Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/140205

The Qualitative Property of Numerical Solution of Third Order Sublinear Neutral-Delay Generalized Difference Equation

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Received: 2 May 2019, Revised: 1 Jun. 2019, Accepted: 2 Jul. 2019
Published online: 1 Mar. 2020

Abstract: Here, we discuss adequate oscillatory conditions for third-order sub linear neutral-delay ℓ-difference equation

$$\Delta_\ell\left(\alpha_2(k)\Delta_\ell\left(\alpha_1(k)\Delta_\ell(x(k)+p(k)x(k-\tau\ell))\right)\right)+q(k)x^\beta(k-\sigma\ell)=0,$$

we apply Riccati transformation technique in deriving enough considerations to make sure that every result of this equation is oscillatory. We provide suitable examples to validate our results.

Keywords: Generalized difference operator, Oscillation, Nonoscillation.

1 Introduction

Recently, many researchers focused their study on the oscillatory, rotatory as well as the asymptotic properties of numerical and exact solutions of certain type of neutral, delay and neutral-delay difference equations, see [1,2,3]. This area of study witnessed the publication of hundreds of research articles and many monographs, see for example [4,5,6,7]. Researchers showed greater involvement on the work of the oscillatory properties of solutions of higher order, particularly linear and nonlinear second order delay, neutral and neutral difference equations. Even though some applications of certain type of third-order delay, neutral delay difference equations are very evident in the study of mathematical biology, economics and many other areas in mathematics [8,9,10, 11,12,13,14,15,16,17,18,19], it received only less importance in the literature.

All these authors have handled various types of difference equations taking into consideration the conventional forward difference operator Δ defined as

$$\Delta x(t)=x(t+1)-x(t),\quad t\in\mathbb{N}.$$

Even though many authors [4,6,7,20] suggested an alternative definition for Δ as

$$\Delta x(t)=x(t+h)-x(t),\quad t\in\mathbb{R},\ell\in\mathbb{R}-\{0\},\tag{1}$$

for decades together, no significant contribution is available in the literature of difference equation based on the definition of Δ given in (1). As of late, Thandapani et. al., [21], studied the operator Δ which is expressed in (1) and generalized theory of difference equations in a new dimension. To make it suitable, the operator Δ given in (1) is renamed as Δ_ℓ by replacing h by ℓ and t by k and by obtaining its inverse Δ_ℓ^{-1}, numerous elating outcomes and applications were obtained in number theory. In order to increase the scope of the study of complex solutions of the difference equations $\Delta_\ell x(k)=y(k)$, certain behavior of the numerical and exact solutions having the nature say, spiral, rotation, shrinking, expand and web-like have been developed for the equations containing Δ_ℓ [22], an application of difference equations in maneuvering target

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tracking, see [23].

We present here some of the results already available relevant to our study which may serve the readers and motivate the contents of this paper. Tang et. al., [24], studied the first-order non-linear delay difference equation of the form

$$x_{n+1} - x_n + p_n x_{n-k}^{\alpha} = 0.$$

Oscillation criteria for non-linear delay homogeneous equation

$$\Delta\left(u_n\left(\Delta(x_n + q_n x_{n-\tau})\right)^{\gamma}\right) + f(n, x_{n-\sigma}) = 0,$$

have been discussed by Later Saker [17] and obtained oscillation criteria for the above mentioned equations. One can refer to the oscillatory behavior of superlinear, quasilinear, sub-linear, and halflinear difference equations [18,25,26,27]. Recently Chandrasekar and Jaison [28] studied the oscillation of second kind generalized sub-linear neutral-delay difference equation as shown

$$\Delta_\ell\left(\alpha(k)\left(\Delta_\ell(q(k)v(k-\tau\ell) + v(k))\right)\right) + p(k)v^{\gamma}(k-\sigma\ell) = 0.$$

This paper aims to get Riccati type transformation to arrive at certain criteria of oscillatory and asymptotic behavior for the generalized third order difference equation

$$\Delta_\ell\left(\alpha_2(k)\Delta_\ell\left(\alpha_1(k)\Delta_\ell z(k)\right)\right) + q(k)x^{\beta}(k-\sigma\ell) = 0, \quad (2)$$

where $z(k) = x(k) + p(k)x(k-\tau\ell)$, $\beta \in (0,1)$ is the odd positive quotient, σ and τ are constant non-negative integers, with the following condition

$$\sum_{s=k}^{\infty} \frac{1}{a_i(s)} = \infty, \quad \text{for } i = 1,2,$$
$$p(k) \in [0,1), \forall k \in (0,\infty), q(k) \geq 0. \quad (3)$$

This paper is structured as follows: Few standard definitions and preliminaries are discussed in section 2. Section 3 deals with new oscillation results for (2) and in Section 4, we provide suitable examples to demonstrate the main findings.

2 Preliminaries

The following notations are used throughout this paper.
(a) $\mathbb{N}_\ell(b) = \{b, b+\ell, b+2\ell, \ldots\}$.
(b) $\lceil y \rceil$ represents the upper integer and $[y]$ represents the integer part of y.
(c) $n = \max\{\tau\ell, \sigma\ell\}$.
(d) $j = k - k_0 - [(k-k_0)/\ell]\ell$. $\bar{k_i} = k_i + j$.

Definition 1.[1] *Consider a real valued function $y(k)$ with real variable k. The operator $\Delta_\ell y(k) = y(k+\ell) - y(k)$, $k \in [0,\infty)$, $\ell \in (0,\infty)$, and if $\Delta_\ell x(k) = y(k)$, then $x(n) = \Delta_\ell^{-1} y(k) + c_j$. where the real number c_j can be obtained by substituting lower limit j, for all $k \in \{j, j+\ell, j+2\ell, \cdots\}$.*

Definition 2.[21] *For any positive interger λ, the ℓ-polynomial factorial function can be given as*

$$k_\ell^{(\lambda)} = \prod_{i=0}^{\lambda-1}(k - i\ell). \quad (4)$$

Lemma 1.[21] *Let $\ell \in [0,\infty)$. then $\Delta_\ell(k_\ell^{(\lambda)}) = (\lambda\ell)k_\ell^{(\lambda-1)}$*

Lemma 2.[21] *For the given two functions $u(k)$ and $v(k)$, we have*

$$\begin{aligned}\Delta_\ell\{u(k)v(k)\} &= u(k+\ell)\Delta_\ell u(k) + v(k)\Delta_\ell v(k) \\ &= v(k+\ell)\Delta_\ell u(k) + u(k)\Delta_\ell v(k).\end{aligned}$$

Definition 3.[26] *Let $f(k, x_1, x_2, \ldots, x_m)$ be a real-valued function. This function can be called as strongly sub-linear if $\exists$ a real number $\beta \in (0,1)$, β is an odd positive quotient and $d > 0$ with $|x|^{-\beta}|f(k, x_1, x_2, \cdots, x_m)|$ is non-increasing in $|x|$ for $|x| \in [0,d)$.*

Definition 4.[21] *Let $y(k)$ be a real valued function, Then for $k \in [k_0, \infty)$,*

$$x(k) = x(k_0 + j) + \sum_{r=0}^{\left[\frac{k-k_0-j-\ell}{\ell}\right]} y(k_0 + j + r\ell),$$

for $k \in \mathbb{N}_\ell(j)$, where $j = k - k_0 - \left[\frac{k-k_0}{\ell}\right]\ell$.

3 Riccati transformation in generalized third-order sublinear neutral delay difference equation

For this particular work, the following notations are introduced.

$$E_0(k) = z(k), \quad E_i(k) = a_i(k)\Delta_\ell E_{i-1}(k), \quad i = 1,2$$

$$R_n(k) = \frac{1}{a_1(k)} \sum_{s=0}^{\left[\frac{k-n-\ell-j}{\ell}\right]} \frac{1}{a_2(n+j+s\ell)} \quad \text{and}$$

$$\overline{R_n}(k) = \sum_{s=0}^{\left[\frac{k-n-\ell-j}{\ell}\right]} R_n(n+j+s\ell).$$

Theorem 1.*Let $\rho(k)$ be a positive function which satisfies the condition (3) and such that for every $M \geq \ell$,*

$$\limsup_{k\to\infty} \sum_{s=0}^{\left[\frac{k-\bar{k_0}-\ell}{\ell}\right]} \left[\rho(\bar{k_0}+s\ell)\phi(\bar{k_0}+s\ell) - \frac{\left(\Delta_\ell\rho(\bar{k_0}+s\ell)\right)^2}{4\psi(\bar{k_0}+s\ell)}\right] = \infty, \quad (5)$$

where $\phi(k) = p(k)(1 - q(k-\sigma\ell))^{\beta}$. Then $\psi(k) = \dfrac{\beta\rho(k)R_n(k-\sigma\ell)}{(M(k-\sigma\ell+\ell))^{1-\beta}}$, satisfies equation (2).

*Proof.*Let $\{y(k)\}$ be a positive solution of (2) $\forall k \geq k_0$. Since $y(k) = E_0(k)$, we obtain $z(k) \geq y(k) > 0$ and $y(k-\tau\ell) > 0 \; \forall k \geq k_1 \geq k_0$, and also from (2), we have

$$\Delta_\ell E_2(k) = -q(k)y^\beta(k-\sigma\ell) \leq 0.$$

We know that $E_2(k)$ is a decreasing function on $[k_1,\infty)$ and it is eventually positive or negative. We can find that $E_2(k) > 0$ for $k \geq k_1$. If not, $\exists$ a positive real $N_1 > 0$ with the condition

$$\Delta_\ell E_1(k) < -\frac{N_1}{a_2(k)} < 0, \;\; \text{for} \;\; k \geq k_2 \geq k_1.$$

Hence, by Definition 4

$$E_1(k) \leq E_1(\bar{k_2}) - N_1 \sum_{s=0}^{\left[\frac{k-\bar{k_2}-\ell}{\ell}\right]} \frac{1}{a_2(\bar{k_2}+s\ell)}.$$

Letting $k \to \infty$ and using (3) we have $\lim_{k\to\infty} E_1(k) = -\infty$. Then there exists $k_3 \geq k_2$ also, constant $N_2 > 0$ such that

$$a_1(k)\Delta_\ell z(k) < -N_2, \;\; \text{for} \;\; k \geq k_3.$$

If we divide the above expression by $\alpha_1(k)$ and adding from k_3 to k, we get

$$z(k) < z(\bar{k_3}) - N_2 \sum_{s=0}^{\left[\frac{k-\bar{k_3}-\ell}{\ell}\right]} \frac{1}{\alpha_2(\bar{k_3}+s\ell)},$$

Allowing $k \to \infty$ and using (3) , we see that $z(k) \to -\infty$. That is $z(k) < 0$ eventually which is contradictory with $z(k) > 0$. Hence, we find

$$z(k) > 0, E_1(k) > 0 \;\; \text{and} \;\; \Delta_\ell E_1(k) > 0 \quad \text{for} \;\; k \geq k_0 \quad (6)$$

and then from $z(k)$ and (6) we have $x(k) = z(k) - x(k-\tau\ell)p(k) \geq z(k)(1-p(k))$ which yields, for $k \geq k_1 = k_0 + \sigma\ell$,

$$x(k-\sigma\ell) \geq z(k-\sigma\ell)(1-p(k-\sigma\ell)).$$

Thus, by (2), we arrive

$$\begin{aligned}\Delta_\ell E_2(k) &\leq -q(k)x^\beta(k-\sigma\ell)\\ &\leq -q(k)(1-p(k-\sigma\ell))^\beta z^\beta(k-\sigma\ell) < 0. \quad (7)\end{aligned}$$

Now, from (6), there exists $n \geq k_1$ such that

$$E_1(k) = E_1(n+j) + \sum_{s=0}^{\left[\frac{k-n-\ell-j}{\ell}\right]} \frac{E_2(n+j+s\ell)}{a_2(n+j+s\ell)}.$$

Since $\Delta_\ell E_2(k) < 0$, we obtain

$$E_1(k) \leq E_2(k) \sum_{s=0}^{\left[\frac{k-n-\ell-j}{\ell}\right]} \frac{1}{a_2(n+j+s\ell)}.$$

This implies that

$$\Delta_\ell z(k) \geq E_2(k)R_n(k). \quad (8)$$

The above equation can be written as

$$\Delta_\ell z(k-\sigma\ell) \geq E_2(k)R_n(k-\sigma\ell). \quad (9)$$

Define $\omega(k)$ by the Riccati substitution

$$\omega(k) = \rho(k)\frac{E_2(k)}{z^\beta(k-\sigma\ell)}. \quad (10)$$

We see that $\omega(k) > 0$ and satisfies

$$\Delta_\ell \omega(k) = E_2(k+\ell)\Delta_\ell\left(\frac{\rho(k)}{z^\beta(k-\sigma\ell)}\right) + \frac{\rho(k)\Delta_\ell E_2(k)}{z^\beta(k-\sigma\ell)}.$$

Thus, from (7) and (9), we derive

$$\begin{aligned}\Delta_\ell \omega(k) \leq& -\rho(k)q(k)(1-p(k-\sigma\ell))^\beta + \frac{\Delta_\ell \rho(k)}{\rho(k+\ell)}\omega(k+\ell)\\ &- \frac{\rho(k)E_2(k+\ell)\Delta_\ell z^\beta(k-\sigma\ell)}{z^\beta(k-\sigma\ell+\ell)z^\beta(k-\sigma\ell)}. \quad (11)\end{aligned}$$

Since $z(k) = E_0(k)$ and $0 < z(k-\sigma\ell) \leq z(k-\sigma\ell+\ell)$, this implies that

$$\begin{aligned}\Delta_\ell \omega(k) \leq& -\rho(k)\phi(k) + \frac{\Delta_\ell \rho(k)}{\rho(\ell+k)}\omega(\ell+k)\\ &- \frac{E_2(\ell+k)\Delta_\ell z^\beta(k-\sigma\ell)\rho(k)}{\left(z^\beta(k-\sigma\ell+\ell)\right)^2}. \quad (12)\end{aligned}$$

With the inequality given in ([5]),

$$(u^\beta - v^\beta) < \beta v^{\beta-1}(u-v)$$

for all $0 < v \leq u$ and $0 < \beta \leq 1$, we find that

$$\Delta_\ell z^\beta(k-\sigma\ell) < \beta\left(z^{\beta-1}(k-\sigma\ell+\ell)\right)\Delta_\ell z(k-\sigma\ell). \quad (13)$$

Substituting (13) in (12), we arrive

$$\begin{aligned}\Delta_\ell \omega(k) \leq& -\rho(k)\phi(k) + \frac{\Delta_\ell \rho(k)}{\rho(k+\ell)}\omega(k+\ell)\\ &- \frac{\rho(k)E_2(k+\ell)\beta\left(z^{\beta-1}(k-\sigma\ell+\ell)\right)\Delta_\ell z(k-\sigma\ell)}{\left(z^\beta(k-\sigma\ell+\ell)\right)^2}. \quad (14)\end{aligned}$$

Using (9) in (14), we derive

$$\begin{aligned}\Delta_\ell \omega(k) \leq& -\phi(k)\rho(k) + \frac{\Delta_\ell \rho(k)}{\rho(k+\ell)}\omega(k+\ell)\\ &- \frac{\rho(k)E_2^2(k+\ell)\beta\left(z^{\beta-1}(k-\sigma\ell+\ell)\right)R_n(k-\sigma\ell)}{\left(z^\beta(k-\sigma\ell+\ell)\right)^2}. \quad (15)\end{aligned}$$

By (10) in (14) becomes

$$\Delta_\ell \omega(k) \leq -\phi(k)\rho(k) + \frac{\Delta_\ell \rho(k)}{\rho(\ell+k)}\omega(\ell+k) - \beta \frac{\rho(k)\omega^2(\ell+k)R_n(k-\sigma\ell)}{(\rho(\ell+k))^2\left(z^{1-\beta}(k-\sigma\ell+\ell)\right)}. \quad (16)$$

Hence, form (6), it is easy to obtain

$$z(k) \leq z(k_0) + \Delta_\ell z(k_0)(k-k_0), \quad k \geq k_0$$

and consequently $\exists$, $k_1 \geq k_0$ and an suitable constant $M \geq \ell$ with the condition $z(k) \leq M$ $\forall k_1 \leq k$, and yields $z(k-\sigma\ell+\ell) \leq M(k-\sigma\ell+\ell)$ $\forall k \geq k_2 = k_1 - \sigma\ell + \ell$, which yields

$$\frac{1}{z^{1-\beta}(k-\sigma\ell+\ell)} \geq \frac{1}{(M(k-\sigma\ell+\ell))^{1-\beta}}$$

Using the above inequality in (16),

$$\Delta_\ell \omega(k) \leq -\phi(k)\rho(k) + \frac{\Delta_\ell \rho(k)\omega(\ell+k)}{\rho(\ell+k)} - \frac{\beta\rho(k)R_n(k-\sigma\ell)\omega^2(\ell+k)}{\rho^2(\ell+k)\,(M(k-\sigma\ell+\ell))^{1-\beta}} \quad (17)$$

The above equation can also be written as

$$\Delta_\ell \omega(k) \leq -\phi(k)\rho(k) + \frac{(\Delta_\ell \rho(k))^2}{4\psi(k)} - \left[\frac{\sqrt{\psi(k)}\ \omega(\ell+k)}{\rho(k+\ell)} - \frac{\Delta_\ell(\rho(k))}{2\sqrt{\psi(k)}}\right]^2 \quad (18)$$

Then, we arrive

$$\Delta_\ell \omega(k) \leq -\left[\phi(k)\rho(k) - \frac{(\Delta_\ell \rho(k))^2}{4\psi(k)}\right]. \quad (19)$$

Summing (19) for $k = k_2, k_2+\ell, k_2+2\ell, \cdots$ we derive

$$-\omega(\bar{k_2}) < \omega(k+\ell) - \omega(\bar{k_2}) < -\sum_{s=0}^{\left[\frac{k-\bar{k_2}-\ell}{\ell}\right]}\left[\rho(\bar{k_2}+s\ell)\phi(\bar{k_2}+s\ell) - \frac{(\Delta_\ell \rho(\bar{k_2}+s\ell))^2}{4\psi(\bar{k_2}+s\ell)}\right]$$

which yields

$$\sum_{s=0}^{\left[\frac{k-\bar{k_2}-\ell}{\ell}\right]}\left[\rho(\bar{k_2}+s\ell)\phi(\bar{k_2}+s\ell) - \frac{(\Delta_\ell \rho(\bar{k_2}+s\ell))^2}{4\psi(\bar{k_2}+s\ell)}\right] < C,$$

which contradicts (5), when k is large k. Thus we hence the proof.

Remark. Let $\rho(k) = k^\lambda$, where k is greater than k_0, $\lambda > 1$. Theorem 1 gives different conditions for oscillatary solution of (2) if (3) holds for different options of $\rho(k)$.

Corollary 1. *Assume that Theorem 1 holds with all conditions and we replace condition (3) with*

$$\limsup_{k\to\infty} \sum_{s=0}^{\left[\frac{k-\bar{k_0}-\ell}{\ell}\right]}\left[(\bar{k_0}+s\ell)^\lambda \phi(\bar{k_0}+s\ell) - \frac{\left(\Delta_\ell(\bar{k_0}+s\ell)^\lambda\right)^2}{4\psi(\bar{k_0}+s\ell)}\right] = \infty, \quad (20)$$

where $\phi(k) = q(k)(1-p(k-\sigma\ell))^\beta$ *and* $\psi(k) = \dfrac{\beta\rho(k)R_n(k-\sigma\ell)}{(M(k-\sigma\ell+\ell))^{1-\beta}}$. *Then each and every solution of (2) is oscillatory.*

Theorem 2. *Let $y(k)$ be a solution of (2). Assume that (3) holds, and $\exists$ a real valued function $\{G(n,k) : n \geq k \geq 0\}$ with the condition*

(i) $G(n,n) = 0$ for $n \geq 0$,
(ii) $G(n,k) > 0$ for $n > k \geq 0$,
(iii) $\Delta_{\ell 2} G(n,k) = G(n,k+\ell) - G(n,k) \leq 0$.

If

$$\limsup_{n\to\infty} \frac{1}{G(n,0)} \sum_{s=0}^{\left[\frac{k-\bar{k_2}-\ell}{\ell}\right]}\left[G(n,\bar{k_2}+s\ell)\rho(\bar{k_2}+s\ell)\phi(\bar{k_2}+s\ell) - \frac{\rho^2(\bar{k_2}+s\ell+\ell)}{4\psi(\bar{k_2}+s\ell)}\left(g(n,\bar{k_2}+s\ell) - \frac{\Delta_\ell \rho(\bar{k_2}+s\ell)}{\rho(\bar{k_2}+s\ell+\ell)}\sqrt{G(n,\bar{k_2}+s\ell)}\right)^2\right] = \infty \quad (21)$$

where $\Delta_{\ell 2} G(n,k) = -g(n,k)\sqrt{G(n,k)}$, for $n > k \geq 0$, then every solution of (2) oscillates.

Proof. If a non-oscillatory solution exists for the difference equation (2), as in the of Theorem 1, we arrive (17) $\forall k \geq k_2$. From (17) and the condition $\Delta_\ell \rho(k) \leq 0$, we have for $k \geq k_2$

$$\Delta_\ell \omega(k) \leq -\rho(k)\phi(k) + \frac{\Delta_\ell \rho(k)}{\rho(k+\ell)}\omega(k+\ell) - \frac{\psi(k)}{\rho^2(k+\ell)}\omega^2(k+\ell). \quad (22)$$

The above equation will take the form

$$\rho(k)\phi(k) \leq -\Delta_\ell \omega(k) + \frac{\Delta_\ell \rho(k)}{\rho(k+\ell)}\omega(k+\ell) - \frac{\psi(k)}{\rho^2(k+\ell)}\omega^2(k+\ell). \quad (23)$$

Multiplying (23) by $G(n,k)$ and summing from k_2 to $k-\ell$, we obtain

$$
\begin{aligned}
&\sum_{s=0}^{\left[\frac{k-\bar{k}_2-\ell}{\ell}\right]} G(n,\bar{k}_2+s\ell)\rho(\bar{k}_2+s\ell)\phi(\bar{k}_2+s\ell)\\
&\le -\sum_{s=0}^{\left[\frac{k-\bar{k}_2-\ell}{\ell}\right]} G(n,\bar{k}_2+s\ell)\Delta_\ell\omega(\bar{k}_2+s\ell)\\
&+\sum_{s=0}^{\left[\frac{k-\bar{k}_2-\ell}{\ell}\right]} \frac{G(n,\bar{k}_2+s\ell)\Delta_\ell\rho(\bar{k}_2+s\ell)}{\rho(\bar{k}_2+s\ell+\ell)}\omega(\bar{k}_2+s\ell+\ell)\\
&-\sum_{s=0}^{\left[\frac{k-\bar{k}_2-\ell}{\ell}\right]} \frac{G(n,\bar{k}_2+s\ell)\psi(\bar{k}_2+s\ell)}{\rho^2(\bar{k}_2+s\ell+\ell)}\omega^2(\bar{k}_2+s\ell+\ell), \qquad (24)
\end{aligned}
$$

which yields, after summing by parts,

$$
\begin{aligned}
&\sum_{s=0}^{\left[\frac{k-\bar{k}_2-\ell}{\ell}\right]} G(n,\bar{k}_2+s\ell)\rho(\bar{k}_2+s\ell)\phi(\bar{k}_2+s\ell)\\
&\le G(n,\bar{k}_2)w(\bar{k}_2)\\
&\quad+\sum_{s=0}^{\left[\frac{k-\bar{k}_2-\ell}{\ell}\right]} \omega(\bar{k}_2+s\ell+\ell)\Delta_\ell G(n,\bar{k}_2+s\ell)\\
&\quad+\sum_{s=0}^{\left[\frac{k-\bar{k}_2-\ell}{\ell}\right]} \frac{G(n,\bar{k}_2+s\ell)\Delta_\ell\rho(\bar{k}_2+s\ell)}{\rho(\bar{k}_2+s\ell+\ell)}\omega(\bar{k}_2+s\ell+\ell)\\
&\quad-\sum_{s=0}^{\left[\frac{k-\bar{k}_2-\ell}{\ell}\right]} \frac{G(n,\bar{k}_2+s\ell)\psi(\bar{k}_2+s\ell)}{\rho^2(\bar{k}_2+s\ell+\ell)}\omega^2(\bar{k}_2+s\ell+\ell)\\
&= G(n,\bar{k}_2)w(\bar{k}_2)\\
&\quad-\sum_{s=0}^{\left[\frac{k-\bar{k}_2-\ell}{\ell}\right]} g(m,\bar{k}_2+s\ell)\sqrt{G(n,\bar{k}_2+s\ell)}\,\omega(\bar{k}_2+s\ell+\ell)\\
&\quad+\sum_{s=0}^{\left[\frac{k-\bar{k}_2-\ell}{\ell}\right]} \frac{\Delta_\ell\rho(\bar{k}_2+s\ell)}{\rho(\bar{k}_2+s\ell+\ell)}G(n,\bar{k}_2+s\ell)\omega(\bar{k}_2+s\ell+\ell)\\
&\quad-\sum_{s=0}^{\left[\frac{k-\bar{k}_2-\ell}{\ell}\right]} \frac{\psi(\bar{k}_2+s\ell)}{\rho^2(\bar{k}_2+s\ell+\ell)}G(n,\bar{k}_2+s\ell)\omega^2(\bar{k}_2+s\ell+\ell)\\
&= G(n,\bar{k}_2)w(\bar{k}_2)\\
&\quad+\sum_{s=0}^{\left[\frac{k-\bar{k}_2-\ell}{\ell}\right]} \frac{\rho^2(\bar{k}_2+s\ell+\ell)}{4\psi(\bar{k}_2+s\ell)}\Bigg(g(n,\bar{k}_2+s\ell)\\
&\quad-\frac{\Delta_\ell\rho(\bar{k}_2+s\ell)}{\rho(\bar{k}_2+s\ell+\ell)}\sqrt{G(n,\bar{k}_2+s\ell)}\Bigg)^2\\
&\quad-\sum_{s=0}^{\left[\frac{k-\bar{k}_2-\ell}{\ell}\right]} \Bigg[\frac{\sqrt{G(n,\bar{k}_2+s\ell)\psi(\bar{k}_2+s\ell)}}{\rho(\bar{k}_2+s\ell+\ell)}\omega(\bar{k}_2+s\ell+\ell)\\
&\quad+\frac{\rho(\bar{k}_2+s\ell+\ell)}{2\sqrt{G(n,\bar{k}_2+s\ell)\psi(\bar{k}_2+s\ell)}}\Bigg(g(n,\bar{k}_2+s\ell)\\
&\quad\sqrt{G(n,\bar{k}_2+s\ell)}-\frac{\Delta_\ell\rho(\bar{k}_2+s\ell)}{\rho(\bar{k}_2+s\ell+\ell)}G(n,\bar{k}_2+s\ell)\Bigg)\Bigg]^2
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\sum_{s=0}^{\left[\frac{k-\bar{k}_2-\ell}{\ell}\right]} \Bigg[G(n,\bar{k}_2+s\ell)\rho(\bar{k}_2+s\ell)\phi(\bar{k}_2+s\ell)\\
&\quad-\frac{\rho^2(\bar{k}_2+s\ell+\ell)}{4\psi(\bar{k}_2+s\ell)}\Bigg(g(n,\bar{k}_2+s\ell)\\
&\quad-\frac{\Delta_\ell\rho(\bar{k}_2+s\ell)}{\rho(\bar{k}_2+s\ell+\ell)}\sqrt{G(n,\bar{k}_2+s\ell)}\Bigg)^2\Bigg]\\
&< G(n,\bar{k}_2)w(\bar{k}_2) \le G(n,0)w(\bar{k}_2)
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{G(n,0)}\sum_{s=0}^{\left[\frac{k-\bar{k}_2-\ell}{\ell}\right]} \Bigg[G(n,\bar{k}_2+s\ell)\rho(\bar{k}_2+s\ell)\\
&\phi(\bar{k}_2+s\ell)-\frac{\rho^2(\bar{k}_2+s\ell+\ell)}{4\psi(\bar{k}_2+s\ell)}\Bigg(g(n,\bar{k}_2+s\ell)\\
&\quad-\frac{\Delta_\ell\rho(\bar{k}_2+s\ell)}{\rho(\bar{k}_2+s\ell+\ell)}\sqrt{G(n,\bar{k}_2+s\ell)}\Bigg)^2\Bigg] < \infty
\end{aligned}
$$

which is a contradiction to the expression (21). Hence the proof.

*Remark.*Several oscillation criteria for the equation (2) can be obtained by the choice of $G(n,k)$. The identity $G(n,n)=0$ for $n\ge 0$ and $G(n,k)>0$ and $\Delta_{\ell 2}G(n,k)\le 0$ for $n>k>0$ follows by the choice of $G(n,k)=(n-k)^{(\lambda)}_\ell$ or $\left(\log\frac{n+\ell}{k+\ell}\right)^\lambda$, where $\lambda\ge 1$ and $n\ge k\ge 0$.

Corollary 2.*Assume the conditions given in Theorem 2 and (21) is replaced by*

$$
\begin{aligned}
&\limsup_{m\to\infty}\frac{1}{(m)^{(\lambda)}_\ell}\sum_{s=0}^{\left[\frac{k-\bar{k}_2-\ell}{\ell}\right]} \Bigg[(m-\bar{k}_2-s\ell)^{(\lambda)}_\ell\rho(\bar{k}_2+s\ell)\phi(\bar{k}_2+s\ell)\\
&\quad-\frac{\rho^2(\bar{k}_2+s\ell+\ell)}{4\psi(\bar{k}_2+s\ell)}\Bigg(\lambda\ell(m-\bar{k}_2-s\ell)^{(\frac{\lambda}{2}-1)}_\ell\\
&\quad-\frac{\Delta_\ell\rho(\bar{k}_2+s\ell)}{\rho(\bar{k}_2+s\ell+\ell)}\sqrt{(m-\bar{k}_2+s\ell)^{(\lambda)}_\ell}\Bigg)^2\Bigg]=\infty \qquad (25)
\end{aligned}
$$

where $\phi(k) = q(k)(1 - p(k - \sigma\ell))^{\beta}$ *and* $\psi(k) = \dfrac{\beta\rho(k)R_n(k-\sigma\ell)}{(M(k-\sigma\ell+\ell))^{1-\beta}}$, *then every solution of (2) oscillates.*

Corollary 3. *Assume the hypothesis given in Theorem 2, (21) is replaced by*

$$\limsup_{t\to\infty} \frac{1}{(\log(m+\ell))^{\lambda}} \sum_{s=0}^{\left[\frac{k-\bar{k_2}-\ell}{\ell}\right]} \left[\left(\log\left(\frac{m+\ell}{\bar{k_2}+s\ell+\ell}\right)\right)^{\lambda} \rho(\bar{k_2}+s\ell)\phi(\bar{k_2}+s\ell) - \frac{\rho^2(\bar{k_2}+s\ell+\ell)}{4\psi(\bar{k_2}+s\ell)}\left(\frac{\ell}{\bar{k_2}+s\ell+\ell}\left(\log\left(\frac{m+\ell}{\bar{k_2}+s\ell+\ell}\right)\right)^{(\frac{\lambda}{2}-1)} - \frac{\Delta_\ell\rho(\bar{k_2}+s\ell)}{\rho(\bar{k_2}+s\ell+\ell)}\sqrt{\left(\log\left(\frac{m+\ell}{\bar{k_2}+s\ell+\ell}\right)\right)^{\lambda}}\right)^2\right] = \infty \tag{26}$$

where $\phi(k) = q(k)(1 - p(k - \sigma\ell))^{\beta}$ *and* $\psi(k) = \dfrac{\beta\rho(k)R_n(k-\sigma\ell)}{(M(k-\sigma\ell+\ell))^{1-\beta}}$, *then every solution of (2) oscillates.*

4 Examples

Example 1. Consider the third-order generalized sublinear neutral delay difference equation

$$\Delta_\ell\left(\frac{1}{k}\Delta_\ell\left(\frac{1}{k}\Delta_\ell\left(x(k)+\frac{1}{2}x(k-2\ell)\right)\right)\right) + \frac{3(4k^3+10k^2\ell+7k\ell^2+2\ell^3)}{k^2(k+\ell)^2(k+2\ell)}x^{\frac{1}{3}}(k-2\ell) = 0,\ k \geq 2\ell \tag{27}$$

Here $a_1(k) = a_2(k) = \frac{1}{k}$, $p(k) = \frac{1}{2}$, $q(k) = \dfrac{3(4k^3+10k^2\ell+7k\ell^2+2\ell^3)}{k^2(k+\ell)^2(k+2\ell)}$, $\beta = \frac{1}{3}$ and $\tau = \sigma = 2$. By taking $\rho(k) = k^2(k+\ell)^2(k+2\ell)$, it is easy to see that condition (5) is satisfied. Hence by Theorem 1, the equation (27) is oscillatory.

Example 2. Consider the third-order generalized sublinear neutral delay difference equation

$$\Delta_\ell^2\left(\frac{1}{k}\Delta_\ell\left(x(k)+\frac{1}{3}x(k-4\ell)\right)\right) + \frac{16(2k^2+4k\ell+\ell^2)}{3k(k+\ell)(k+2\ell)}x^{\frac{1}{5}}(k-2\ell) = 0, \quad k \geq 4\ell \tag{28}$$

Here $a_1(k) = \frac{1}{k}$, $a_2(k) = 1$, $p(k) = \frac{1}{3}$, $q(k) = \dfrac{16(2k^2+4k\ell+\ell^2)}{3k(k+\ell)(k+2\ell)}$, $\tau = 4$, $\sigma = 2$ and $\beta = \frac{1}{5}$. By taking $\rho(k) = k^2$ for $k = 1,2,3,\cdots$ and $M = 10$. Then,

$$R_n(k) = \frac{k-n}{\ell},$$
$$\overline{R_n}(k) = \frac{(k-n-\ell)(k-n)(2k-n-\ell)}{6\ell^2}.$$
$$\phi(k) = \frac{16(2k^2+4k\ell+\ell^2)}{3k(k+\ell)(k+2\ell)}\left(\frac{2}{3}\right)^{\frac{1}{5}},$$
$$\psi(k) = \frac{k^2(k-n-2\ell)(k-n-3\ell)(2k+n-5\ell)}{300\ell^2(k-\ell)^{\frac{1}{5}}},$$

which implies

$$\limsup_{k\to\infty}\sum_{s=0}^{\left[\frac{k-\bar{k_0}-\ell}{\ell}\right]}\left[\frac{16(\bar{k_0}+s\ell)(2(\bar{k_0}+s\ell)^2+4(\bar{k_0}+s\ell)\ell+\ell^2)}{3(\bar{k_0}+s\ell+\ell)(\bar{k_0}+s\ell+2\ell)}\left(\frac{2}{3}\right)^{\frac{1}{5}} - \frac{75\ell^4(2(\bar{k_0}+s\ell)+\ell)^2(\bar{k_0}+s\ell-\ell)^{\frac{1}{5}}}{(\bar{k_0}+s\ell)^2(\bar{k_0}+(s-2)\ell-k)(\bar{k_0}+(s-3)\ell-n)(2\bar{k_0}+(2s-5)\ell+n)}\right] = \infty.$$

It follows from Corollary 1 that every solution of equation (28) is oscillatory. In fact $\{x(k)\} = \{(-1)^{\left[\frac{5k}{\ell}\right]}\}$ is one such oscillatory solution of equation (28).

Example 3. Here, we discuss the following equation

$$\Delta_\ell^3\left(x(k)+\frac{1}{3}x(k-\ell)\right) + \frac{16}{3}x^{\frac{1}{3}}(k-2\ell) = 0. \tag{29}$$

Here $a_1(k) = a_2(k) = 1$, $p(k) = \frac{1}{3}$, $q(k) = \frac{16}{3}$, $\tau = 1$, $\sigma = 2$ and $\beta = \frac{1}{3}$. By taking $\lambda = 2$, $\rho(k) = 1$ for $k = 1,2,3,\cdots$ and $M > \ell$. Then,

$$R_n(k) = \frac{k-n}{\ell},$$
$$\overline{R_n}(k) = \frac{(k-n)(k-n-\ell)}{2\ell^2}.$$
$$\phi(k) = \frac{16}{3}\left(\frac{2}{3}\right)^{\frac{1}{3}},$$
$$\psi(k) = \frac{(k-n-2\ell)(k-n-3\ell)}{6M\ell^2(k-\ell)^{\frac{1}{3}}},$$

which implies

$$\limsup_{k\to\infty}\frac{1}{(m)_\ell^{(2)}}\sum_{s=0}^{\left[\frac{k-\bar{k_2}-\ell}{\ell}\right]}\left[(m-\bar{k_2}-s\ell)_\ell^{(2)}\frac{16}{3}\left(\frac{2}{3}\right)^{\frac{1}{3}}\right] = \infty.$$

It follows from Corollary 2 that all the solution of (29) is oscillatory. One such oscillatory solution is $\{x(k)\} = \{(-1)^{\left[\frac{3k}{\ell}\right]}\}$.

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