begin{aligned} \|TX - TV\|_\Lambda &= \sup_t \int_{-\infty}^t |y(s)-v(s)|e^{-(t-s)\alpha}ds \\ &\leq \|X-V\|_\Lambda \sup_t \int_{-\infty}^t e^{-(t-s)\alpha}ds \\ &\leq \frac{1}{\alpha}\|X-V\|_\Lambda. \end{aligned}$$

Consequently,

$$\|TX - TV\|_\Lambda \leq \frac{1}{\alpha}\|X-V\|_\Lambda.$$

Case 2.

$$\|TX - TV\|_\Lambda = \sup_t \int_{-\infty}^t |z(s)-w(s)|e^{-(t-s)\beta}ds.$$

Similarly of case 1, we obtain

$$\|TX - TV\|_\Lambda \leq \frac{1}{\beta}\|X-V\|_\Lambda.$$

Case 3.

$$\begin{aligned} \|TX - TV\|_\Lambda &= \sup_t \int_{-\infty}^t e^{-\int_s^t \psi(u)du}\Big|\varphi(s)-\phi(s)\Big|ds \\ &\leq \|X-V\|_\Lambda(\sup_\xi\Big|(\alpha+\beta)(a(\xi)-\alpha) \\ &\quad -\beta^2 - b(\xi)(1+\alpha) \\ &\quad +\alpha^2(a(\xi)-\alpha) - nL_g\Big|)\int_{-\infty}^t e^{-(t-s)\underline{\psi}}ds \\ &\leq \mu\|X-V\|_\Lambda. \end{aligned}$$

Which proves that $T$ is a contraction. Consequently, $T$ has a unique fixed point $X^* \in \Lambda$.

### *4.2 Schauder's, Leray Schauder's fixed point Theorem*

Now we introduce the Theorem of Schauder and Leray-Schauder fixed point which are used to prove the existence of a pseudo almost periodic solution.

Let us recall the Schauder's second Theorem.

**Theorem 3.***[23] Let $\mathscr{M}$ be a non-empty convex subset of a normed space $\mathscr{B}$. Let $T$ be a continuous mapping of $\mathscr{M}$ into a compact set $\mathscr{H} \subset \mathscr{M}$. Then $T$ has a fixed point.*

Let us denote by $K = \max(\frac{1}{\alpha}, \frac{1}{\beta}, \mu)$.

**Theorem 4.** *Suppose that (H1)-(H3) hold and if the operator $T : B \to B$, where $B = \{X \in \Lambda : \|X\|_\Lambda \le M\}$ is continuous and compact operator. Then the equation (2) has a fixed pseudo almost periodic solution.*

*Proof.*Let $B = \{X \in \Lambda : \|X\|_\Lambda \le M\}$, clearly that $B$ is a closed convex subset of $\Lambda$. For all $(x,y,z) \in \Lambda$, $t \in \mathbb{R}$

$$T(x,y,z)(t) = \Big(\int_{-\infty}^{t} y(s)e^{-(t-s)\alpha}ds, \int_{-\infty}^{t} z(s)e^{-(t-s)\beta}ds, \int_{-\infty}^{t} \varphi(s)e^{-\int_s^t \psi(u)du}\Big).$$

It is clear that, by Lemma (2) and (3) the operator $T$ is a pseudo almost periodic function.

Now, we have to prove that $T$ is continuous. Let $(X_n)_n$ be a sequence of $\Lambda$, such that $X_n \to X$ in $\Lambda$ as $n \to \infty$, i.e. $x_n \to x$, $y_n \to y$ and $z_n \to z$ as $n \to \infty$. Given $\delta > 0$, there exists $N$ such that, for $n \ge N$ we have $\|X_n - X\|_\Lambda < \delta$. Hence, for $t \in \mathbb{R}$;

$$\begin{aligned}|(TX_n)(t) - (TX)(t)| = |\Big(&\int_{-\infty}^{t}(y_n(s) - y(s))e^{-(t-s)\alpha}ds,\\ &\int_{-\infty}^{t}(z_n(s) - z(s))e^{-(t-s)\beta}ds,\\ &\int_{-\infty}^{t}(\varphi_n(s) - \varphi(s))e^{-\int_s^t \psi(u)du}\Big)|\end{aligned}$$

we have

$$\begin{aligned}&\int_{-\infty}^{t}|y_n(s) - y(s)|e^{-(t-s)\alpha}ds \le \frac{1}{\alpha} \le K\|X_n - X\|_\Lambda,\\ &\int_{-\infty}^{t}|z_n(s) - z(s)|e^{-(t-s)\beta}ds \le \frac{1}{\beta} \le K\|X_n - X\|_\Lambda,\\ &\int_{-\infty}^{t}|\varphi_n(s) - \varphi(s)|e^{-\int_s^t \psi(u)du} \le \mu \le K\|X_n - X\|_\Lambda.\end{aligned}$$

Clearly, that $|(TX_n)(t) - (TX)(t)| \le K\|X_n - X\|_\Lambda$. Consequently, $TX_n \to TX$ as $n \to \infty$ which follows that $T$ is continuous.

Now, we have to show that $T(B) \subset B$. For $X \in B$ and $t \in \mathbb{R}$,

$$\begin{aligned}(TX)(t) = \Big(&\int_{-\infty}^{t} y(s)e^{-(t-s)\alpha}ds,\\ &\int_{-\infty}^{t} z(s)e^{-(t-s)\beta}ds, \int_{-\infty}^{t}\varphi(s)e^{-\int_s^t \psi(u)du}\Big).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\int_{-\infty}^{t}|y(s)|e^{-(t-s)\alpha}ds \le \frac{1}{\alpha}\|X\|_\Lambda \le \|X\|_\Lambda\\ &\int_{-\infty}^{t}|z(s)|e^{-(t-s)\beta}ds \le \frac{1}{\beta}\|X\|_\Lambda \le \|X\|_\Lambda,\end{aligned}$$

and

$$\begin{aligned}&\int_{-\infty}^{t}|\varphi(s)|e^{-\int_s^t \psi(u)du}\\ &= \int_{-\infty}^{t}|-p(s) - [(\alpha+\beta)(a(s)-\alpha) - \beta^2 - b(s)]y(s)\\ &\quad +[\alpha^2(a(s)-\alpha) - \alpha b(s)]x(s)\\ &\quad +\sum_{i=1}^{n} g_i(s, x(s - r_i(s)))|e^{-\int_s^t \psi(u)du}ds\\ &\le \int_{-\infty}^{t}|-[(\alpha+\beta)(a(s)-\alpha) - \beta^2 - b(s)]y(s)\\ &\quad +[\alpha^2(a(s)-\alpha) - \alpha b(s)]x(s)\\ &\quad +\sum_{i=1}^{n} g_i(s, x(s - r_i(s)))|e^{-\int_s^t \psi(u)du}ds\\ &\le \int_{-\infty}^{t}|[(\alpha+\beta)(a(s)-\alpha) - \beta^2 - b(s)](-y(s))\\ &\quad +[\alpha^2(a(s)-\alpha) - \alpha b(s)]x(s)\\ &\quad +\sum_{i=1}^{n} g_i(s, x(s - r_i(s))) - g_i(0,0)|e^{-\int_s^t \psi(u)du}ds\\ &\le \int_{-\infty}^{t}|[(\alpha+\beta)(a(s)-\alpha) - \beta^2 - b(s)](-y(s))\\ &\quad +[\alpha^2(a(s)-\alpha) - \alpha b(s)]x(s)\\ &\quad +\sum_{i=1}^{n} L_g|x(s - r_i(s))|e^{-\int_s^t \psi(u)du}ds\\ &\le \int_{-\infty}^{t}|[(\alpha+\beta)(a(s)-\alpha) - \beta^2 - b(s)]y(s)\\ &\quad +[\alpha^2(a(s)-\alpha) - \alpha b(s)](-x(s))\\ &\quad -nL_g\|X\|_\Lambda|e^{-\int_s^t \psi(u)du}ds\\ &\le \|X\|_\Lambda \sup_{\xi}\big|[(\alpha+\beta)(a(\xi)-\alpha) - \beta^2 - b(\xi)]\\ &\quad +\alpha^2(a(\xi)-\alpha) - \alpha b(\xi) - nL_g|\int_{-\infty}^{t}|e^{-\int_s^t \psi(u)du}ds\\ &\le \mu\|X\|_\Lambda \le \|X\|_\Lambda.\end{aligned}$$

Therefore

$$|(TX)(t)| \le \|X\|_\Lambda.$$

Consequently, $|(TX)(t)| \le M$. Then $T$ is a self mapping.

Now we have to prove the following

1. $\{(TX)(t) : X \in B\}$ is a relatively compact subset of $\Lambda$ for each $t \in \mathbb{R}$.
2. $\{TX : X \in B\}$ is equi-continuous.

1. Let $Y_n(t) = (TX_n)(t)$ be a sequence of $\{(TX)(t) : X \in B\}$. $X_n \in B$, i.e. $X_n \in \Lambda$ and $\|X_n\|_\Lambda \le M$. Therefore $(X_n)_n$ is a bounded sequence, then there exists a subsequence $(X_{n_k})$ of $(X_n)$ in $B$, such that $X_{n_k} \to X$ as $n_k \to \infty$ in $B$. Since $T$ is continuous, then $TX_{n_k} \to TX$ as $n_k \to \infty$ in $B$, i.e. $\sup_t \max_{1\le i\le 3}|(TX_{n_k})(t) - (TX)(t)| < \varepsilon$. Then there

exists a subsequence $Y_{n_k}(t)$ of $Y_n(t)$ such that $Y_{n_k}(t) \to Y(t)$ as $n_k \to \infty$. Consequently, $\{(TX)(t) : X \in B\}$ is relatively compact in $\Lambda$.

2.For $X \in B$, such that $t_1 < t_2$ and $|t_2 - t_1| < \delta$. Let

$$\begin{aligned}
&(TX)(t_2) - (TX)(t_1)\\
&= ((e^{-(t_2-t_1)\alpha} - 1)\int_{-\infty}^{t_1} y(s)e^{-(t_1-s)\alpha}ds\\
&+e^{-(t_2-t_1)\alpha}\int_{t_1}^{t_2} y(s)e^{-(t_1-s)\alpha}ds,\\
&(e^{-(t_2-t_1)\beta} - 1)\int_{-\infty}^{t_1} z(s)e^{-(t_1-s)\beta}ds\\
&+e^{-(t_2-t_1)\beta}\int_{t_1}^{t_2} z(s)e^{-(t_1-s)\beta}ds\\
&(e^{-\int_{t_1}^{t_2}\psi(u)du} - 1)\int_{-\infty}^{t_1} \varphi(s)e^{-\int_s^{t_1}\psi(u)du}\\
&+\int_{t_1}^{t_2} \varphi(s)e^{-\int_s^{t_1}\psi(u)du})
\end{aligned}$$

We have to show the result by term:

$$\begin{aligned}
&\Big|\ ( e^{-(t_2-t_1)\alpha} - 1)\int_{-\infty}^{t_1} y(s)e^{-(t_1-s)\alpha}ds\\
&+e^{-(t_2-t_1)\alpha}\int_{t_1}^{t_2} y(s)e^{-(t_1-s)\alpha}ds\Big|\\
&\le 2\|X\|_\Lambda \frac{|1-e^{-(t_2-t_1)\alpha}|}{\alpha}.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\Big|\ ( e^{-(t_2-t_1)\beta} - 1)\int_{-\infty}^{t_1} z(s)e^{-(t_1-s)\beta}ds\\
&+e^{-(t_2-t_1)\beta}\int_{t_1}^{t_2} z(s)e^{-(t_1-s)\beta}ds\Big|\\
&\le 2\|X\|_\Lambda \frac{|1-e^{-(t_2-t_1)\beta}|}{\beta}.
\end{aligned}$$

Moreover

$$\begin{aligned}
&\Big|(e^{-\int_{t_1}^{t_2}\psi(u)du} - 1)\int_{-\infty}^{t_1} \varphi(s)e^{-\int_s^{t_1}\psi(u)du}\\
&+\int_{t_1}^{t_2} \varphi(s)e^{-\int_s^{t_2}\psi(u)du}\Big|\\
&\le \frac{|1-e^{-(t_2-t_1)\underline{\psi}}|}{\underline{\psi}}\|\varphi\|_\Lambda \int_{-\infty}^{t_1} e^{-(t_1-s)\underline{\psi}}\\
&+\|\varphi\|_\Lambda\Big|\int_{t_1}^{t_2} e^{-(t_2-s)\underline{\psi}}\Big|\\
&\le 2|\frac{1-e^{-(t_2-t_1)\underline{\psi}}}{|}\underline{\psi}\|\varphi\|_\Lambda.
\end{aligned}$$

For $|t_2 - t_1| < \delta$, we have the following estimates

$$\frac{|1-e^{-(t_2-t_1)\alpha}|}{\alpha} < \frac{\varepsilon}{2\|X\|_\Lambda},$$
$$\frac{|1-e^{-(t_2-t_1)\beta}|}{\beta} < \frac{\varepsilon}{2\|X\|_\Lambda},$$
$$\frac{1-e^{-(t_2-t_1)\underline{\psi}}}{\underline{\psi}} < \frac{\varepsilon}{2\|\varphi\|_\Lambda}.$$

Hence, $|(TX)(t_2) - (TX)(t_1)| < \varepsilon$ which shows that $\{TX : X \in B\}$ is equicontinuous.

Consequently, $T : B \to B$ is relatively compact.

Next, we have to prove that $T$ is compact. Denote by $\bar{co}T(B)$ the closed convex of $T(B)$. Since $T(B) \subset B$ and $B$ is closed convex, $\bar{co}T(B) \subset B$. Further $T(\bar{co}T(B)) \subset T(B) \subset \bar{co}T(B)$.

Clearly that $\{(TX)(t) : X \in \bar{co}T(B)\}$ is relatively compact in $\Lambda$ for every $t \in \mathbb{R}$, and $\bar{co}T(B)$ is uniformly bounded and equicontinuous. By the Arzela Ascoli theorem the restriction of $\bar{co}T(B)$ to every compact subset $K$ of $\mathbb{R}$, i.e. $\{(TX)(t) : X \in \bar{co}T(B)\}_{t\in K}$ is relatively compact in $\Lambda$. Then $T : \bar{co}T(B) \to \bar{co}T(B)$ is a compact operator, by Schauder's fixed point Theorem $T$ has a fixed point $X$.

**Lemma 4.***(Leray-Schauder Alternative Theorem,[20]) Let $D$ be a closed convex subset of a Banach space $X$ such that $0 \in D$. Let $\mathfrak{F} : D \to D$ be a completely continuous map. Then the set $\{x \in D : x = \lambda\mathfrak{F}(x), 0 < \lambda < 1\}$ is unbounded or the map $\mathfrak{F}$ has a fixed point in $D$.*

**Theorem 5.** *Suppose that (H1)-(H3) hold and if the operator $T : \Lambda \to \Lambda$ is completely continuous. Then the equation (2) has a fixed pseudo almost periodic solution.*

*Proof.*For all $(x,y,z) \in \Lambda$, $t \in \mathbb{R}$ the operator $T$ is defined as follows

$$\begin{aligned}
T(x,y,z)(t) &= (\int_{-\infty}^{t} y(s)e^{-(t-s)\alpha}ds, \int_{-\infty}^{t} z(s)e^{-(t-s)\beta}ds,\\
&\int_{-\infty}^{t} \varphi(s)e^{-\int_s^t \psi(u)du}).
\end{aligned}$$

It is clear that, by Lemma (2) and (3) the operator $T$ is a mapping of $\Lambda$ into itself. Then $T$ is a mapping of $(BC(\mathbb{R},\mathbb{R}))^3$ into itself.

Now, we have to prove that $T$ is continuous. Let $(X_n)_n$ be a sequence of $\Lambda$, such that $X_n \to X$ in $\Lambda$ as $n \to \infty$, i.e. $x_n \to x$, $y_n \to y$ and $z_n \to z$ as $n \to \infty$. Given $\delta > 0$, there exists $N$ such that, for $n \ge N$ we have $\|X_n - X\|_\Lambda < \delta$. Hence, for $t \in \mathbb{R}$;

$$|(TX_n)(t) - (TX)(t)| \le K\|X_n - X\|_\Lambda.$$

Consequently, $TX_n \to TX$ as $n \to \infty$ which follows that $T$ is continuous. Next, let us prove that $T$ is completely continuous. Let $B(0,r)$ be the closed ball with center 0 and radius r in the space $(BC(\mathbb{R},\mathbb{R}))^3$. Let $V = T(B(0,r))$ and $v = TX$ for $X \in B(0,r)$. We have to prove that $V$ is relatively compact, and we prove this in two steps.

Step1:$V(t)$ is relatively compact subset of $(BC(\mathbb{R},\mathbb{R}))^3$ for each $t\in\mathbb{R}$.

$$|v(t)| = |(TX)(t)| = \left|\left(\int_{-\infty}^t y(s)e^{-(t-s)\alpha}ds,\right.\right.$$
$$\int_{-\infty}^t z(s)e^{-(t-s)\beta}ds,$$
$$\left.\left.\int_{-\infty}^t \varphi(s)e^{-\int_s^t \psi(u)du}ds\right)\right|.$$

Similarly, by the proof of theorem (4), we have

$$|v(t)| \le K\|X\|_\Lambda < \|X\|_\Lambda,$$

then, $|v(t)|\le r$. Consequently, $v(t)\in B(0,r)$. Hence, $V(t)$ is relatively compact.

Step2:$V$ is equi-continuous. In fact, for each $\varepsilon>0$, $h>0$ $\|X(t+h)-X(t)\|<\frac{\varepsilon}{K}$

$$v(t+h)-v(t) = (\int_{-\infty}^t (y(s+h)-y(s))e^{-(t-s)\alpha}ds,$$
$$\int_{-\infty}^t (z(s+h)-z(s))e^{-(t-s)\beta}ds,$$
$$\int_{-\infty}^t (\varphi(s+h)-\varphi(s))e^{-\int_s^t \psi(u)du}ds),$$

hence,

$$|v(t+h)-v(t)| \le K\|X(t+h)-X(t)\| < \varepsilon.$$

(i)Now, we have to prove that, the set $E=\{u^\lambda : u^\lambda = \lambda Tu^\lambda, \lambda\in(0,1)\}$ is bounded. If $u^\lambda$ is a solution of the equation $u^\lambda=\lambda Tu^\lambda$ for some $0<\lambda<1$, then

$$\begin{aligned}|u^\lambda(t)| &= |\lambda Tu^\lambda|\\ &\le \lambda|Tu^\lambda|\\ &< |Tu^\lambda|\\ &\le K\|X\|_\Lambda\\ &\le Kr.\end{aligned}$$

Hence $\|u^\lambda\|\le Kr$, we conclude that the set $E=\{u^\lambda : u^\lambda=\lambda Tu^\lambda, \lambda\in(0,1)\}$ is bounded.

T is a mapping of $\Lambda$ into itself, and $T:\bar{\Lambda}\to\bar{\Lambda}$ is completely continuous. By the above Lemma and (i), $T$ has a fixed point $X\in\bar{\Lambda}$. Let $(X_n)$ be a sequence in $\Lambda$ such that it converge to $X\in(BC(\mathbb{R},\mathbb{R}))^3$. For $\varepsilon>0$, let $\eta>0$, there exists $n_0\in\mathbb{N}$ such that $\|X_n-X\|\le\eta$ for all $n\ge n_0$. For $n\ge n_0$;

$$\|TX_n-TX\|_\Lambda \le K\|X_n-X\|.$$

Consequently $(TX_n)_n$ converge to $TX=X$ uniformly, which implies that $X\in\Lambda$.

### 4.3 Krasnoselskii's fixed point Theorem

**Theorem 6.**[21] *Let $\Omega$ be a closed convex nonempty subset of a Banach space $(S,\|.\|)$, Suppose that $A$ and $B$ map $\Omega$ into $S$ such that*

*–$Ax+By\in\Omega$ ($\forall x,y\in\Omega$).*
*–$A$ is continuous on $\Omega$ and $A(\Omega)$ is a relatively compact subset of $S$.*
*–$B$ is a contraction mapping.*

*Then there exists $y\in\Omega$ such that $Ay+By=y$.*

**Theorem 7.** *Suppose that (H1)-(H3) are satisfied, then the operator $T:\Omega\to\Omega$ has a fixed pseudo almost periodic solution, where $\Omega=\{X\in\Lambda; \|X\|_\Lambda\le M\}$.*

*Proof.*Clearly, that $\Omega$ is a convex subset of $\Lambda$. For all $(x,y,z)\in\Lambda$, $t\in\mathbb{R}$ the operator $T$ is defined as follows

$$T(x,y,z)(t) = (\int_{-\infty}^t y(s)e^{-(t-s)\alpha}ds, \int_{-\infty}^t z(s)e^{-(t-s)\beta}ds,$$
$$\int_{-\infty}^t \varphi(s)e^{-\int_s^t \psi(u)du}).$$

It is clear that, by Lemma (2) and (3) the operator $T$ is a pseudo almost periodic function. Besides, one can write $T$ as follows, for $X\in\Omega$, and for $t\in\mathbb{R}$

$$(TX)(t) = (AX)(t)+(BX)(t),$$

where

$$A(x,y,z)(t) = (\int_{-\infty}^t y(s)e^{-(t-s)\alpha}ds, \int_{-\infty}^t z(s)e^{-(t-s)\beta}ds, 0)$$
$$\begin{aligned}B(x,y,z)(t) = (0,0,\int_{-\infty}^t e^{-\int_s^t \psi(u)du}\Big[&((\alpha+\beta)(a(s)-\alpha)\\ &-\beta^2-b(s))y(s)\\ &-(\alpha^2(a(s)-\alpha)-\alpha b(s))x(s)\\ &-\sum_{i=1}^n g_i(s,x(s-r_i(s)))+p(s)\Big]ds).\end{aligned}$$

Let us prove that $AX+BY\in\Omega,\forall X,Y\in\Omega$. For $t\ge0$,

$$Z(t) = (AX)(t)+(BY)(t).$$

Hence,

$$\begin{aligned}\|Z\|_\Lambda = \sup_t\max|(&\int_{-\infty}^t y(s)e^{-(t-s)\alpha}ds, \int_{-\infty}^t z(s)e^{-(t-s)\beta}ds,\\ &\int_{-\infty}^t e^{-\int_s^t \psi(u)du}\Big[((\alpha+\beta)(a(s)-\alpha)\\ &-\beta^2-b(s))v(s)-(\alpha^2(a(s)-\alpha)-\alpha b(s))u(s)\\ &-\sum_{i=1}^n g_i(s,u(s-r_i(s)))+p(s)\Big]ds)|\end{aligned}$$

Since

$$\int_{-\infty}^t |y(s)|e^{-(t-s)\alpha}ds \le \frac{\|X\|_\Lambda}{\alpha},$$
$$\int_{-\infty}^t |z(s)|e^{-(t-s)\beta}ds \le \frac{\|X\|_\Lambda}{\beta},$$

and

$$\int_{-\infty}^{t} e^{-\int_s^t \psi(u)du} \Big| ((\alpha+\beta)(a(s)-\alpha)-\beta^2-b(s))v(s) -(\alpha^2(a(s)-\alpha)-\alpha b(s))u(s) -\sum_{i=1}^{n} g_i(s,u(s-r_i(s)))+p(s)\Big| ds \leq \|Y\|_\Lambda \mu,$$

hence, we need to prove the inequality desired in three steps.

–If $\max(\frac{\|X\|_\Lambda}{\alpha}, \frac{\|X\|_\Lambda}{\beta}, \|Y\|_\Lambda \mu) = \frac{\|X\|_\Lambda}{\alpha}$, then

$$|Z(t)| \leq \frac{\|X\|_\Lambda}{\alpha} < \|X\|_\Lambda \leq M.$$

–If $\max(\frac{\|X\|_\Lambda}{\alpha}, \frac{\|X\|_\Lambda}{\beta}, \|Y\|_\Lambda \mu) = \frac{\|X\|_\Lambda}{\beta}$, we give

$$|Z(t)| \leq \frac{\|X\|_\Lambda}{\beta} < \|X\|_\Lambda \leq M.$$

–If $\max(\frac{\|X\|_\Lambda}{\alpha}, \frac{\|X\|_\Lambda}{\beta}, \|Y\|_\Lambda \mu) = \|Y\|_\Lambda \mu$, we obtain

$$|Z(t)| \leq \|Y\|_\Lambda \mu < \|Y\|_\Lambda \leq M$$

Consequently, $\|Z\|_\Lambda \leq M$.

Now we have to prove that $Z \in \Lambda$. For $t \geq 0$, $Z(t) = (AX)(t) + (BY)(t)$, from Lemma (2), the function $t \to (\int_{-\infty}^t y(s)e^{-(t-s)\alpha}ds, \int_{-\infty}^t z(s)e^{-(t-s)\beta}ds)$ belongs to $PAP(\mathbb{R},\mathbb{R})$, then $(AX)(.)$ belongs to $\Lambda$, and from Lemma (3) the function $t \to \int_{-\infty}^t e^{-\int_s^t \psi(u)du}\Big[((\alpha+\beta)(a(s)-\alpha)-\beta^2-b(s))v(s)-(\alpha^2(a(s)-\alpha)-\alpha b(s))u(s)-\sum_{i=1}^n g_i(s,u(s-r_i(s)))+p(s)\Big]ds$ belongs to $PAP(\mathbb{R},\mathbb{R})$, hence $(BY)(.)$ belongs to $\Lambda$. Besides, $Z \in \Omega$.

Now, we have to prove $B$ is a contraction mapping. Let $X=(x,y,z), U=(u,v,w) \in \Omega$

$$(BX)(t)-(BU)(t) = (0,0,\int_{-\infty}^t e^{-\int_s^t \psi(u)du}(\varphi(s)-\phi(s))ds).$$

Since

$$|(BX)(t)-(BU)(t)\| = \sup_t \max|(0,0, \int_{-\infty}^t e^{-\int_s^t \psi(u)du}(\varphi(s)-\phi(s))ds)|,$$

then

$$\|(BX)(t)-(BU)(t)\| = \sup_t |\int_{-\infty}^t e^{-\int_s^t \psi(u)du}(\varphi(s)-\phi(s))ds|.$$

Consequently,

$$\begin{aligned}&| \ (BX)(t)-(BU)(t)| \\ &\leq \|X-U\|_\Omega[\sup_\xi|((\alpha+\beta)(a(\xi)-\alpha)-\beta^2-b(\xi)) \\ &\quad +(\alpha^2(a(\xi)-\alpha)-\alpha b(\xi))-nL_g|]\int_{-\infty}^t e^{-(t-s)\underline{\psi}}ds.\end{aligned}$$

By hypothesis (P4.2), we have $|(BX)(t)-(BU)(t)| < \|X-U\|_\Omega$.

Then $B$ is a contraction mapping.

Now, we have to see that $A$ is continuous. Fix $X=(x,y,z), U=(u,v,w) \in \Omega$ with $\|X-U\|_\Omega < \eta$, $\|X\|_\Omega \leq M$. Then, for $t \geq 0$ we have

$$\begin{aligned}&| \ (AX)(t)-(AU)(t)| \\ &= |(\int_{-\infty}^t y(s)e^{-(t-s)\alpha}ds - \int_{-\infty}^t v(s)e^{-(t-s)\alpha}ds, \\ &\quad \int_{-\infty}^t z(s)e^{-(t-s)\beta}ds - \int_{-\infty}^t w(s)e^{-(t-s)\beta}ds, 0))| \\ &= |(\int_{-\infty}^t (y(s)-v(s))e^{-(t-s)\alpha}ds, \\ &\quad \int_{-\infty}^t (z(s)-w(s))e^{-(t-s)\beta}ds, 0)|.\end{aligned}$$

Since $\|(AX)(t)-(AU)(t)\| = \sup_t \max|(\int_{-\infty}^t (y(s)-v(s))e^{-(t-s)\alpha}ds, \int_{-\infty}^t (z(s)-w(s))e^{-(t-s)\beta}ds, 0)|$, and

$$\int_{-\infty}^t |y(s)-v(s)|e^{-(t-s)\alpha}ds \leq \frac{1}{\alpha}\|X-U\|_\Lambda < \|X-U\|_\Lambda$$

$$\int_{-\infty}^t (z(s)-w(s))e^{-(t-s)\beta}ds \leq \frac{1}{\beta}\|X-U\|_\Lambda < \|X-U\|_\Lambda,$$

then

$\|(AX)(t)-(AU)(t)\| < \|X-U\|_\Lambda$.

Next we show that the operator $A\Omega$ is relatively compact. We prove this in two steps

Step1:$\{AX : X \in \Omega\}$ is equi-continuous. Note that if $s<t$, we have

$$\begin{aligned}&| \ (AX)(t)-(AX)(s)| \\ &= |(\int_{-\infty}^t y(s)e^{-(t-s)\alpha}ds - \int_{-\infty}^s y(s)e^{-(t-s)\alpha}ds, \\ &\quad \int_{-\infty}^t z(s)e^{-(t-s)\beta}ds - \int_{-\infty}^s z(s)e^{-(t-s)\beta}ds, 0))| \\ &= (\int_{-\infty}^{t_1} y(s)e^{-(t_2-s)\alpha}ds + \int_{t_1}^{t_2} y(s)e^{-(t_2-s)\alpha}ds \\ &\quad - \int_{-\infty}^{t_1} y(s)e^{-(t_1-s)\alpha}ds, \\ &\quad \int_{-\infty}^{t_1} z(s)e^{-(t_2-s)\alpha}ds + \int_{t_1}^{t_2} z(s)e^{-(t_2-s)\alpha}ds \\ &\quad - \int_{-\infty}^{t_1} z(s)e^{-(t_1-s)\alpha}ds, 0) \\ &= |((e^{-(t_2-t_1)\alpha}-1)\int_{-\infty}^{t_1} y(s)e^{-(t_1-s)\alpha}ds \\ &\quad + e^{-(t_2-t_1)\alpha}\int_{t_1}^{t_2} y(s)e^{-(t_1-s)\alpha}ds, \\ &\quad (e^{-(t_2-t_1)\beta}-1)\int_{-\infty}^{t_1} z(s)e^{-(t_1-s)\beta}ds \\ &\quad + e^{-(t_2-t_1)\beta}\int_{t_1}^{t_2} z(s)e^{-(t_1-s)\beta}ds, 0)|.\end{aligned}$$

Since

$$| \ ((e^{-(t_2-t_1)\alpha}-1)\int_{-\infty}^{t_1} y(s)e^{-(t_1-s)\alpha}ds$$
$$+e^{-(t_2-t_1)\alpha}\int_{t_1}^{t_2} y(s)e^{-(t_1-s)\alpha}ds|$$
$$\leq 2\|X\|_\Lambda \frac{|1-e^{-(t_2-t_1)\alpha}|}{\alpha}$$
$$| \ (e^{-(t_2-t_1)\beta}-1)\int_{-\infty}^{t_1} z(s)e^{-(t_1-s)\beta}ds$$
$$+e^{-(t_2-t_1)\beta}\int_{t_1}^{t_2} z(s)e^{-(t_1-s)\beta}ds|$$
$$\leq 2\|X\|_\Lambda \frac{|1-e^{-(t_2-t_1)\beta}|}{\beta}.$$

then, we have to prove $|(AX)(t)-(AX)(s)| \to 0$ as $t \to s$ in two cases.

1.If $\max(2\|X\|_\Lambda \frac{|1-e^{-(t_2-t_1)\alpha}|}{\alpha}, 2\|X\|_\Lambda \frac{|1-e^{-(t_2-t_1)\beta}|}{\beta}) = 2\|X\|_\Lambda \frac{|1-e^{-(t_2-t_1)\beta}|}{\beta}$, then

$$|(AX)(t)-(AX)(s)| \leq 2\|X\|_\Lambda \frac{|1-e^{-(t_2-t_1)\beta}|}{\beta},$$

letting $s \to t$, we obtain $1-e^{\beta(s-t)} \to 0$, and $|(AX)(t)-(AX)(s)| \to 0$.

2.If $\max(2\|X\|_\Lambda \frac{|1-e^{-(t_2-t_1)\alpha}|}{\alpha}, 2\|X\|_\Lambda \frac{|1-e^{-(t_2-t_1)\beta}|}{\beta}) = 2\|X\|_\Lambda \frac{|1-e^{-(t_2-t_1)\alpha}|}{\alpha}$, then

$$|(AX)(t)-(AX)(s)| \leq 2\|X\|_\Lambda \frac{|1-e^{-(t_2-t_1)\alpha}|}{\alpha},$$

letting $s \to t$ , we obtain $1-e^{\alpha(s-t)}$, $1-e^{\beta(s-t)} \to 0$, and $|(AX)(t)-(AX)(s)| \to 0$.

Step2:$\{AX(t) : X \in \Omega\}$ is relatively compact subset of $\Lambda$ for each $t \in \mathbb{R}$.
let $Y_n(t) = (AX_n)(t)$ be a sequence of $\{(AX)(t) : X \in \Omega\}$. $X_n \in \Omega$, i.e. $X_n \in \Lambda$ and $\|X_n\|_\Lambda \leq M$. Therefore $(X_n)_n$ is a bounded sequence, then there exists a subsequence $(X_{n_k})$ of $(X_n)$ in $\Omega$, such that $X_{n_k} \to X$ as $n_k \to \infty$ in $\Omega$. Since $A$ is continuous, then $AX_{n_k} \to AX$ as $n_k \to \infty$ in $\Omega$, i.e. $\sup_t \max_{1\leq i\leq 3} |(AX_{n_k})(t)-(AX)(t)| < \varepsilon$. Then there exists a subsequence $Y_{n_k}(t)$ of $Y_n(t)$ such that $Y_{n_k}(t) \to Y(t)$ as $n_k \to \infty$. Consequently, $\{(AX)(t) : X \in \Omega\}$ is relatively compact in $\Lambda$.

This completes the proof of Theorem. Hence, by Krasnoselskii's Theorem $TX^* = X^*$ and the equation (2) has a fixed pseudo almost periodic solution in $\Omega$.

## 5 Stability of the pseudo almost periodic solution

**Theorem 8.** *Suppose that assumptions $(H1)-(H3)$ hold. Then the unique pseudo almost periodic solution $X^*(t)$ of equation (2) in Theorem (2) is globally attractive.*

*Proof.*Let $X(t)$ be a solution of system (2). We need to prove $\lim_{t\to\infty} |X(t)-X^*(t)| = 0$. By contradiction, we pose $\limsup |X^*(t)-X(t)| = \sigma > 0$.

Case1.$\limsup |X^*(t)-X(t)| = \limsup |x^*(t)-x(t)| = \sigma > 0$. Then

$$|x^*(t)-x(t)| = \left|\int_{-\infty}^{t} |y^*(s)-y(s)|e^{-(t-s)\alpha}ds\right|$$
$$\leq \frac{1}{\alpha}\|y^*-y\|_\infty \leq \frac{\sigma}{\alpha},$$

then $\sigma \leq \frac{\sigma}{\alpha}$, which is a contradiction, then $\limsup |x^*(t)-x(t)| = 0$.

Case 2.$\limsup |X^*(t)-X(t)| = \limsup |y^*(t)-y(t)| = \sigma$. The same of case 1, $\sigma = \limsup |y^*(t)-y(t)| \leq \frac{\sigma}{\beta}$. Then $\limsup |y^*(t)-y(t)| = 0$.

Case 3.$\limsup |X^*(t)-X(t)| = \limsup |z^*(t)-z(t)| = \sigma$. Then

$$|z^*(t)-z(t)| \leq \sup_\xi \Big|(\alpha+\beta)(a(\xi)-\alpha)-\beta^2-b(\xi)$$
$$+\alpha^2(a(\xi)-\alpha)-\alpha b(\xi)$$
$$-\sum_{i=1}^{n} L_{g_i}\Big| \left|\int_{-\infty}^{t} \|X^*-X\|_\Lambda e^{-\int_s^t \psi(u)du}ds\right.$$
$$\leq \|X^*-X\|_\Lambda \mu.$$

Besides,

$$\sigma = \limsup |z^*(t)-z(t)| \leq \sigma\mu,$$

which is a contradiction. Hence $\limsup |z^*(t)-z(t)| = 0$. Consequently, $\limsup |X^*(t)-X(t)| = 0$.

## 6 Example

In this section we give an example in order to illustrate the validity of Theorem (2). Let us consider the following pseudo almost periodic Third-order differential equation

$$x^{(3)} + (10-\frac{6}{1+\frac{1}{2}(\cos(\pi t)^2+\cos(\sqrt{2}t)^2)})x^{(2)}$$
$$+(13-\frac{6}{1+\frac{1}{2}(\cos(\pi t)^2+\cos(\sqrt{2}t)^2)})x^{(1)}$$
$$+ e^{|\cos(\sqrt{2}t)|}\sin(x(t-0.2))$$
$$= \cos(\pi t)+\cos(\sqrt{2}t)+e^{-(t\cos t)^2}. \quad (3)$$

For

$$a(t) = 10-\frac{6}{1+\frac{1}{2}(\cos(\pi t)^2+\cos(\sqrt{2}t)^2)},$$
$$b(t) = 13-\frac{6}{1+\frac{1}{2}(\cos(\pi t)^2+\cos(\sqrt{2}t)^2)},$$
$$n = 1,$$

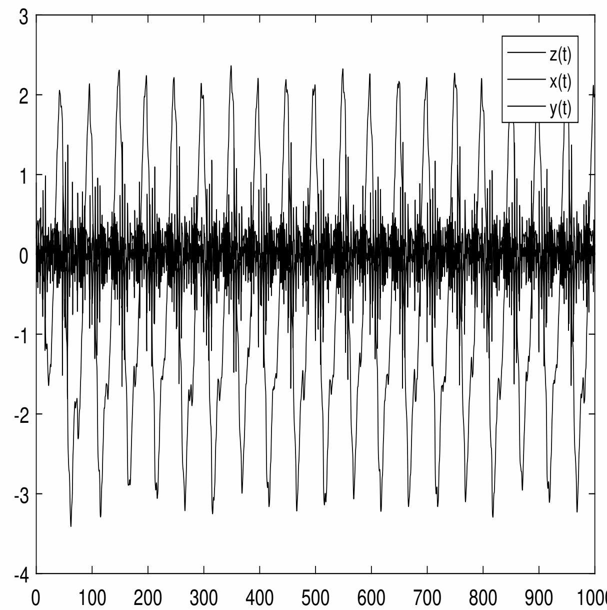


**Fig. 1:** Curves of the pseudo almost periodic solution of equation (3) above with multiple delays.

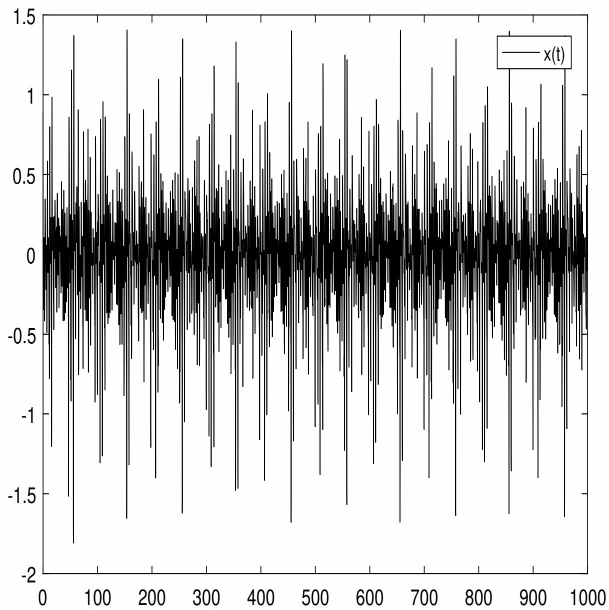


**Fig. 2:** Curve of the x(t) with initial time t=0.

and

$$g(t,x(t-r(t))) = e^{|\cos(\sqrt{2}t)|}\sin(x(t-0.2)),$$

$$p(t) = \cos(\pi t)+\cos(\sqrt{2}t)+e^{-(t\cos t)^2}.$$

in addition to that $\alpha=\beta=1.02$, the equation (3) satisfies $(H1)-(H2)$ and

$$\frac{\sup_t\left(|(\alpha+\beta)(a(t)-\alpha)-\beta^2+\alpha^2(a(t)-\alpha)-b(t)(1+\alpha)-L_g|\right)}{\inf_t\{a(t)-\alpha-\beta\}}\leq 0.168<1.$$

Then the equation (3) has a unique pseudo almost periodic solution.

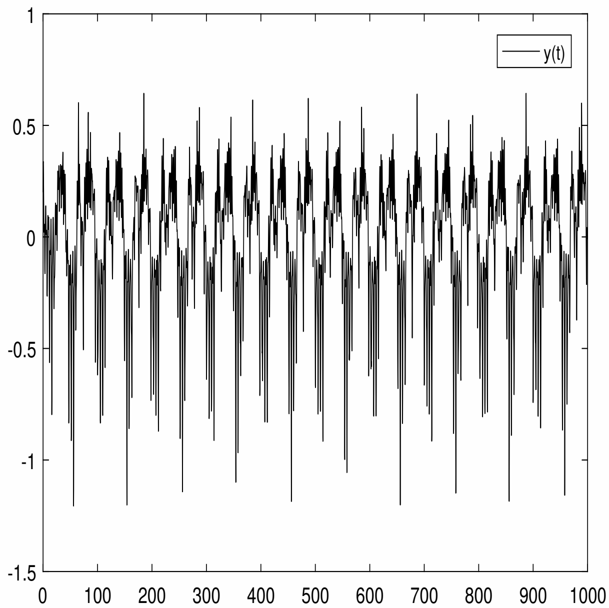


**Fig. 3:** Curve of the y(t) with initial time t=1.5.

## 7 Conclusion

Nonlinear differential differential equations of higher order have been extensively studied with high degree of generality. In particular, boundedness, uniform boundedness, ultimate boundedness, uniform ultimate boundedness and asymptotic behaviour of solutions have been recently discussed by many authors (one can see [5],[7],[8],[9],[10],[11],[13]). In this paper a third-order differential equation with multiple delays is studied. By new and sufficient conditions we prove the uniformly-bounded solutions. In addition, we establish the existence, uniqueness of the pseudo almost periodic solutions, which is done by the use of different fixed point theorems (Banach, Schauder, Leray-Schauder and Krasnoselskii). Furthermore, the global attractivity of the pseudo almost periodic solutions is proved. Finally, we show the validity of our result by an example.

## Acknowledgments.

The authors would like to thank the anonymous reviewers and the editor for their constructive comments, which greatly improved the quality of this paper.

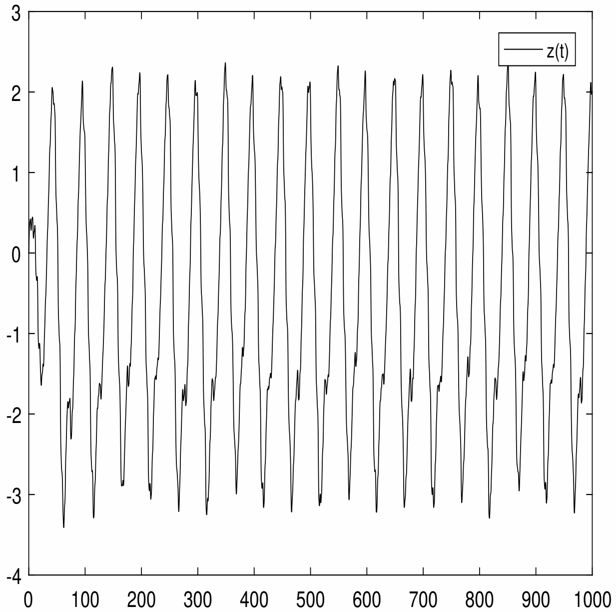


**Fig. 4:** Curve of the z(t) with initial time t=2.

## References


[1] A.T. Ali, M.M.A. Khater, R. A.M. Attia, A. Abdel-Aty and Dianchen Lu, Abundant numerical and analytical solutions of the generalized formula of Hirota-Satsuma coupled KdV system, Chaos, Solitons & Fractals, 131,(*109473*),2020.

[2] A.T. Elgendy, A. Abdel-Aty, Amr A. Youssef, M. A. A. Khder, Kh. Lotfy and S. Owyed, Exact solution of Arrhenius equation for non-isothermal kinetics at constant heating rate and n-th order of reaction, Journal of Mathematical Chemistry, https://doi.org/10.1007/s10910-019-01056-7, 2019.

[3] G.M. Ismail, H.R. Abdl-Rahim, A. Abdel-Aty, R. Kharabsheh, W.Alharbi and M. Abdel-Aty, An analytical solution for fractional oscillator in a resisting medium, Chaos, Solitons & Fractals, 130, ( *109395*), 2020.

[4] S. Owyed, M.A. Abdou, A. Abdel-Aty and S. Saha Ray, New Optical Soliton Solutions of Nonlinear Evolution Equation Describing Nonlinear Dispersion, Communications in Theoretical Physics, 71, (*1063-1068*), 2019.

[5] A.I. Sadek, Stability and Boundedness of a Kind of Third-Order Delay Differential System,*Applied Mathematics Letters*, 16, (*657–662*), 2003.

[6] A. S. C. Sinha, On Stability of Solutions of Some Third and Fourth Order Delay-Differential Equations,Information and Control, 23, (*165–172*), 1973.

[7] A. T. Ademola, Existence and uniqueness of a periodic solution to certain third order nonlinear delay differential equation with multiple deviating arguments, Acta Univ. Sapientiae Mathematica, 5, (*113–131*), 2013.

[8] C. Tunç, On the stability and boundedness of solutions to third order nonlinear differential equations with retarded argument, Nonlinear Dynamics, 57, (*97–106*), 2009.

[9] E. N. Chukwu, On the boundedness and the existence of a periodic solution of some nonlinear third order delay differential equation, Atti della Accademia Nazionale dei Lincei, Classe di Scienze Fisiche, Matematiche Naturali, 64,(*440–447*), 1978.

[10] M. O. Omeike, Stability and boundedness of solutions of a certain system of third order nonlinear delay differential equations, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math, 54, (*109–119*), 2015.

[11] C. Tunç, E. Tunç, New ultimate Boundedness and periodicity results for certain third order nonlinear vector differential equations, Math. J. Okayama Univ, 48, (*159–172*), 2006.

[12] C. Tunç, Existence of periodic solutions to Nonlinear differential equations of third order with multiple deviating Arguments, International Journal of Differential Equations, 5, (*13*), 2012.

[13] R. Yazgan, C. Tunç, On the Weighted Pseudo Almost Periodic Solutions of Nonlinear Functional Duffing Equation, Applied Mathematics and Information Sciences, 11,(*1609–1614*), 2017.

[14] Z.Chuanyi, Almost Periodic Type Functions and Ergodicity, Kluwer Academic Publishers, London, 2003.

[15] D. Toka, Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces, Springer International Publishing Switzerland, 2013.

[16] B. Ammar, F. Chérif, A. M. Alimi, Existence and uniqueness of pseudo almost periodic solutions of recurrent neural networks with time-varying coefficients and mixed delays, IEEE Transactions on neural networks and Learning systems, 23, (*109-118*), 2012.

[17] F. Chérif, Pseudo almost periodic solutions for a class of fourth-order nonlinear differential equations with a deviating argument, Differ Equ Dyn Syst, 24, (*81-97*), 2014.

[18] M.Amdouni, F.Chérif, The pseudo almost periodic solutions of the new class of Lotka-Volterra recurrent neural Networks with mixed delays, Chaos, Solitons and Fractals, 113, (*79-88*), 2018.

[19] S. Rihani, A. Kessab, F. Chérif, Pseudo Almost Periodic Solutions for a Lasota-Wazewska model, Electronic Journal of Differential Equations, 62, (*1-17*), 2016.

[20] G. Andrzej, D. James. Fixed Point Theory, Springer-Verlag, New York, 2003.

[21] M. A. Krasnosel'skii, Some problems of nonlinear analysis, Amer. Math. Soc. Transl. Ser, 9, (*57-114*), 1954.

[22] T. Yoshizawa, Stability Theory by Liapunov's Second Method,Publications of the Mathematical Society of Japan, Tokya, 1966.

[23] D. R. Smart, Fixed Point Theorems, Cambridge University Press, 1980.

**Manel Amdouni** received the M.S. degree from the University of Tunis El Manar, Tunis, and the Ph.D. degree from the University of Ecole Superieure des Sciences et de Technologie de Hammam Sousse , Sousse, Tunis, all in applied mathematics/mathematics, in 2014 and 2019 respectively.

Her major research interests include artificial neural networks, ecology, almost periodic solutions, pseudo almost periodic solutions, almost automorphic solutions, pseudo almost automorphic solutions, stability.

**Farouk Chérif** received the M.S. degree from the University of Paris 7, France, and the Ph.D. degree from the University of Paris 1 Panthéon-Sorbonne, Paris, France, all in mathematics/applied mathematics, in 1992 and 1995 respectively. He was with CERMCEM, University of Paris 1 Panthéon-Sorbonne from 1992 to 1996. In 1996, he joined the Department of Mathematics and Computer Science, Military Academy, Sousse, Tunisia. He is currently a Professor with the Department of Computer science, the higher institute of applied sciences and technology Sousse, University of Sousse. Farouk is the author or co-author of more than 40 journal papers and book chapters. His research was first nonlinear analysis and especially the qualitative study of Hamiltonian systems. Later, he studied the nonlinear dynamics and the delayed differential equations and different applications in various fields such as artificial neural networks (RNNs, CNNs, SICNNs). Dr. Farouk, currently, serves as a reviewer of several international journals and a Program Committee for various international conferences and workshops. Also Farouk chérif is a member of European neural networks society