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Properties and Hermite–Hadamard Type Inequalities for the (m,h_1,h_2)–HA–Convex Functions.

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Abstract: In the paper, I introduce the definition of the (m,h_1,h_2)–HA-convex functions, present some properties of these new class of functions and establish some inequalities of type Hermite–Hadamard for these functions and an application these inequalities for special means.

Keywords: Harmonically convex functions; Hermite–Hadamard inequalities; special means.

1 Introduction

The convex functions are important and provide a basis for constructing literature on mathematical inequalities. A function $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$ is called convex if

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y),$$

where $t \in [0,1]$ and $x,y \in I$.

A large number of inequalities have been developed for convex functions and their generalizations see [1,2,3,4,5]. A classical inequality for convex functions is the Hermite–Hadamard inequality, this is given as follows:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2},$$

where $f : I \to \mathbb{R}$ is a convex function and $a,b \in I$ with $a<b$.

The authors in [6] gives the definition of (m,h_1,h_2)–convex functions.

Definition 1. *Assume $f : I \subset \mathbb{R}_0 = [0,+\infty) \to \mathbb{R}$, $h_1,h_2 : [0,1] \to \mathbb{R}_0$, and $m \in (0,1]$. Then f is said to be (m,h_1,h_2)–convex if the inequality*

$$f(tx+m(1-t)f(y)) \le h_1(t)f(x)+mh_2(1-t)f(y)$$

holds for all $x,y \in I$ and $t \in [0,1]$.

One of the recent generalizations of convexity was introduced by İ. İşcan in [7]. İşcan gave the following definition of harmonically convex functions:

Definition 2. *Let I be an interval in $\mathbb{R}\setminus\{0\}$. A function $f : I \to \mathbb{R}$ is said to be* harmonically convex *on I if the inequality*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le tf(y)+(1-t)f(x), \tag{1}$$

holds, for all $x,y \in I$ and $t \in [0,1]$.

The following result of the Hermite-Hadamard type for harmonically convex functions holds.

Theorem 1([7]). *Let $f : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a harmonically convex function and $a,b \in I$ with $a<b$. If $f \in L[a,b]$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{f(a)+f(b)}{2}. \tag{2}$$

In [8], the authors gave the definition of harmonic s–convexity in the second sense as follows.

Definition 3. *A function $f : I \subset \mathbb{R}_+ = (0,+\infty) \to \mathbb{R}$ is said to be harmonically s–convex in the second sense and $s \in (0,1]$ if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le t^s f(y)+(1-t)^s f(x), \tag{3}$$

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is valid for all $x,y \in I$ *and* $t \in [0,1]$.

In [9], they gave the following definition

Definition 4. *Let* $h : [0,1] \subset J \to \mathbb{R}_0 = [0,+\infty)$ *be a function. A function* $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ *is said to be harmonically h–convex function, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq h(t)f(y)+h(1-t)f(x), \tag{4}$$

for all $x,y \in I$ *and* $t \in [0,1]$.

Remark. It is obvious that for $h(t) = t$ and $h(t) = t^s$ in Definition 4, we have the definitions of harmonically convex functions and harmonically s–convex functions of second sense respectively.

In [10], they defined m–harmonic-arithmetically convex functions.

Definition 5. *Let* $f : (0,b] \subset \mathbb{R}_+ \to \mathbb{R}$ *and* $m \in (0,1]$ *be a constant. If*

$$f\left(\frac{xy}{ty+m(1-t)x}\right) \leq tf(x)+m(1-t)f(y),$$

for all $x,y \in (0,b]$ *and* $t \in [0,1]$, *then* f *is said to be an m–harmonic-arithmetically convex (or m–HA–convex) function.*

Some authors introduced a new introduce a new concept of the harmonic convex functions with respect to an arbitrary non-negative function.

Definition 6([11]). *Let* $h : [0,1] \to \mathbb{R}$ *be a nonnegative function. A function* $f : [a,b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *is said to be relative harmonic m–convex function, where* $m \in (0,1]$, *if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq mh(1-t)f(xm)+h(t)f(y), \tag{5}$$

for all $x,y \in [a,b]$ *and* $t \in (0,1)$.

The HermiteHadamard type inequalities for relative harmonic m–convex functions were investigated in [11]. In the same paper the following two theorems were proved:

Theorem 2. *Let* $f : [a,b] \subset \mathbb{R} \setminus (\not{\vdash}) \to \mathbb{R}$ *be relative harmonic m–convex function, where* $m \in (0,1]$. *If* $f \in L[a,b]$, *then*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \min\{f(b)+mf(ma), f(a)+mf(mb)\}\int_0^1 h(t)dt. \tag{6}$$

Theorem 3. *Let* $f : [a,b] \subset \mathbb{R} \setminus 0 \to \mathbb{R}$ *be relative harmonic m–convex function, where* $m \in (0,1]$. *If* $f \in L[a,b]$, *then*

$$\frac{1}{h(1/2)}f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)+mf(xm)}{x^2}dx \leq \frac{1}{2}\Big\{f(a)+f(b)+2m[f(am)+f(bm)]+m^2[f(am^2) +f(bm^2)]\Big\} \times \int_0^1 h(t)dt. \tag{7}$$

Theorem 4. *Let* $f,g : [a,b] \subset \mathbb{R} \setminus 0 \to \mathbb{R}$ *be relative harmonic* m_1*–convex function and relative harmonic* m_2*–convex function respectively, where* $m_1,m_2 \in (0,1]$. *If* $f \in L[a,b]$, *then*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}dx \leq \min\{M_1(a,b), M_2(a,b)\}, \tag{8}$$

where

$$\begin{aligned} M_1(a,b) &= [m_1m_2f(am_1)g(am_2)+f(b)g(b)]\int_0^1 [h(t)]^2dt \\ &\quad +[m_1f(am_1)g(b)+m_2f(b)g(am_2)]\int_0^1 h(t)h(1-t)dt, \\ M_2(a,b) &= [m_1m_2f(bm_1)g(bm_2)+f(a)g(a)]\int_0^1 [h(t)]^2dt \\ &\quad +[m_2f(a)g(bm_2)+m_1f(bm_1)g(a)]\int_0^1 h(t)h(1-t)dt. \end{aligned}$$

2 Main results

We now introduce the concept of (m,h_1,h_2)–HA–convex functions.

Definition 7. *Let* $h_i : [0,1] \to \mathbb{R}_0 = [0,+\infty)$ *and* $m \in (0,1]$ *such that* $h_i \not\equiv 0$, *for* $i = 1,2$. *A function* $f : \mathbb{R}_+ \to \mathbb{R}_0$ *is said* (m,h_1,h_2)*–HA–convex function, if*

$$f\left(\frac{xy}{ty+m(1-t)x}\right) = f\left(\frac{1}{t\frac{1}{x}+m(1-t)\frac{1}{y}}\right) \leq h_1(t)f(x)+mh_2(1-t)f(y), \tag{9}$$

for all $x,y \in I$ *and* $t \in [0,1]$.

Remark.
1. If $h_1(t) = h_2(t) = h(t)$, for all $t \in [0,1]$, then $f : \mathbb{R}_+ \to \mathbb{R}_0$ is (h,m)–HA–convex.
2. If $f : \mathbb{R}_+ \to \mathbb{R}_0$ is an (h,m)–HA–convex, then f is a relative harmonic m–convex function.
3. Let $f : \mathbb{R}_+ \to \mathbb{R}_0$ be a harmonically (h,m)–convex function and $m \in [0,1]$. When $h(t) = t$ for $t \in [0,1]$, the function f is said to be m–harmonic-arithmetically convex (or m–HA–convex).

4. If $f:\mathbb{R}_+\to\mathbb{R}_0$ is harmonically $(h,1)$–convex function, then it is harmonically h–convex.

Example 1. Let $f:\mathbb{R}_+\to\mathbb{R}$ be a function defined as: $f(x)=\frac{1}{x^p}$, for $p\geq 1$ fixed, and $m\in(0,1]$. Let $h_1(t)=t^{p_1}$ and $h_2(t)=t^{p_2}$ for $t\in(0,1]$ and $0<p_1,p_2\leq 1$. Then f is (m,h_1,h_2)–HA–convex function.

In effect, let $x,y\in\mathbb{R}_+$, $t\in[0,1]$ and $m\in(0,1]$, such that

$$\begin{aligned} f\left(\frac{xy}{tx+m(1-t)y}\right) &= \left[\frac{tx+m(1-t)y}{xy}\right]^p \\ &= \frac{[tx+m(1-t)y]^p}{x^py^p} \\ &\leq \frac{tx^p+(1-t)(my)^p}{x^py^p} \\ &\leq \frac{t^{p_1}x^p+(1-t)^{p_2}m^py^p}{x^py^p} \\ &= t^{p_1}\frac{1}{y^p}+m^p(1-t)^{p_2}\frac{1}{x^p} \\ &= h_1(t)f(y)+mh_2(1-t)f(x). \end{aligned}$$

Thus, f is (m,h_1,h_2)–HA–convex.

Now we present some properties of (m,h_1,h_2)–HA–convex functions.

Theorem 5. *Let $h_i:[0,1]\to\mathbb{R}_0$, such that, $h_i\not\equiv 0$, for $i=1,2$ and $f:I\subset\mathbb{R}_+\to\mathbb{R}_+$ is a (m,h_1,h_2)–HA–convex function, if $m=1$, then $h_1(t)+h_2(t)\geq 1$, $\forall t\in[0,1]$.*

Proof. Since f is a (m,h_1,h_2)–HA–convex, and $m=1$, then for all $x\in I$ and $t\in[0,1]$, we obtain

$$\begin{aligned} f(x) &= f\left(\frac{x^2}{tx+(1-t)x}\right) \\ &\leq h_1(t)f(x)+h(1-t)f(x)=[h_1(t)+h_2(1-t)]f(x). \end{aligned}$$

Thus, $h_1(t)+h_2(1-t)\geq 1$, for all $t\in[0,1]$.

Theorem 6. *Let $h_i:[0,1]\to\mathbb{R}_0$, for $i=1,2,3,4$, $f:I\subset\mathbb{R}_+\to\mathbb{R}_+$ and $m\in(0,1]$. If f is a (m,h_1,h_2)–HA–convex function on I, $h_1(t)\leq h_3(t)$ and $h_2(t)\leq h_4(t)$, for $t\in[0,1]$, then f is a (m,h_3,h_4)–HA–convex function on I.*

Proof. Since f is a (m,h_1,h_2)–HA–convex function on I, then for $x,y\in I$ and $t\in[0,1]$, we get

$$\begin{aligned} f\left(\frac{xy}{tx+m(1-t)y}\right) &\leq h_1(t)f(y)+mh_2(1-t)f(x) \\ &\leq h_3(t)f(y)+mh_4(1-t)f(x). \end{aligned}$$

Hence, the proof of Theorem 6 is complete.

Corollary 1. *Let $h_i:[0,1]\to\mathbb{R}_0$ and $f_i:I\subset\mathbb{R}_+\to\mathbb{R}$ and $m\in(0,1]$. If $h(t)=\max_{1\leq j\leq n}\{h_i(t)\}$, for $t\in[0,1]$ and f_i is a (h_i,m)–HA–convex on I, for $i=1,\ldots,n$, then $\sum_{i=1}^n f_i$ is a (h,m)–HA–convex on I.*

Proof. This follows from Theorem 6 and induction on n.

Theorem 7. *Let $h_i:[0,1]\to\mathbb{R}_0$ such that $h_i\not\equiv 0$ for $i=1,2$, $f:I\subset\mathbb{R}_+\to\mathbb{R}_+$, $g:J\subset\mathbb{R}_+\to g(J)\subset I$ and $m\in(0,1]$. If f is nondecreasing and (m,h_1,h_2)–convex function with respect to function g on I and if g is a m–HA–convex function on J, then $f\circ g$ is a (m,h_1,h_2)–HA–convex function on J;*

Proof. Since g is a m–HA–convex function, for any $x,y\in J$ and $t\in[0,1]$ and $m\in(0,1]$, we obtain,

$$g\left(\frac{xy}{ty+m(1-t)x}\right)\leq tg(x)+m(1-t)g(y)$$

In addition, f is a nondecreasing function and is a (m,h_1,h_2)–convex function, therefore

$$\begin{aligned} f\left(g\left(\frac{xy}{ty+m(1-t)x}\right)\right) &\leq f\left(tg(x)+m(1-t)g(y)\right) \\ &\leq h_1(t)f(g(x))+mh_2(1-t)f(g(y)). \end{aligned}$$

Thus, $f\circ g$ is (m,h_1,h_2)–HA–convex.

We establish some new Hermite–Hadamard type inequalities for (m,h_1,h_2)–HA-convex functions, which is the main motivation of this paper.

Theorem 8. *Let $h_i:[0,1]\to\mathbb{R}_0$, $h_i\not\equiv 0$, for $i=1,2$, $m\in(0,1]$ and $f:\mathbb{R}_+\to\mathbb{R}_0$ be an (m,h_1,h_2)–HA–convex function on $\mathbb{R}_+$ such that $f\in L_1([a,b])$ and $h_1,h_2\in L_1([0,1])$, for $0<a<b$, then*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \frac{abh_1(1/2)}{b-a}\int_a^b\frac{f(x)}{x^2}dx \\ &\quad+\frac{abmh_2(1/2)}{b-a}\int_a^b\frac{f(mx)}{x^2}dx. \end{aligned} \tag{10}$$

Proof. Since

$$\frac{2ab}{a+b}=\frac{1}{\frac{1}{2}\frac{1}{\frac{ab}{ta+(1-t)b}}+\frac{1}{2}\frac{1}{\frac{ab}{tb+(1-t)a}}},$$

for $t\in[0,1]$, from the $(m,h_1,h_2)-HA$ convexity of f, we obtain

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq h_1(1/2)f\left(\frac{ab}{ta+(1-t)b}\right) \\ &\quad+mh_2(1/2)f\left(\frac{ab}{tb+(1-t)a}\right). \end{aligned} \tag{11}$$

If replacing $\frac{ab}{ta+(1-t)b}$ and $\frac{ab}{tb+(1-t)a}$ for $0\le t\le 1$ by x, then

$$\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt=\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx \tag{12}$$

$$\int_0^1 f\left(\frac{abm}{tb+(1-t)a}\right)dt=\frac{ab}{b-a}\int_a^b\frac{f(mx)}{x^2}dx \tag{13}$$

Substituting (12) and (13) in (11), we obtain (10). Thus proof of Theorem 8 is complete.

Theorem 9. *Let $h_i:[0,1]\to\mathbb{R}_0$, $h_i\not\equiv 0$, for $i=1,2$, $m\in(0,1]$ and $f:\mathbb{R}_+\to\mathbb{R}_0$ to be a $(m,h_1,h_2)-$ HA–convex function on $\mathbb{R}_+$ such that $f\in L_1([a,b])$ and $h_1,h_2\in L_1([0,1])$, for $0<a<b$, then*

$$\begin{aligned}&\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx \qquad (14)\\ &\le\min\left\{f(a)\int_0^1h_1(t)dt+mf(mb)\int_0^1h_2(1-t)dt,\right.\\ &\qquad\left.f(b)\int_0^1h_1(t)dt+mf(ma)\int_0^1h_2(1-t)dt\right\}.\end{aligned}$$

Proof. Let $x,y\in\mathbb{R}_+$ and f an (m,h_1,h_2)–HA–convex function, then

$$f\left(\frac{xy}{ty+(1-t)x}\right)\le h_1(t)f(x)+mh_2(1-t)f(my). \tag{15}$$

Substituting $x=a$ and $y=b$ in (15), we obtain

$$f\left(\frac{ab}{tb+(1-t)a}\right)\le h_1(t)f(a)+mh_2(1-t)f(mb).$$

Integrating on $[0,1]$ the above inequality, we get

$$\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\le f(a)\int_0^1h_1(t)dt+mf(mb)\int_0^1h_2(1-t)dt. \tag{16}$$

Now similarly substituting $x=b$ and $y=a$ in (15), we have

$$\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\le f(b)\int_0^1h_1(t)dt+mf(ma)\int_0^1h_2(1-t)dt. \tag{17}$$

Thus, from (16) and (17),

$$\begin{aligned}&\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\\ &\le\min\left\{f(a)\int_0^1h_1(t)dt+mf(mb)\int_0^1h_2(1-t)dt,\right.\\ &\qquad\left.f(b)\int_0^1h_1(t)dt+mf(ma)\int_0^1h_2(1-t)dt\right\}.\end{aligned}$$

Corollary 2. *If $h_1(t)=h_2(t)=h(t)$, for all $t\in[0,1]$, from 9, we have the inequality (6).*

Corollary 3. *Let $h_1(t)=t^{s_1}$, $h_2(t)=t^{s_2}$, for all $t\in[0,1]$, $s_1,s_2\in(-1,1]$ and $m\in(0,1]$, and let $f:\mathbb{R}_+\to\mathbb{R}_0$ be a (m,h_1,h_2)–HA–convex function, such that $f\in L_1(\mathbb{R}_+)$. Then for $0<a<b$, we have*

$$\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\le\min\left\{\frac{f(a)}{s_1+1}+\frac{mf(mb)}{s_2+1},\frac{f(b)}{s_1+1}+\frac{mf(ma)}{s_2+1}\right\}.$$

Theorem 10. *Let $h_i:[0,1]\to\mathbb{R}_0$, $h_i\not\equiv 0$, for $i=1,2$, $m\in(0,1]$ and $f:\mathbb{R}_+\to\mathbb{R}_0$ be a $(m,h_1,h_2)-$HA–convex function on $\mathbb{R}_+$ such that $f\in L_1([a,b])$ and $h_1,h_2\in L_1([0,1])$, for $0<a<b$, then*

$$\begin{aligned}&f\left(\frac{ab}{a+b}\right) \qquad (18)\\ &\le h_1\left(\frac12\right)\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx+mh_2\left(\frac12\right)\frac{ab}{b-a}\int_a^b\frac{f(mx)}{x^2}dx\\ &\le\min\left\{\left[h_1\left(\frac12\right)f(b)+mh_2\left(\frac12\right)f(ma)\right]\int_0^1h_1(t)dt\right.\\ &\qquad+m\left[h_1\left(\frac12\right)f(ma)+mh_2\left(\frac12\right)f(m^2b)\right]\int_0^1h_2(t)dt,\\ &\qquad\left[h_1\left(\frac12\right)f(a)+mh_2\left(\frac12\right)f(mb)\right]\int_0^1h_1(t)dt\\ &\qquad\left.+m\left[h_1\left(\frac12\right)f(mb)+mh_2\left(\frac12\right)f(m^2a)\right]\int_0^1h_2(t)dt\right\}.\end{aligned}$$

Proof. From the (m,h_1,h_2)–HA convexity of f, we obtain

$$\begin{aligned}f\left(\frac{2ab}{a+b}\right)&\le h_1\left(\frac12\right)f\left(\frac{ab}{ta+(1-t)b}\right)\\ &\quad+mh_2\left(\frac12\right)f\left(\frac{abm}{tb+(1-t)a}\right)\\ &\le h_1\left(\frac12\right)[h_1(t)f(b)+mh_2(1-t)f(ma)] \qquad (19)\\ &\quad+mh_2\left(\frac12\right)\left[h_1(t)f(ma)+mh_2(1-t)f(m^2b)\right].\end{aligned}$$

On the other hand,

$$\begin{aligned}f\left(\frac{2ab}{a+b}\right)&\le h_1\left(\frac12\right)f\left(\frac{ab}{tb+(1-t)a}\right)\\ &\quad+mh_2\left(\frac12\right)f\left(\frac{abm}{ta+(1-t)b}\right)\\ &\le h_1\left(\frac12\right)[h_1(t)f(a)+mh_2(1-t)f(mb)] \qquad (20)\\ &\quad+mh_2\left(\frac12\right)\left[h_1(t)f(mb)+mh_2(1-t)f(m^2a)\right].\end{aligned}$$

Integrating on both sides of the above inequalities (19) with respect to $t\in[0,1]$ and making changes of appropriate variables lead to

$$f\left(\frac{2ab}{a+b}\right) \tag{21}$$
$$\leq h_1\left(\frac{1}{2}\right)\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx + mh_2\left(\frac{1}{2}\right)\frac{ab}{b-a}\int_a^b \frac{f(mx)}{x^2}dx$$
$$\leq \left[h_1\left(\frac{1}{2}\right)f(b) + mh_2\left(\frac{1}{2}\right)f(ma)\right]\int_0^1 h_1(t)dt$$
$$+\left[h_1\left(\frac{1}{2}\right)f(ma) + mh_2\left(\frac{1}{2}\right)f(m^2b)\right]\int_0^1 h_2(t)dt.$$

Similarly, integrating on both sides of the above inequalities (20), we get

$$f\left(\frac{2ab}{a+b}\right) \tag{22}$$
$$\leq h_1\left(\frac{1}{2}\right)\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx + mh_2\left(\frac{1}{2}\right)\frac{ab}{b-a}\int_a^b \frac{f(mx)}{x^2}dx$$
$$\leq \left[h_1\left(\frac{1}{2}\right)f(a) + mh_2\left(\frac{1}{2}\right)f(mb)\right]\int_0^1 h_1(t)dt$$
$$+\left[h_1\left(\frac{1}{2}\right)f(mb) + mh_2\left(\frac{1}{2}\right)f(m^2a)\right]\int_0^1 h_2(t)dt.$$

Therefore, from (21) and (22), we obtain

$$f\left(\frac{ab}{a+b}\right)$$
$$\leq h_1\left(\frac{1}{2}\right)\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx + mh_2\left(\frac{1}{2}\right)\frac{ab}{b-a}\int_a^b \frac{f(mx)}{x^2}dx$$
$$\leq \min\left\{\left[h_1\left(\frac{1}{2}\right)f(b) + mh_2\left(\frac{1}{2}\right)f(ma)\right]\int_0^1 h_1(t)dt\right.$$
$$+m\left[h_1\left(\frac{1}{2}\right)f(ma) + mh_2\left(\frac{1}{2}\right)f(m^2b)\right]\int_0^1 h_2(t)dt,$$
$$\left[h_1\left(\frac{1}{2}\right)f(a) + mh_2\left(\frac{1}{2}\right)f(mb)\right]\int_0^1 h_1(t)dt$$
$$\left.+m\left[h_1\left(\frac{1}{2}\right)f(mb) + mh_2\left(\frac{1}{2}\right)f(m^2a)\right]\int_0^1 h_2(t)dt\right\}.$$

Thus the demonstration is completed.

Corollary 4. *If $h_1(t) = h_2(t) = h(t)$, with $h \not\equiv 0$, for all $t \in [0,1]$, from 19, we get the inequality (7).*

Proof. Let $h_1(t) = h_2(t) = h(t)$ for all $t \in [0,1]$ and applying the Theorem 10 and we get the desired result.

Corollary 5. *Let $h_1(t) = t^{s_1}$, $h_2(t) = t^{s_2}$, for all $t \in [0,1]$, $s_1, s_2 \in (-1,1]$ and $m \in (0,1]$, and let $f : \mathbb{R}_+ \to \mathbb{R}_0$ be a (m,h_1,h_2)–HA–convex function, such that $f \in L_1(\mathbb{R}_+)$. Then for $0 < a < b$, we have*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \min\left\{\frac{f(a)}{s_1+1} + \frac{mf(mb)}{s_2+1}, \frac{f(b)}{s_1+1} + \frac{mf(ma)}{s_2+1}\right\}.$$

Theorem 11. *Let $h_i : [0,1] \to \mathbb{R}_0$, $h_i \not\equiv 0$, for $i = 1,2$, $m \in (0,1]$ and $f, g : \mathbb{R}_+ \to \mathbb{R}_0$ be a $(m,h_1,h_2) - HA$–convex function on $\mathbb{R}_+$ such that $fg \in L_1([a,b])$, for $0 < a < b$, then*

$$\frac{a-b}{ab}f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right) \tag{17}$$
$$\leq \left[h_1\left(\frac{1}{2}\right)\right]^2\int_a^b \frac{f(x)g(x)}{x^2}dx + mh_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\int_a^b \frac{f(x)g(mx)}{x^2}dx$$
$$+mh_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\frac{ab}{b-a}\int_a^b \frac{f(mx)g(x)}{x^2}dx$$
$$+\left[h_2\left(\frac{1}{2}\right)\right]^2\int_a^b \frac{f(mx)g(mx)}{x^2}dx.$$

Proof. Using the $(m,h_1,h_2) - HA$–convexity of f and g on $\mathbb{R}_+$, we obtain

$$f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)$$
$$\leq \left[h_1\left(\frac{1}{2}\right)f\left(\frac{abm}{ta+(1-t)b}\right) + mh_2\left(\frac{1}{2}\right)f\left(\frac{abm}{tb+(1-t)a}\right)\right]$$
$$\times\left[h_1\left(\frac{1}{2}\right)g\left(\frac{ab}{ta+(1-t)b}\right) + mh_2\left(\frac{1}{2}\right)g\left(\frac{ab}{tb+(1-t)a}\right)\right]$$
$$= \left[h_1\left(\frac{1}{2}\right)\right]^2 f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)$$
$$+mh_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{abm}{tb+(1-t)a}\right)$$
$$+mh_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)f\left(\frac{abm}{tb+(1-t)a}\right)g\left(\frac{ab}{ta+(1-t)b}\right)$$
$$+\left[mh_2\left(\frac{1}{2}\right)\right]^2 f\left(\frac{abm}{tb+(1-t)a}\right)g\left(\frac{abm}{tb+(1-t)a}\right).$$

Integrating the above inequality on $[0,1]$ with respect to t and making changes of appropriate variables, we obtain the inequality (17). The Theorem 11 is thus proved.

Theorem 12. *Let $h_i : [0,1] \to \mathbb{R}_0$, $h_i \not\equiv 0$, for $i = 1,2$, $m_1, m_2 \in (0,1]$ and $f, g : \mathbb{R}_+ \to \mathbb{R}_0$. If f is a (m_1,h_1,h_2) – HA–convex function, g is a (m_2,h_1,h_2) – HA–convex function such that $fg \in L_1(\mathbb{R}_+)$, and $h_1^2, h_2^2 \in L_1([0,1])$ for $0 < a < b$, then*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}dx \tag{18}$$
$$\leq f(b)g(b)\int_0^1 [h_1(t)]^2 dt + m_1m_2 f(m_1a)g(m_2a)\int_0^1 [h_2(t)]^2 dt$$
$$+[m_2 f(b)g(m_2a) + m_1 f(m_1a)g(b)]\int_0^1 h_1(t)h_2(1-t)dt.$$

Proof. Let $x = \dfrac{ab}{ta+(1-t)b}$ for $t\in[0,1]$. By the (m,h_1,h_2)–HA convexity of f and g, we have

$$\begin{aligned}
&\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}dx\\
&=\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)dt\\
&\leq \int_0^1 [h_1(t)f(b)+m_1h_2(1-t)f(m_1a)]\\
&\quad\times[h_1(t)g(b)+m_2h_2(1-t)g(m_2a)]\,dt\\
&=f(b)g(b)\int_0^1 [h_1(t)]^2\,dt\\
&\quad+m_1m_2f(m_1a)g(m_2a)\int_0^1 [h_2(t)]^2\,dt\\
&\quad+[m_2f(b)g(m_2a)+m_1f(m_1a)g(b)]\int_0^1 h_1(t)h_2(1-t)dt.
\end{aligned}$$

Thus the proof of Theorem 12 is complete.

Corollary 6. *Under the conditions of Theorem 12, if $h_1(t)=h_2(t)=h(t)$ for all $t\in[0,1]$, then we obtain the inequality (8).*

3 Application for special means

Let us recall the following special means of two numbers $a,b\in\mathbb{R}$ (see [8]):

1. The geometric mean
$$G(a,b):=\sqrt{ab}.$$
2. The harmonic mean
$$H(a,b):=\frac{2ab}{a+b}.$$
3. The p–logarithmic mean
$$L_p(a,b):=\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}},$$
with $p\in\mathbb{R}\setminus\{0\}$ and $a<b$.

The following theorem is a result in which we present the relationship between the means defined above.

Theorem 13. *Let $0<a<b$. Then we have the following inequality*

$$2^{p_1+p_2}G^2(a^p,b^p)H^{-p}(a,b)\leq\left[2^{p_2}m^{p-1}+2^{p_1}\right]L_p^p(a,b) \qquad (19)$$

with $p\geq 1$, $m\in(0,1]$ and $p_1,p_2\in(0,1)$.

Proof. By the example 1, we have $f:\mathbb{R}_+\to\mathbb{R}$ defined by $f(x)=\dfrac{1}{x^p}$, for $p\geq 1$ the function is (m,h_1,h_2)–HA–convex, where $h_1(t)=t^{p_1}$, $h_2(t)=t^{p_2}$ with $m\in(0,1]$ and $p_1,p_2\in(0,1)$ and using the Theorem 8, we get

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{abh_1(1/2)}{b-a}\int_a^b\frac{f(x)}{x^2}dx \qquad (20)$$

$$+\frac{abmh_2(1/2)}{b-a}\int_a^b\frac{f(mx)}{x^2}dx. \qquad (21)$$

Solving each of expressions present in the above inequalities,

$$f\left(\frac{2ab}{a+b}\right)=\left(\frac{2ab}{a+b}\right)^{-p}=H^{-p}(a,b). \qquad (22)$$

$$h_1\left(\frac{1}{2}\right)=\frac{1}{2^{p_1}}\quad\text{and}\quad h_2\left(\frac{1}{2}\right)=\frac{1}{2^{p_2}} \qquad (23)$$

$$\begin{aligned}
\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx&=\frac{ab}{b-a}\int_a^b\frac{1}{x^{p+2}}dx \qquad (24)\\
&=\frac{ab}{-(b-a)(p+1)}\left[b^{-(p+1)}-a^{-(p+1)}\right]\\
&=\frac{ab}{(b-a)(p+1)}\left[\frac{1}{a^{p+1}}-\frac{1}{b^{p+1}}\right]\\
&=\frac{ab}{(b-a)(p+1)}\frac{b^{p+1}-a^{p+1}}{a^{p+1}b^{p+1}}\\
&=\frac{1}{a^pb^p}\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\\
&=\frac{1}{G^2(a^p,b^p)}\left\{\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}\right\}^p\\
&=\frac{1}{G^2(a^p,b^p)}L_p^p(a,b).
\end{aligned}$$

$$\begin{aligned}
\frac{abm}{b-a}\int_a^b\frac{f(mx)}{x^2}dx&=\frac{ab}{b-a}\int_a^b\frac{1}{m^px^{p+2}}dx \qquad (25)\\
&=\frac{ab}{-(b-a)(p+1)m^{p-1}}\left[b^{-(p+1)}-a^{-(p+1)}\right]\\
&=\frac{ab}{(b-a)(p+1)m^{p-1}}\left[\frac{1}{a^{p+1}}-\frac{1}{b^{p+1}}\right]\\
&=\frac{ab}{(b-a)(p+1)m^{p-1}}\frac{b^{p+1}-a^{p+1}}{a^{p+1}b^{p+1}}\\
&=\frac{1}{a^pb^pm^{p-1}}\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\\
&=\frac{1}{G^2(a^p,b^p)m^{p-1}}\left\{\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}\right\}^p\\
&=\frac{1}{m^{p-1}G^2(a^p,b^p)}L_p^p(a,b).
\end{aligned}$$

Substituting (22)–(25) in (20), we get

$$\begin{aligned} &H^{-p}(a,b)\\ &\leq \frac{1}{2^{p_1}}\frac{1}{G^2(a^p,b^p)}L_p^p(a,b)+\frac{1}{2^{p_2}}\frac{1}{m^{p-1}G^2(a^p,b^p)}L_p^p(a,b)\\ &=\frac{2^{p_2}m^{p-1}+2^{p_1}}{2^{p_1+p_2}m^{p-1}}\frac{1}{G^2(a^p,b^p)}L_p^p(a,b)\end{aligned}$$

This implies,

$$2^{p_1+p_2}m^{p-1}G^2(a^p,b^p)H^{-p}(a,b) \leq \left[2^{p_2}m^{p-1}+2^{p_1}\right]L_p^p(a,b)$$

Therefore we get the inequality (19).

4 Conclusion

The principal contribution of this paper has been the introduction of a new class of functions of generalized convexity, we present some examples and properties. We have shown that these class contain some previously known classes as special cases as well as Hermite–Hadamard's inequalities type for these functions and applied these inequalities to special means. We expect that the ideas and techniques used in this paper may inspire interested readers to explore some new applications of these newly introduced functions in various fields of pure and applied sciences.

References

[1] M. Bracamonte, J. Giménez, and J. Medina, *Hermite–Hadamard and Fejér type inequalities for strongly harmonically convex functions*, MATUA. Vol. III, No. 2, 33– 46, 2016.

[2] M. Bracamonte, J. Medina, and N. Merentes, *Some inequalities for harmonically s–convex functions*, Boletín de la Asociación Matemática Venezolana, Vol. XXIII, No. 2, 105–116, 2016.

[3] M. Bracamonte, J. Medina, and M. Vivas, *Some inequalities for reciprocally (s,m)–convex in the second sense functions and applications to special means*, Accepted for publication in the Journal of Mathematical & Inequalities, 2019.

[4] F. Chen and S.Wu, *Hermite–Hadamard type inequalities for harmonically s–convex functions*, Sci.World J. no. 7, Article ID 279158, 2014.

[5] S. S. Dragomir, *Inequalities of Hermite–Hadamard type for h–convex functions on linear spaces*, Proyecciones 34, no. 4, 323–341, 2015.

[6] D.-P. Shi, B.-Y. Xi, and F. Qi, *Hermite–Hadamard Type Inequalities for (m,h_1,h_2)–Convex Functions Via Riemann–Liouville Fractional Integrals*, Turkish Journal of Analysis and Number Theory, Vol. 2, No. 1, 23–28, 2014.

[7] İ. İşcan, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacettepe Journal of Mathematics and Statistics Volume 43 (6), 935–942, 2014.

[8] İ. İşcan, *Ostrowski type inequalities for harmonically sconvex functions*, Konuralp Journal of Mathematics, Volume 3 No. 1, 63–74, 2015.

[9] M. A. Noor, K. I. Noor, M. U. Awan, and S. Costache, *Some integral inequalities for harmonically h–convex functions*, Konuralp Journal of Mathematics, U.P.B. Sci. Bull., Series A, Vol. 77, Iss. 1, 5–16, 2015.

[10] B.-Y. Xi, F. Qi, and T.-Y. Zhang, *Some inequalities of Hermite–Hadamard type for m–harmonic-arithmetically convex functions*, ScienceAsia 41, 357–361, 2015.

[11] M. A. Noor, K. I. Noor, and S. Ifitikhar, *Relative Harmonic m–convex functions and integral inequalities*, J. Adv. Math. Stud. Vol. 10 No. 2, 231–242, 2017.

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