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An Attractive Analytic-Numeric Approach for the Solutions of Uncertain Riccati Differential Equations using Residual Power Series

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Abstract: In this paper, the analytic-approximate solution for a class of quadratic Riccati differential equation under uncertainty is obtained using a modified residual power series (RPS) expansion algorithm. The proposed method is a well-known efficient precise algorithm to address numerous issues in physics and engineering. The RPS is a systematic tool based on the use of the Taylor approach and residual error concept by minimizing error functions to determine the values of the coefficients of the PS according to given initial data of symmetric triangular fuzzy numbers. To interpret the solutions of fuzzy quadratic Riccati equations, the strongly-generalized differentiability sense is implemented. This algorithm provides an approximate series of solutions within a radius suitable for the desired domain. Numerical applications are introduced to clarify the compatibility and reliability of the RPS algorithm. The gained results confirm that the suggested simulated is highly reliable, simple and can be implemented to other classes of nonlinear uncertain natural problems.

Keywords: Fuzzy differential equations, Riccati initial value problems, power series expansion, strongly generalized differentiability

1 Introduction

Differential equations under uncertainty have attracted much attention due to the fundamental and critical role for modeling different real-world fuzzy problems arising in sciences with many physical applications, including civil engineering, population models, acoustics, modeling hydraulic, and quantum optics [1,2,3,4,5,6]. Particular interest is utilized in electronic mechanisms including air conditioners, vacuum cleaners, washing machines, and electronic-controlled pattern systems, with further applications to transmission systems. The fuzzy topic has been investigated since the 1920s, Zadeh, however, introduced the term fuzzy derivative within 1972 [7]. Following, the concept of the fuzzy differential calculus has been presented using Zadeh's extension principle by Dubois and Prade in [8]. Furthermore, during the recent years, several scholars have shown other notations and results for fuzzy mappings over a crisp interval [9,10,11,12]. On the other hand, the strongly-generalized differentiability (SGD) for fuzzy-valued function is discussed perfectly by Bede in [13]. Anyhow, it's difficult generally to get exact solution, in most cases, to these uncertainties because of the complexities involved, so we need to apply reliable numerical and approximate techniques to deal with those situations. For example, the finite element, residual power series, fuzzy Picard, and reproducing kernel are some of these techniques [14,15,16].

Quadratic Riccati differential equations (QRDEs) constitute a particular nonlinear model for describing specific class of physical systems with applications in optimal control, diffusion process, artificial intelligence and optical networks. Nowadays, numerous analytical and numerical methods are implemented to obtain the solution of the QRDEs, for instance, the solution for such

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QRDEs has been studied in [17] by using the homotopy analysis method. While the homotopy perpetuation method (HPM) has been proposed in [18], as well as the method of differential transformation has been applied in [19]. Also, a solution to QRDEs has been discussed in [20] by utilizing Adomian decomposition method. These mentioned studies dealt with the crisp sense of the variables and parameters. In this regard, uncertainties make the issue more provocative and challenging. However, numerical investigations for fuzzy QRDEs are rare in literature. In [21], the HPM has been applied for solving fuzzy QRDEs with various types of fuzzy environment. The Euler method has been used for solving fuzzy QRDEs under SGD [22]. Meanwhile, the fuzzy Picard technique is investigated for solving fuzzy QRDEs and PainleveI equation [14]. However, other categories of advanced numerical methods for different topics can be found in [23]-[32].

The target of this paper is to extend the application of the RPS algorithm for determining the PS solution of fuzzy QRDEs along with suitable fuzzy constraint under strongly-generalized differentiability. More precisely, we consider the following nonlinear fuzzy QRDE:

$$\hat{f}'(t) = p(t) + q(t)\hat{f} + r(t)\hat{f}^2(t), t > 0 \tag{1}$$

with the fuzzy initial condition

$$\hat{f}(0) = \hat{c}_0, \tag{2}$$

where $p(t)$, $q(t)$ and $r(t)$ are given functions, $\hat{c}_0$ is arbitrary fuzzy number, and $\hat{f}(t)$ is unknown fuzzy function of the crisp variable t. Anyhow, assuming that the IVPs (1) and (2) has a unique fuzzy solution at each $t > 0$. During this article, $\mathbb{R}_F$ stands to the set of all fuzzy numbers defined on $\mathbb{R}$.

The RPS algorithm is a novel numeric scheme that developed to study and interpret the series solution of first- and second-order uncertain IVPs. This method is used effectively to provide the power series and fractional power series solutions to several problems which arise in engineering and science area. The proposed approach aims at building a solution of a PS expansion as well as minimizing residual error functions for computing the unknown coefficients of PS by applying a certain differential operator without linearization or limitation on the problem structure [33]-[37]. On the other hand, we refer to [38]-[45] to see many characteristics of modeling and simulation of some advanced methods to deal with different issues that occur in natural phenomena.

This paper is structured as follows. In Section 2, fundamental concepts and definitions for the theory of fuzzy calculus are given briefly. In Section 3, formulation of quadratic fuzzy Riccati differential equation (QFRDE) is presented under the concept of SGD. The RPS algorithm is described in Section 4 for handling the quadratic fuzzy Riccati problem. Some examples are also presented in Section 5 to clarify the proposed method. In the end, some conclusions are presented.

2 Preliminaries

The necessary definitions and properties concerning the theory of fuzzy calculus are briefly recalled in this section. In general, a fuzzy number ω is a fuzzy subset of $\mathbb{R}$ with normal, convex, and upper semi-continuous membership function of bounded support [10,11,12].

Definition 2.1 [10] The fuzzy number ω is a mapping $\omega : \mathbb{R} \to [0,1]$ such that:

1. $\omega(\lambda s + (1-\lambda)t) \geq \min\{\omega(s), \omega(t)\}$ for all $s,t \in \mathbb{R}$ and $0 \leq \lambda \leq 1$, i.e. ω is fuzzy convex.
2. If $\exists s \in \mathbb{R}$, with $\omega(s) = 1$ then ω is normal.
3. ω is upper semi continues.
4. The closure of $supp(\omega) = \{s \in \mathbb{R} : \omega(s) > 0\}$ is compact, where $supp(\omega)$ is called the support of ω .

For each $\gamma \in (0,1]$, set $[\omega]^\gamma = \{s \in \mathbb{R} \backslash \omega(s) \geq \gamma\}$ and $[\omega]^0 = \overline{\{s \in \mathbb{R} \backslash \omega(s) > 0\}}$, whereas $\overline{\{.\}}$ denote to the closure of $\{.\}$. Thus, clearly that ω is a fuzzy number whenever $[\omega]^\gamma$ is convex compact set of $\mathbb{R}$ at each $\gamma \in [0,1]$ and $[\omega]^1 \neq \phi$. Consequently, if ω is a fuzzy number, then $[\omega]^\gamma = [\omega_1(\gamma), \omega_2(\gamma)]$, where $\omega_1(\gamma) = \min\{s : s \in [\omega]^\gamma\}$ and $\omega_2(\gamma) = \max\{s : s \in [\omega]^\gamma\}$ for each $\gamma \in [0,1]$. Here, $[\omega]^\gamma$ is called the γ-level representation of ω. Furthermore, the parametric form or the γ-level can be given by $[\omega]^\gamma = \{s \in \mathbb{R} \setminus \omega(s) \geq \gamma\}$ as soon as $\gamma \in (0,1]$, and $[\omega]^\gamma = \overline{supp(\omega)}$ when $\gamma = 0$. Obviously, ω in its parametric form is closed and bounded interval $[\omega_1(\gamma), \omega_2(\gamma)]$ where ω_1 and ω_2 are the lower and upper γ-level representations, respectively, for the fuzzy number ω. Anyhow, the following definition is an equivalent characterization of the parametric form of ω.

Theorem 2.1 [10] Suppose that $\omega_1, \omega_2 : [0,1] \to \mathbb{R}$ are bounded functions satisfying the requirements:

1. ω_1 and ω_2 are increasing and decreasing, respectively, with $\omega_1(1) \leq \omega_2(1)$.
2. ω_1 and ω_2 are right-hand continuous for $\gamma = 0$.
3. ω_1 and ω_2 are left-hand continuous for $\gamma = i$, $\forall i \in (0,1]$.

Thus, $\omega : \mathbb{R} \to [0,1]$ defined as $\omega(s) = sup\{\gamma : \omega_1(\gamma) \leq s \leq \omega_2(\gamma)\}$ is a fuzzy number with parametric form $[\omega_{1\gamma}, \omega_{2\gamma}]$. Furthermore, for any fuzzy number $\omega : \mathbb{R} \to [0,1]$ with parametric form $[\omega_{1\gamma}, \omega_{2\gamma}]$, it can say that the functions ω_1 and ω_2 satisfy the above conditions.

For each $0 \leq \gamma \leq 1$, and ω and $\mathscr{P}$ are two arbitrary fuzzy numbers with $[\omega]^\gamma = [\omega_{1\gamma}, \omega_{2\gamma}]$ and $[\mathscr{P}]^\gamma = [\mathscr{P}_{1\gamma}, \mathscr{P}_{2\gamma}]$ and $c \in \mathbb{R}$, we have the following:

1. $[\omega + \mathscr{P}]^\gamma = [(\omega + \mathscr{P})_{1\gamma}, (\omega + \mathscr{P})_{2\gamma}] = [\omega_{1\gamma} + \mathscr{P}_{1\gamma}, \omega_{2\gamma} + \mathscr{P}_{2\gamma}]$.
2. $[c\omega]^\gamma = \begin{cases} [c\omega_{1\gamma}, c\omega_{2\gamma}] : c \geq 0 \\ [c\omega_{2\gamma}, c\omega_{1\gamma}] : c < 0. \end{cases}$
3. $\omega = \mathscr{P}$ if and only if $\omega_{1\gamma} = \mathscr{P}_{1\gamma}$ and $\omega_{2\gamma} = \mathscr{P}_{2\gamma}$.

Definition 2.2 [11] The complete metric space of $\mathbb{R}_F$ is defied by the mapping $d : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}^+ \cup \{0\}$ with

Hausdorff distance such that

$$d(\omega,\mathscr{P}) = \sup_{0\leq\gamma\leq 1} \max\{|\ \omega_{1\gamma} - \mathscr{P}_{1\gamma}\ |, |\ \omega_{2\gamma} - \mathscr{P}_{2\gamma}\ |\}$$

for arbitrary fuzzy number ω and $\mathscr{P}$.

Definition 2.3 [12] Let ω and $\mathscr{P} \in \mathbb{R}_F$, if $\exists \mathscr{K} \in \mathbb{R}_F$ where $\omega = \mathscr{P} + \mathscr{K}$, then $\mathscr{K}$ is called Hukuhara difference (H-difference) of ω and $\mathscr{P}$, indicated by $\omega \ominus_h \mathscr{P}$.

Definition 2.4 [46] Suppose that $\hat{f} = [a,b] \to \mathbb{R}_F$. For fixed $t_0 \in [a,b]$, $\hat{f}$ is called strongly generalized differentiable at t_0, if there exists an element $\hat{f}'(t_0) \in \mathbb{R}_F$ such that either:

1.The H-difference $\hat{f}(t_0+\varepsilon) \ominus_h \hat{f}(t_0), \hat{f}(t_0) \ominus_h \hat{f}(t_0-\varepsilon)$ exist and $\hat{f}'(t_0) = \lim_{\varepsilon\to 0^+} \frac{\hat{f}(t_0+\varepsilon)\ominus_h \hat{f}(t_0)}{\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{\hat{f}(t_0)\ominus_h \hat{f}(t_0-\varepsilon)}{\varepsilon}$, for all $\varepsilon > 0$ sufficiently near to 0, and the limits in a metric d.

2.The H-difference $\hat{f}(t_0) \ominus_h \hat{f}(t_0+\varepsilon), \hat{f}(t_0)\hat{f}(t_0-\varepsilon) \ominus_h \hat{f}(t_0)$ exist and $\hat{f}'(t_0) = \lim_{\varepsilon\to 0^+} \frac{\hat{f}(t_0)\ominus_h \hat{f}(t_0+\varepsilon)}{-\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{\hat{f}(t_0-\varepsilon)\ominus_h \hat{f}(t_0)}{-\varepsilon}$, for all $\varepsilon > 0$ sufficiently near to 0, and the limits in a metric d.

Theorem 2.2 [47] If $\hat{f} : [a,b] \to \mathbb{R}_F$, set $[\hat{f}(t)]^\gamma = [\hat{f}_{1\gamma}(t), \hat{f}_{2\gamma}(t)]$ for each $\gamma \in [0,1]$, then:

1.If $\hat{f}$ is (1)-differentiable, then $\hat{f}_{1\gamma}$ and $\hat{f}_{2\gamma}$ are differentiable functions and $[\hat{f}'(t)]^\gamma = [\hat{f}'_{1\gamma}(t), \hat{f}'_{2\gamma}(t)]$.

2.If $\hat{f}$ is (2)-differentiable, then $\hat{f}_{1\gamma}$ and $\hat{f}_{2\gamma}$ are differentiable functions and $[\hat{f}'(t)]^\gamma = [\hat{f}'_{2\gamma}(t), \hat{f}'_{1\gamma}(t)]$.

Definition 2.5 [47] Let $\hat{f} : [a,b] \to \mathbb{R}_F$ be a fuzzy-valued function. For fixed $t_0 \in [a,b]$ and $\varepsilon > 0$ if there exist $\delta > 0$ such that $|\ t - t_0\ | < \delta$ which implies $d(\hat{f}(t), \hat{f}(t_0)) < \varepsilon$, then we say that $\hat{f}$ is continuous at t_0.

3 Formulation of QFRDE

The purpose of this section is to discuss the formulation of the QFRDE with fuzzy initial condition in the following form under the concept of SGD:

$$\hat{f}'(t) = A\hat{f}^2(t) + B\hat{f}(t) + C, t > 0, \tag{3}$$

with the fuzzy initial condition

$$\hat{f}(0) = \hat{f}_0. \tag{4}$$

where A, B and $C \in \mathbb{R}$, $\hat{f}(t) = [0,T] \to \mathbb{R}_F$ and $\hat{f}_0 \in \mathbb{R}_F$.

To construct the section of the QFRDE (3) based on the type of differentiability and fuzzy initial condition (4), we consider the γ-level representation of $\hat{f}'(t), \hat{f}(t), \hat{f}^2(t)$ and $\hat{f}(0)$ as $[\hat{f}'_{1\gamma}(t), \hat{f}'_{2\gamma}(t)]$, $[\hat{f}_{1\gamma}(t), \hat{f}_{2\gamma}(t)]$, $[\hat{f}^2_{1\gamma}(t), \hat{f}^2_{2\gamma}(t)]$ and $[\hat{f}_{0,1\gamma}(t), \hat{f}_{0,2\gamma}(t)]$, respectively. Consequently, the QFRDEs (3) and (4) should be rewritten with the parametric form as follows:

$$[\hat{f}'(t)]^\gamma = A[\hat{f}^2(t)]^\gamma + B[\hat{f}(t)]^\gamma + C, \quad t > 0 \tag{5}$$

with the fuzzy initial condition

$$[\hat{f}(0)]^\gamma = [\hat{f}_0]^\gamma. \tag{6}$$

Definition 3.1 [47] Let $\hat{f} : [a,b] \to \mathbb{R}_F$ where $D_{(1)}\hat{f}(t)$ or $D_{(2)}\hat{f}(t)$ exists. If $\hat{f}$ and $D_{(1)}\hat{f}(t)$ satisfy QFRDE (3), then we say $\hat{f}$ is (1)-solution for QFRDE (3). Otherwise, if $\hat{f}$ and $D_{(2)}\hat{f}(t)$ satisfy QFRDE (3), then we say $\hat{f}$ is (2)-solution for QFRDE (3).

Now, the next algorithm presents us the RPS strategy for solving IVPs (5) and (6) in γ- level representation that converted to crisp systems of ODEs.

Algorithm 3.1 To obtain the fuzzy solution $\hat{f}(t)$ for the IVPs (5) and (6), two cases are considered according to the kinds of differentiability, where $\hat{f}(t)$ is either (1)-differentiable or (2)-differentiable.

Case 1: If $\hat{f}(t)$ is (1)-differentiable, then IVPs (5) and (6) can be converted into the following crisp system:

$$\begin{aligned} \hat{f}'_{1\gamma}(t) &= A\hat{f}^2_{1\gamma}(t) + B\hat{f}_{1\gamma}(t) + C, \\ \hat{f}'_{2\gamma}(t) &= A\hat{f}^2_{2\gamma}(t) + B\hat{f}_{2\gamma}(t) + C, \end{aligned} \tag{7}$$

with the initial condition,

$$\begin{aligned} \hat{f}_{1\gamma}(t) &= \hat{f}_{0,1\gamma}, \\ \hat{f}_{2\gamma}(t) &= \hat{f}_{0,2\gamma}. \end{aligned} \tag{8}$$

Consequently, the following actions should be taken:

A1: Solve the system (7) and (8) using the procedure of RPS algorithm.

A2: Ensure that the solution $[\hat{f}_{1\gamma}(t), \hat{f}_{2\gamma}(t)]$ and $[\hat{f}'_{1\gamma}(t), \hat{f}'_{2\gamma}(t)]$ are valid γ-level sets, $\forall\gamma \in [0,1]$.

A3: Obtain the (1)-solution $\hat{f}$ whose γ-level representation is $[\hat{f}_{1\gamma}(t), \hat{f}_{2\gamma}(t)]$.

Case 2: If $\hat{f}(t)$ is (2)-differentiable, then IVPs (5) and (6) can be converted into the following crisp system:

$$\begin{aligned} \hat{f}'_{1\gamma}(t) &= A\hat{f}^2_{2\gamma}(t) + B\hat{f}_{2\gamma}(t) + C, \\ \hat{f}'_{2\gamma}(t) &= A\hat{f}^2_{1\gamma}(t) + B\hat{f}_{1\gamma}(t) + C, \end{aligned} \tag{9}$$

with the initial condition,

$$\begin{aligned} \hat{f}_{1\gamma}(t) &= \hat{f}_{0,1\gamma}, \\ \hat{f}_{2\gamma}(t) &= \hat{f}_{0,2\gamma}. \end{aligned} \tag{10}$$

Consequently, the following actions should be taken:

B1: Solve the system (9) and (10) using the procedure of RPS algorithm.

B2: Ensure that the solution $[\hat{f}_{1\gamma}(t),\hat{f}_{2\gamma}(t)]$ and $[\hat{f'}_{1\gamma}(t),\hat{f'}_{2\gamma}(t)]$ are valid γ-level sets, $\forall\gamma\in[0,1]$.
B3: Obtain the (2)-solution $\hat{f}$ whose γ-level representation is $[\hat{f}_{2\gamma}(t),\hat{f}_{1\gamma}(t)]$.

The previous formulation of IVPs (5) and (6) along with Theorem 2.2 shows us how to deal with numerical solution of such problem allowing the consideration of two cases, where $\hat{f}(t)$ is either (1)-differentiable or (2)-differentiable. For each case, the original QFRDE can be switched to an equivalent crisp system of ODEs. As a result, numerical methods can be used directly to solve the crisp system obtained without having to be formulated in an uncertain sense (see [14,15,16]).

4 The RPS method for the fuzzy QRDE

In this section, we seek to obtain the (1)-solution for the fuzzy QRDE (5) and (6) by employing the procedures of RPS method. Further, same procedure can be followed, whenever $\hat{f}(t)$ is (2)-differentiable, to construct the (2)-solution for the fuzzy QRDE QRDE (5) and (6). To perform so, we assume that $\hat{f}(t)$ is (1)-differentiable, therefore the solutions of system (7) and (8) at $t_0=0$ have the following forms:

$$\begin{aligned}\hat{f}_{1\gamma}(t)&=\sum_{n=0}^{\infty}a_nt^n,\\ \hat{f}_{2\gamma}(t)&=\sum_{n=0}^{\infty}b_nt^n.\end{aligned}\tag{11}$$

By using the initial conditions $\hat{f}_{1\gamma}(0)=\hat{f}_{0,1\gamma}=a_0$ and $\hat{f}_{2\gamma}(0)=\hat{f}_{0,2\gamma}=b_0$ as initial iterative approximations. Then, the expansion of (11) can be written as:

$$\begin{aligned}\hat{f}_{1\gamma}(t)&=\hat{f}_{0,1\gamma}+\sum_{n=1}^{\infty}a_nt^n,\\ \hat{f}_{2\gamma}(t)&=\hat{f}_{0,2\gamma}+\sum_{n=1}^{\infty}b_nt^n.\end{aligned}\tag{12}$$

Consequently, the j^{th}-truncated series solutions of $\hat{f}_{1\gamma}(t)$ and $\hat{f}_{2\gamma}(t)$ can be given by:

$$\begin{aligned}\hat{f}_{j,1\gamma}(t)&=\hat{f}_{0,1\gamma}+\sum_{n=1}^{j}a_nt^n,\\ \hat{f}_{j,2\gamma}(t)&=\hat{f}_{0,2\gamma}+\sum_{n=1}^{j}b_nt^n.\end{aligned}\tag{13}$$

According the the RPS approach, the j^{th}-residual functions of system (7) and (8) are defined by

$$\begin{aligned}Res_{j,1\gamma}(t)&=\hat{f'}_{j,1\gamma}(t)-A\hat{f}^2_{j,1\gamma}(t)-B\hat{f}_{j,1\gamma}(t)-C,\\ Res_{j,2\gamma}(t)&=\hat{f'}_{j,2\gamma}(t)-A\hat{f}^2_{j,2\gamma}(t)-B\hat{f}_{j,2\gamma}(t)-C.\end{aligned}\tag{14}$$

where the ∞^{th}- residual functions are given by

$$\begin{aligned}Res_{\infty,1\gamma}(t)&=\lim_{j\to\infty}Res_{j,1\gamma}(t)\\ &=\hat{f'}_{1\gamma}(t)-A\hat{f}^2_{1\gamma}(t)-B\hat{f}_{1\gamma}(t)-C,\\ Res_{\infty,2\gamma}(t)&=\lim_{j\to\infty}Res_{j,2\gamma}(t)\\ &=\hat{f'}_{2\gamma}(t)-A\hat{f}^2_{2\gamma}(t)-B\hat{f}_{2\gamma}(t)-C.\end{aligned}\tag{15}$$

As in [38,39,48], clearly $Res_{\infty,i\gamma}(t)=0$ for each $t\in[0,R]$, R is radius of convergence and $i=\{1,2\}$, which are infinitely differentiable functions at $t=0$. Further, $\frac{d^{k-1}}{dt^{k-1}}Res_{\infty,i\gamma}(0)=\frac{d^{k-1}}{dt^{k-1}}Res_{k,i\gamma}(0)=0$, for $k=1,2,3,...,j$. Which is considered as a basic fact of RPS algorithm that helps us to determine the parameters a_n and $b_n, n\geq 1$.

In light of RPS algorithm, to find the coefficients a_1 and b_1, substitute $\hat{f}_{1,1\gamma}(t)=\hat{f}_{0,1\gamma}+a_1t$ and $\hat{f}_{1,2\gamma}(t)=\hat{f}_{0,2\gamma}+b_1t$ into the residual functions , $Res_{1,1\gamma}(t)$ and $Res_{1,2\gamma}(t)$, at $j=1$ of (14) such that:

$$\begin{aligned}Res_{1,1\gamma}(t)&=\hat{f'}_{1,1\gamma}(t)-A\hat{f}^2_{1,1\gamma}(t)-B\hat{f}_{1,1\gamma}(t)-C\\ &=(\hat{f}_{0,1\gamma}+a_1t)'-A(\hat{f}_{0,1\gamma}+a_1t)^2\\ &-B(\hat{f}_{0,1\gamma}+a_1t)-C\\ &=a_1-A(\hat{f}_{0,1\gamma}+a_1t)^2-B(\hat{f}_{0,1\gamma}+a_1t)-C,\\ Res_{1,2\gamma}(t)&=\hat{f'}_{1,2\gamma}(t)-A\hat{f}^2_{1,2\gamma}(t)-B\hat{f}_{1,2\gamma}(t)-C\\ &=(\hat{f}_{0,2\gamma}+b_1t)'-A(\hat{f}_{0,2\gamma}+b_1t)^2\\ &-B(\hat{f}_{0,2\gamma}+b_1t)-C\\ &=b_1-A(\hat{f}_{0,2\gamma}+a_1t)^2-B(\hat{f}_{0,2\gamma}+b_1t)-C.\end{aligned}\tag{16}$$

Using the facts that $Res_{1,1\gamma}(0)=0$ and $Res_{1,2\gamma}(0)=0$ in (16) it yields that $a_1=A\hat{f}^2_{0,1\gamma}+B\hat{f}_{0,1\gamma}+C$ and $b_1=A\hat{f}^2_{0,2\gamma}+B\hat{f}_{0,2\gamma}+C$. Therefore, the first RPS approximations are:

$$\begin{aligned}\hat{f}_{1,1\gamma}(t)&=a_0+(Aa_0^2+Ba_0+C)t,\\ \hat{f}_{1,2\gamma}(t)&=b_0+(Ab_0^2+Bb_0+C)t.\end{aligned}$$

For $j=2$, substitute $\hat{f}_{2,1\gamma}(t)=a_0+(Aa_0^2+Ba_0+C)t+a_2t^2$ and $\hat{f}_{2,2\gamma}(t)=b_0+(Ab_0^2+Bb_0+C)t+b_2t^2$ into the residual functions, $Res_{2,1\gamma}(t)$ and $Res_{2,2\gamma}(t)$ of (14) such that

$$\begin{aligned} Res_{2,1\gamma}(t) &= \hat{f}'_{2,1\gamma}(t) - A\hat{f}^2_{2,1\gamma}(t) - B\hat{f}_{2,1\gamma}(t) - C \\ &= ((Aa_0^2 + Ba_0 + C) + 2a_2t) \\ &\quad - A(a_0 + (Aa_0^2 + Ba_0 + C)t + a_2t^2)^2 \\ &\quad - B(a_0 + (Aa_0^2 + Ba_0 + C)t + a_2t^2) - C, \\ Res_{2,2\gamma}(t) &= \hat{f}'_{2,2\gamma}(t) - A\hat{f}^2_{2,2\gamma}(t) - B\hat{f}_{2,2\gamma}(t) - C \\ &= ((Ab_0^2 + Bb_0 + C) + 2b_2t) \\ &\quad - A(b_0 + (Ab_0^2 + Bb_0 + C)t + b_2t^2)^2 \\ &\quad - B(b_0 + (Ab_0^2 + Bb_0 + C)t + b_2t^2) - C. \end{aligned} \tag{17}$$

Now, differentiable both sides of (17) such that

$$\begin{aligned} \frac{d}{dt}Res_{2,1\gamma}(t) &= \frac{d}{dt}\big[((Aa_0^2 + Ba_0 + c) + 2a_2t) \\ &\quad - A\big(a_0 + (Aa_0^2 + Ba_0 + C)t + a_2t^2\big)^2 \\ &\quad - B\big(a_0 + (Aa_0^2 + Ba_0 + C)t + a_2t^2\big) - C\big] \\ &= 2a_2 - 2A\big((Aa_0^2 + Ba_0 + c) + 2a_2t\big) \\ &\quad \big(a_0 + (Aa_0^2 + Ba_0 + C)t + a_2t^2\big) \\ &\quad - B\big((Aa_0^2 + Ba_0 + C) + 2a_2t\big), \\ \frac{d}{dt}Res_{2,2\gamma}(t) &= \frac{d}{dt}\big[((Ab_0^2 + Bb_0 + c) + 2b_2t) \\ &\quad - A\big(b_0 + (Ab_0^2 + Bb_0 + C)t + b_2t^2\big)^2 \\ &\quad - B\big(b_0 + (Ab_0^2 + Bb_0 + C)t + b_2t^2\big) - C\big] \\ &= 2b_2 - 2A\big((Ab_0^2 + Bb_0 + c) + 2b_2t\big) \\ &\quad + \big(b_0(Ab_0^2 + Bb_0 + C)t + b_2t^2\big) \\ &\quad - B\big((Ab_0^2 + Bb_0 + C) + 2b_2t\big). \end{aligned}$$

Finally by using that facts $\frac{d}{dt}Res_{2,1\gamma}(0) = 0$ and $\frac{d}{dt}Res_{2,2\gamma}(0) = 0$, it can be deduced that $a_2 = Aa_0a_1 + \frac{1}{2}Ba_1$ and $b_2 = Ab_0b_1 + \frac{1}{2}Bb_1$. Therefore, the second approximations are:

$$\begin{aligned} \hat{f}_{2,1\gamma}(t) &= a_0 + (Aa_0^2 + Ba_0 + C)t + (Aa_0a_1 + \frac{1}{2}Ba_1)t^2, \\ \hat{f}_{2,2\gamma}(t) &= b_0 + (Ab_0^2 + Bb_0 + C)t + (Ab_0b_1 + \frac{1}{2}Bb_1)t^2. \end{aligned}$$

For $j = 3$, if we substitute $\hat{f}_{3,1\gamma}(t)$ and $\hat{f}_{3,2\gamma}(t)$ into the residual functions $Res_{3,1\gamma}(t)$ and $Res_{3,2\gamma}(t)$ of Eq (14) and then utilize the facts $\frac{d^2}{dt^2}Res_{3,1\gamma}(0) = 0$ and $\frac{d^2}{dt^2}Res_{3,2\gamma}(0) = 0$. Then, the third coefficients a_3 and b_3 are given by

$$\begin{aligned} a_3 &= \frac{1}{3}A(2a_0a_2 + a_1^2) + \frac{1}{3}Ba_2, \\ b_3 &= \frac{1}{3}A(2b_0b_2 + b_1^2) + \frac{1}{3}Bb_2. \end{aligned}$$

Hence, the third RPS approximations can be also given. By continuing with the same procedures until arbitrary coefficients order $j = n$ as well as using the facts $\frac{d^{(n-1)}}{dt^{(n-1)}}Res_{n,1\gamma}(0) = \frac{d^{(n-1)}}{dt^{(n-1)}}Res_{n,2\gamma}(0) = 0$, then the unknown coefficients a_n and b_n can be obtained. Anyhow, more iteration leads to more accurate solutions. Similarly, if $\hat{f}(t)$ is (2)-differentiable, then the (2)-solution for the fuzzy QRDE (5) and (6) can be obtained.

5 Illustrative Example

To test the applicability and accuracy of the suggested algorithm, several examples are tested numerically in this section. The RPS methodology is directly applied without using transformation or discretization. For results and calculations, the numeric computations are implemented using Mathematics 10.

Example 5.1 Consider the following the fuzzy QRDE:

$$\hat{f}'(t) = 2\hat{f}(t) - \hat{f}^2(t) + 1, t > 0, \tag{18}$$

with the fuzzy initial condition

$$[\hat{f}(0)]^\gamma = [\gamma - 1, 1 - \gamma], \gamma \in [0,1]. \tag{19}$$

In particular, for $\gamma = 1$, the exact solution of (18) with crisp initial condition $f(0) = 0$ can be found as follows:

$$f(t) = 1 + \sqrt{2}\tanh\Big(\sqrt{2}t + \frac{1}{2}\log\big(\frac{\sqrt{2}-1}{\sqrt{2}+1}\big)\Big). \tag{20}$$

Using definition 2.4, the fuzzy IVPs (18) and (19) can be reduced to the set of ODEs corresponding to their parametric forms as follows:

$$\begin{aligned} \hat{f}'_{1\gamma}(t) &= 2\hat{f}_{1\gamma}(t) - \hat{f}^2_{1\gamma}(t) + 1, \\ \hat{f}'_{2\gamma}(t) &= 2\hat{f}_{2\gamma}(t) - \hat{f}^2_{2\gamma}(t) + 1, \end{aligned} \tag{21}$$

subject to the initial conditions

$$\begin{aligned} \hat{f}'_{1\gamma}(0) &= \gamma - 1, \\ \hat{f}'_{2\gamma}(0) &= 1 - \gamma. \end{aligned} \tag{22}$$

According the producer of the RPS algorithm presented in the last section and depending on the initial data $\hat{f}'_{0,1\gamma} = \hat{f}'_{1\gamma}(0) = \gamma - 1$ and $\hat{f}'_{0,2\gamma} = \hat{f}'_{2\gamma}(0) = 1 - \gamma$, the proposed RPS solutions $\hat{f}'_{1\gamma}(t)$ and $\hat{f}'_{2\gamma}(t)$ of system (21) can be given by:

$$\begin{aligned} \hat{f}_{j,1\gamma}(t) &= \gamma - 1 + a_1t + a_2t^2 + \ldots + a_jt^j + \ldots, \\ \hat{f}_{j,2\gamma}(t) &= 1 - \gamma + b_1t + b_2t^2 + \ldots + b_jt^j + \ldots. \end{aligned} \tag{23}$$

By utilizing the facts $\frac{d^{(j-1)}}{dt^{(j-1)}}Res_{j,1\gamma}(0) = 0$ and $\frac{d^{(j-1)}}{dt^{(j-1)}}Res_{j,2\gamma}(0) = 0$, for $j = 1, 2, \ldots,$, the first few terms

a_j and b_j are:

$$\begin{aligned}
a_1 &= (-2+4\gamma-\gamma^2)t,\\
a_2 &= (-4+10\gamma-6\gamma^2+\gamma^3)t^2,\\
a_3 &= \frac{1}{3}(-20+64\gamma-64\gamma^2+24\gamma^3-3\gamma^4)t^3,\\
a_4 &= \frac{1}{3}(-32+128\gamma-180\gamma^2+110\gamma^3-30\gamma^4+3\gamma^5)t^4,\\
a_5 &= \frac{1}{15}(-256+1232\gamma-2228\gamma^2+1920\gamma^3\\
&\quad -840\gamma^4+180\gamma^5-15\gamma^6)t^5,\\
a_6 &= \frac{1}{45}(-1232+6920\gamma-15288\gamma^2+17108\gamma^3\\
&\quad -10500\gamma^4+3570\gamma^5-630\gamma^6+45\gamma^7)t^6,\\
a_7 &= \frac{1}{315}(-13840+88832\gamma-231872\gamma^2\\
&\quad +319872\gamma^3-255024\gamma^4+120960\gamma^5\\
&\quad -33600\gamma^6+5040\gamma^7-315\gamma^8)t^7,\\
&\ \ \vdots
\end{aligned}$$

and

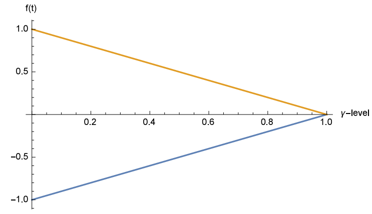
$$\begin{aligned}
b_1 &= (2-\gamma^2)t,\\
b_2 &= (2\gamma-\gamma^3)t^2,\\
b_3 &= \frac{1}{3}(-4+8\gamma^2-3\gamma^4)t^3,\\
b_4 &= \frac{1}{3}(-8\gamma+10\gamma^3-3\gamma^5)t^4,\\
b_5 &= \frac{1}{15}(16-68\gamma^2+60\gamma^4-15\gamma^6)t^5,\\
b_6 &= \frac{1}{45}(136\gamma-308\gamma^3+210\gamma^5-45\gamma^7)t^6,\\
b_7 &= \frac{1}{315}(-272+1984\gamma^2-3024\gamma^4+1680\gamma^6-315\gamma^8)t^7,\\
&\ \ \vdots
\end{aligned}$$

and so on. Particularly, if $\gamma = 1$, then the RPS solution

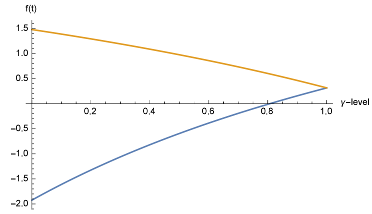
$$f(t) = t+t^2+\frac{1}{3}t^3-\frac{1}{3}t^4-\frac{7}{15}t^5-\frac{7}{45}t^6+\frac{53}{315}t^7+\frac{71}{315}t^8+\ldots$$

which matches the Taylor series expansion of the exact solution of Eq.(20).

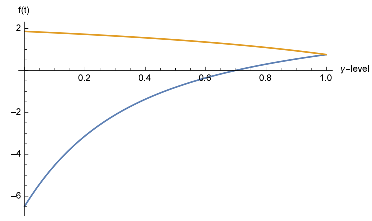
In the light of showing the agreement between the exact solutions and RPS solutions, some numerical results together with the absolute and relative errors at some selected grid points t_i in $[0,1]$ with step-size 0.1 are listed in Table 1 for $n = 51$ and $\gamma = 1$. From the table, it can be noted that the RPS approximate solutions are in good agreement with the exact solutions over the domain of interest. Anyhow, more iteration leads to more accurate solutions. For further analysis, numerical comparison is presented in Table 2 between the 10^{th} RPS solution and other existing numerical methods including 4^{th}-order Runge-Kutta method (RK-4), optimal homotopy asymptotic method (OHA) [49], and multiagent neural network method (MNN) [50]. It can be concluded from the numeric comparisons that the gained results by the RPS method are very well in comparison with those obtained by other methods.



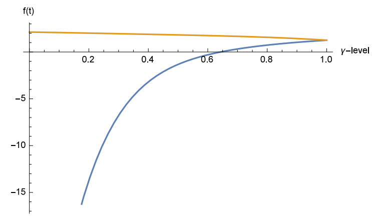
(a) Solution plots for $t = 0$



(b) Solution plots for $t = 0.25$



(c) Solution plots for $t = 0.5$



(d) Solution plots for $t = 0.75$

Fig. 1: Triangular fuzzy solution plots for the $8th$-RPS solutions of Example 5.1.

Figure 1 shows the lower and upper bounds of the triangular fuzzy RPS solutions at $n = 8$ with different values of t such that $t \in \{0.0, 0.25, 0.5, 0.75\}$. While Figure 2 depicts the surface plot of the 10th-RPS approximate solution for all $t \in [0,1]$ and $\gamma \in [0,1]$, where blue and yellow color correspond to the upper and lower bounds of the 10th-RPS fuzzy solution.

Table 1: Numerical result of the RPS solutions for Example 5.1 at $\gamma = 1$.

t_i	Exact solution	Approximate solution	Absolute error	Relative error
0.1	0.110295196916	0.110295196917	1.52656×10^{-16}	1.38406×10^{-15}
0.2	0.241976799621	0.241976799621	5.55112×10^{-17}	2.29407×10^{-16}
0.3	0.395104848660	0.395104848660	2.77556×10^{-16}	7.02486×10^{-16}
0.4	0.567812166293	0.567812166293	3.33067×10^{-16}	5.86579×10^{-16}
0.5	0.756014393431	0.756014393431	3.33067×10^{-16}	4.40556×10^{-11}
0.6	0.953566216472	0.953566216472	1.11022×10^{-16}	1.16429×10^{-16}
0.7	1.152948966979	1.152948966980	5.32907×10^{-14}	4.62212×10^{-14}
0.8	1.346363655368	1.346363655422	5.39830×10^{-11}	4.00954×10^{-11}
0.9	1.526911313280	1.526911336846	2.35658×10^{-8}	1.54336×10^{-8}

Table 2: Approximate solutions of Example 5.1 by various methods at $\gamma = 1$.

t_i	Exact	10^{th} RPSM	OHAM [47]	MNN [48]	RK-4
0.1	0.110295	0.110295	0.110328	0.110295	0.100000
0.2	0.241977	0.241977	0.242273	0.241976	0.219000
0.3	0.395105	0.395105	0.396175	0.395089	0.358004
0.4	0.567812	0.567812	0.570231	0.567660	0.516788
0.5	0.756014	0.756014	0.759555	0.755134	0.693439
0.6	0.953566	0.953566	0.955049	0.949964	0.884041
0.7	1.152949	1.152949	1.142444	1.141423	1.082696
0.8	1.346364	1.346358	1.300569	1.315723	1.282012
0.9	1.526911	1.526814	1.400444	1.456545	1.474059

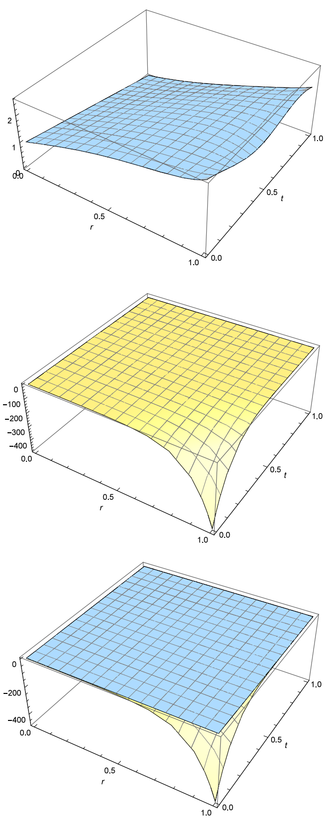


Fig. 2: The 3-dim plot for Example 5.1:blue and yellow are lower and upper bounds of 10th-RPS fuzzy solution.

Example 5.2 Consider the following quadratic fuzzy QRDE:

$$\hat{f}'(t) = 2\hat{f}(t) - \hat{f}^2(t) + 1, t > 0, \tag{24}$$

with the fuzzy initial condition

$$[\hat{f}(0)]^{\gamma} = [0.1 + 0.1\gamma, 0.3 - 0.1\gamma], \gamma \in [0,1]. \tag{25}$$

In particular, for $\gamma = 1$, the exact solution of (24) with crisp initial condition $f(0) = 0.2$ can be found as follows

$$f(t) = 1 + \sqrt{2}\tanh\Big(\sqrt{2}t + \frac{1}{2}\log\big(\frac{5\sqrt{2}-4}{5\sqrt{2}+4}\big)\Big). \tag{26}$$

Using definition 2, the fuzzy IVPs (24) and (25) can be reduced to the set of ODEs corresponding to their parametric forms as follows:

$$\begin{aligned} \hat{f'}_{1\gamma}(t) &= 2\hat{f}_{1\gamma}(t) - \hat{f}_{1\gamma}^2(t) + 1, \\ \hat{f'}_{2\gamma}(t) &= 2\hat{f}_{2\gamma}(t) - \hat{f}_{2\gamma}^2(t) + 1, \end{aligned} \tag{27}$$

which are subject to the initial conditions

$$\begin{aligned} \hat{f'}_{1\gamma}(0) &= 0.1 + 0.1\gamma, \\ \hat{f'}_{2\gamma}(0) &= 0.3 - 0.1\gamma. \end{aligned} \tag{28}$$

Following the RPS algorithm and depending on the initial data of system (27) and (28), the first few terms a_n and b_n of the truncated RPS solution described in Eq. (13)

are given by:

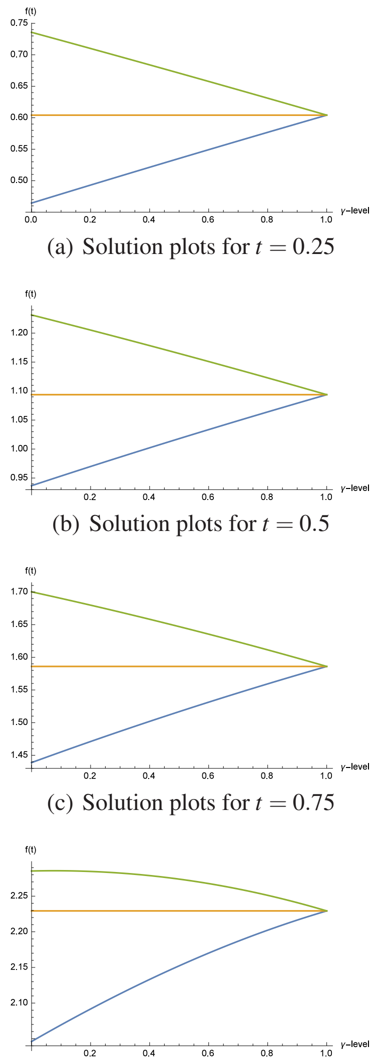
$$
\begin{aligned}
a_1 &= \frac{1}{10^2}(119+18\gamma-\gamma^2)t,\\
a_2 &= \frac{1}{10^3}(1071+43\gamma-27\gamma^2+\gamma^3)t^2,\\
a_3 &= \frac{1}{3\times 10^4}(5117-5652\gamma-658\gamma^2+108\gamma^3-3\gamma^4)t^3,\\
a_4 &= \frac{1}{3\times 10^5}(-168147-64585\gamma+5130\gamma^2+1430\gamma^3\\
&\quad -135\gamma^4+3\gamma^5)t^4,\\
a_5 &= \frac{1}{3\times 10^6}(-1537123+11682\gamma+151955\gamma^2+540\gamma^3\\
&\quad -2445\gamma^4+162\gamma^5-3\gamma^6)t^5,\\
a_6 &= \frac{1}{9\times 10^7}(695079+18187783\gamma+2825739\gamma^2-719285\gamma^3\\
&\quad -40635\gamma^4+11109\gamma^5-567\gamma^6+9\gamma^7)t^6,\\
&\ \vdots
\end{aligned}
$$

and

$$
\begin{aligned}
b_1 &= \frac{1}{10^2}(151-4\gamma-\gamma^2)t,\\
b_2 &= \frac{1}{10^3}(1057+53\gamma-21\gamma^2-\gamma^3)t^2,\\
b_3 &= \frac{1}{3\times 10^4}(-8003+7084\gamma-82\gamma^2-84\gamma^3-3\gamma^4)t^3,\\
b_4 &= \frac{1}{3\times 10^5}(-267421+30985\gamma+10710\gamma^2-470\gamma^3\\
&\quad -105\gamma^4-3\gamma^5)t^4,\\
b_5 &= \frac{1}{3\times 10^6}(-935747-560126\gamma+108755\gamma^2+13020\gamma^3\\
&\quad -1005\gamma^4-126\gamma^5-3\gamma^6)t^5,\\
b_6 &= \frac{1}{9\times 10^7}(42289513-20342887\gamma-1706523\gamma^2\\
&\quad +685685\gamma^3+38955\gamma^4-5061\gamma^5-441\gamma^6-9\gamma^7)t^6,\\
&\ \vdots
\end{aligned}
$$

and so on. Particularly, if $\gamma = 1$, then the RPS solution $f(t) = \frac{1}{5} + \frac{34}{25}t + \frac{136}{125}t^2 - \frac{68}{1875}t^3 - \frac{7072}{9365}t^4 - \frac{21488}{46875}t^5 + \frac{163744}{703125}t^6 + \frac{11459632}{24609375}t^7 + \frac{16217728}{123046875}t^8 + \dots$, which matches the Taylor series expansion about $t = 0$ of the exact solution in Eq.(26).

To show the effectiveness of the proposed algorithm, numerical outcomes of lower and upper bounds of the fuzzy solutions for system (27) and (28) at some selected grid points with step size 0.1 are summarized in Tables 3 and 4 for $\gamma = 1$ and $n = 51$. From these tables, it is interesting to observe that the lower and upper bounds of the RPS-solutions are the same for $\gamma = 1$, which precisely agree with the crisp solution. Figure 3 shows the lower and upper bounds of the triangular fuzzy RPS solutions at $n = 8$ with different values of t such that $t = 0.25, 0.5, 0.75$ and 1, where the midline represents the center (crisp) solution for $\gamma = 1$. While Figure 4 depicts the surface plot of the 10th-RPS approximate solution for all $t \in [0,1]$ and $\gamma \in [0,1]$ in which the blue and orange color correspond to upper and lower bounds of the 10th-RPS fuzzy solution.



(a) Solution plots for $t = 0.25$

(b) Solution plots for $t = 0.5$

(c) Solution plots for $t = 0.75$

(d) Solution plots for $t = 1$

Fig. 3: Triangular fuzzy solution plots for the $8th$-RPS solutions of Example 5.2.

Table 3: Numerical result of lower bound RPS-solution of Example 5.2

t_i	Exact solution	Approximate solution	Absolute error	Relative error
0.1	0.34676399505	0.346763995059	1.11022×10^{-16}	3.20167×10^{-16}
0.2	0.51389727584	0.513897275841	1.11022×10^{-16}	2.16040×10^{-16}
0.3	0.69799087251	0.697990872514	2.22045×10^{-16}	3.18120×10^{-16}
0.4	0.89347216024	0.893472160244	3.33067×10^{-16}	3.72778×10^{-16}
0.5	1.09313499306	1.093134993060	2.22045×10^{-16}	2.03126×10^{-16}
0.6	1.28913717194	1.289137171947	2.22045×10^{-16}	1.72243×10^{-16}
0.7	1.47419437696	1.474194376963	1.05871×10^{-12}	7.18161×10^{-13}
0.8	1.64260136255	1.642601633577	1.02182×10^{-9}	6.22073×10^{-10}
0.9	1.79079791833	1.790798347506	4.29173×10^{-7}	2.39655×10^{-7}

Table 4: Numerical result of upper bound RPS-solution of Example 5.2

t_i	Exact solution	Approximate solution	Absolute error	Relative error
0.1	0.34676399505	0.346763995059	1.66533×10^{-16}	4.80250×10^{-16}
0.2	0.51389727584	0.513897275841	1.11022×10^{-16}	2.16040×10^{-16}
0.3	0.69799087251	0.697990872514	2.22045×10^{-16}	3.18120×10^{-16}
0.4	0.89347216024	0.893472160244	3.33067×10^{-16}	3.72778×10^{-16}
0.5	1.09313499306	1.093134993060	2.22045×10^{-16}	2.03126×10^{-16}
0.6	1.28913717194	1.289137171947	2.22045×10^{-16}	1.72243×10^{-16}
0.7	1.47419437696	1.474194376963	1.05871×10^{-12}	7.18161×10^{-13}
0.8	1.64260136255	1.642601633577	1.02182×10^{-9}	6.22073×10^{-10}
0.9	1.79079791833	1.790798347506	4.29173×10^{-7}	2.39655×10^{-7}

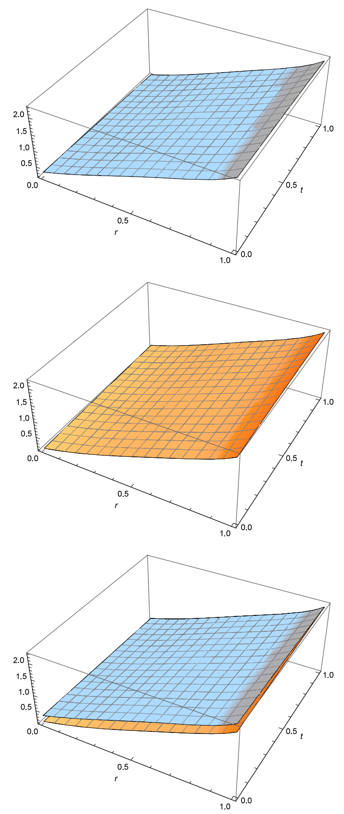


Fig. 4: The 3-dim plot for Example 5.2:blue and orange are lower and upper bounds of 10th-RPS fuzzy solution.

Example 5.3 Consider the following fuzzy QRDE

$$\hat{f}'(t) = -\hat{f}^2(t) + 1, t > 0, \tag{29}$$

with the fuzzy initial conation

$$[\hat{f}(0)]^\gamma = [\gamma - 1, 1 - \gamma], \gamma \in [0,1]. \tag{30}$$

In particular, for $\gamma = 1$, the exact solution of Eq.(29) with crisp initial condition $f(0) = 0$ is given as:

$$f(t) = (e^{2t} - 1)(e^{2t} + 1)^{-1}. \tag{31}$$

Using definition 2, the fuzzy IVPs (29) and (30) can be reduced to the set of ODEs corresponding to their parametric forms as follows:

$$\begin{aligned} \hat{f}'_{1\gamma}(t) &= -\hat{f}^2_{1\gamma}(t) + 1, \\ \hat{f}'_{2\gamma}(t) &= -\hat{f}^2_{2\gamma}(t) + 1, \end{aligned} \tag{32}$$

subject to the initial conditions

$$\begin{aligned} \hat{f}'_{1\gamma}(0) &= \gamma - 1, \\ \hat{f}'_{2\gamma}(0) &= 1 - \gamma. \end{aligned} \tag{33}$$

Following the RPS algorithm and depending on the initial data (32)and (33), the first few terms a_n and b_n of the truncated RPS solution described in (13) are given by:

$$a_1 = (2\gamma - \gamma^2)t,$$
$$a_2 = (-2\gamma + 3\gamma^2 - \gamma^3)t^2,$$
$$a_3 = \frac{1}{3}(4\gamma - 14\gamma^2 + 12\gamma^3 - 3\gamma^4)t^3,$$
$$a_4 = \frac{1}{3}(-2\gamma + 15\gamma^2 - 25\gamma^3 + 15\gamma^4 - 3\gamma^5)t^4,$$
$$a_5 = \frac{1}{15}(4\gamma - 62\gamma^2 + 180\gamma^3 - 195\gamma^4 + 90\gamma^5 - 15\gamma^6)t^5,$$
$$a_6 = \frac{1}{45}(-4\gamma + 126\gamma^2 - 602\gamma^3 + 1050\gamma^4 - 840\gamma^5 + 315\gamma^6 - 45\gamma^7)t^6,$$
$$\vdots$$

and

$$b_1 = (2\gamma - \gamma^2)t,$$
$$b_2 = \gamma(2 - 3\gamma + \gamma^2)t^2,$$
$$b_3 = \frac{1}{3}(4\gamma - 14\gamma^2 + 12\gamma^3 - 3\gamma^4)t^3,$$
$$b_4 = \frac{1}{3}\gamma(2 - 15\gamma + 25\gamma^2 - 15\gamma^3 + 3\gamma^4)t^4,$$
$$b_5 = \frac{1}{15}(4\gamma - 62\gamma^2 + 180\gamma^3 - 195\gamma^4 + 90\gamma^5 - 15\gamma^6)t^5,$$
$$b_6 = \frac{1}{45}\gamma(4 - 126\gamma + 602\gamma^2 - 1050\gamma^3 + 840\gamma^4 - 315\gamma^5 + 45\gamma^6)t^6,$$
$$\vdots$$

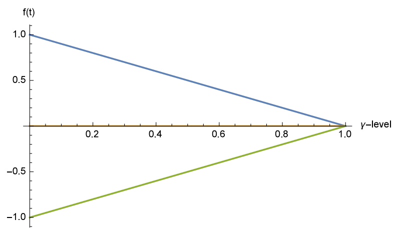
Particularly, if $\gamma = 1$, the terms a_n and b_n vanish as soon as n is even integer and then the RPS solution $f(t) = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \frac{62}{2835} + \ldots$, coincides precisely with the Taylor expansion about $t = 0$ of the exact solution given in (31).

Table 5 shows the error analysis of the proposed algorithm for system (32) and (33) at some nodes t in $[0,1]$ with step size 0.16 for $\gamma = 1$ and $n = 51$. Anyhow, Table 6 shows numerical comparison between the 10^{th} RPS solution and other existing numerical methods including RK-4, OHAM [49], and MNN [50]. From the table, it can be observed that the numerical results obtained by the RPS method are in good agreement with those obtained by other methods.

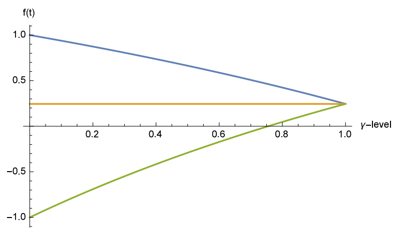
Figure 5 depicts the lower and upper bounds of the triangular fuzzy RPS solutions at $n = 8$ for $t = 0.25, 0.5, 0.75$ and 1, where the midline represents the center (crisp) solution at $\gamma = 1$. Figure 6 depicts the surface plot of the 10th-RPS approximate solution for all t and γ over the interval $[0,1]$ in which blue and orange color correspond to upper and lower bounds of the fuzzy solution.

6 Concluding remarks

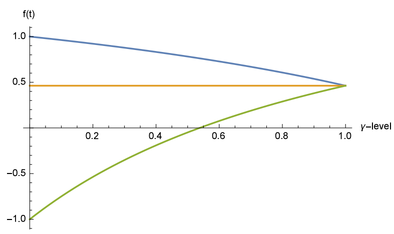
In this article, the RPS algorithm has been applied to investigate the series solution to fuzzy QRDE under strongly generalized differentiability. This method can be used directly by choosing appropriate initial guesses without being linearized, discretized or exposed to perturbation. Numerical results have shown the performance and reliability of the present approach. The results indicate that the RPS method is very efficient and powerful to nonlinear fuzzy differential equations with less calculations, time and effort. From the numerical comparison, it can be concluded that the RPS solutions are very well in comparison with those obtained by other methods.



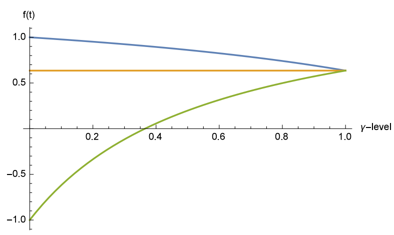
(a) Solution plots for $t = 0$



(b) Solution plots for $t = 0.25$



(c) Solution plots for $t = 0.5$



(d) Solution plots for $t = 0.75$

Fig. 5: Triangular fuzzy solution plots for the $8th$-RPS solutions of Example 5.3.

Table 5: Numerical result of the RPS solution of Example 5.3 at $\gamma = 1$

t_i	Exact solution	Approximation	Absolute error	Relative error
0.16	0.158648504297	0.158648504297	5.55112×10^{-17}	3.49900×10^{-16}
0.32	0.309506921213	0.309506921213	5.55112×10^{-17}	1.79354×10^{-16}
0.48	0.446243610249	0.446243610249	5.55112×10^{-17}	1.24397×10^{-16}
0.64	0.564899552846	0.564899552846	0.00000	0.00000
0.80	0.664036770268	0.664036770268	3.33067×10^{-16}	5.01579×10^{-16}
0.96	0.744276867362	0.744276867358	4.29645×10^{-12}	5.77265×10^{-12}

Table 6: Numerical comparison of Example 5.3 at $\gamma = 1$.

t_i	Exact	10^{th} RPSM	OHAM [47]	MNN [48]	RK-4
0.1	0.099668	0.099668	0.099668	0.099668	0.100000
0.2	0.197375	0.197375	0.197376	0.197375	0.199000
0.3	0.291313	0.291313	0.291315	0.291313	0.295040
0.4	0.379949	0.379949	0.379949	0.379949	0.386335
0.5	0.462117	0.462121	0.462092	0.462121	0.471410
0.6	0.537050	0.537078	0.536910	0.537077	0.549187
0.7	0.604368	0.604514	0.603815	0.604513	0.619026
0.8	0.664037	0.664641	0.662245	0.664640	0.680707
0.9	0.716298	0.718392	0.711287	0.718390	0.734371

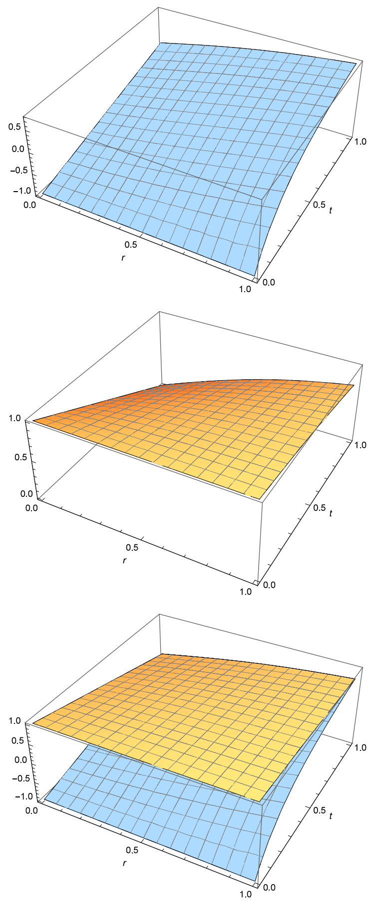


Fig. 6: The 3-dim plot for Example 5.3: blue and orange are lower and upper bounds of 10th-RPS fuzzy solution.

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