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Reduced Order Hybrid Dislocated Synchronization of Complex Fractional Order Chaotic Systems

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Abstract: In this manuscript, we have analyzed the complex Lorenz and complex Duffing systems of fractional order of different dimension. Here, we have blended the ideas of reduced order synchronization with dislocated synchronization schemes. Using stability theory of Lyapunov, sufficient conditions have been derived for accomplishing reduced order hybrid dislocated synchronization. Numerical simulations have been performed in MATLAB to validate the efficacy of the method proposed. The results showed the usefulness and suitability of the method to achieve the synchronization.

Keywords: Complex fractional order chaotic systems, hybrid synchronization, reduced order synchronization, dislocated synchronization.

1 Introduction

Chaos synchronization [1] is a process of having two or more chaotic systems(identical or non-identical) that follow the same path. The dynamics of one system is locked into the other and thereby causes their synchronization in the sense that the state of one asymptotically approach to the other. Until 1990, synchronization between chaotic systems was considered impractical because of the chaotic nature of the individual systems. However it was because of the pioneering work of Pecora and Caroll [2], the synchronization between chaotic systems came into existence and an interesting area of research emerged. Chaos is the inherent property of nonlinear systems and has various applications such as viscoelasticity [3], dielectric polarization, electromagnetic waves [4], diffusion, signal processing, mathematical biology and many other disciplines. The nonlinear systems which show such type of behaviour are known as chaotic systems. Various methods are used to determine the chaotic behaviour of a system, some of them are by plotting phase portrait, poincare section or by finding the lyapunov exponents. Various studies were conducted in the last two decades. Different methods have also been designed for synchronization of chaotic systems such as adaptive feedback control, optimal control, linear and nonlinear feedback synchronization [5], active control [6], sliding mode control [7], adaptive sliding mode technique [8], time delay feedback approach [9], tracking control [10], backstepping design method, etc. Due to the increased interest in chaos synchronization various kinds of synchronization schemes, such as lag synchronization [11], complete synchronization [12], phase and anti-phase synchronization [13], anti-synchronization [14], hybrid synchronization, projective synchronization [15], hybrid funchtion projective synchronization [16], generalised synchronization [17] etc. have been proposed.

Many attempts have been made to synchronize similar systems with different techniques. Moreover the non-identical systems have also been synchronized by many researchers. In this manuscript we have tried to synchronize the two non-identical system of fractional order. We have also introduced dislocated synchronization due to which the number of choices of switching increases and enhances the analysis of our study. In dislocated [18], the slave system states are synchronized with the desired state of the master system. In the process of hybrid synchronization, coexistence of complete and anti-synchronization occurs. This co-existence of synchronization is also referred to as mixed synchronization. We believe that it is the first kind of study addressing the problem of fractional order dislocated hybrid

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synchronization of chaotic systems.

Researchers have done a lot of work on reduced order synchronization. The study of synchronization of chaotic systems of different orders is very significant from the view of usable application in real life problems and practical application [19]. Dislocated synchronization was proposed to increase the security of transmission via synchronization based on active control mechanism. In dislocated synchronization, different states of the slave system are synchronized with the desired state of the master system in a different manner. Due to the number of choices of switching, unpredictability increases and enhances the security in secure communication. In the process of hybrid synchronization, concurrence of complete and anti-synchronization occurs. This co-existence would be very fruitful in secure communication. During our studies, we have synchronized the 3D complex Lorenz system and 2D complex Duffing system. Based on the stability theory of Lyapunov, sufficient conditions are obtained to achieve the desired synchronization among two non-identical chaotic systems. Numerical simulations have been performed using MATLAB to validate the suitability of the proposed method. Results obtained showed the feasibility and effectiveness of the used technique.

2 Preliminaries

2.1 Definition:

As various definitions have been available for fractional order derivative, we have considered Caputo's definition:

$$ {}_aD_x^\alpha g(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{g^{(n)}(\tau)d\tau}{(x-\tau)^{\alpha-n+1}} $$

where n is integer, α lis real number and $(n-1) \leq \alpha < n$ and $\Gamma(.)$ are the Gamma function.

2.2 Problem formulation

Let us consider a complex fractional order chaotic master system as:

$$ \frac{d^qX(t)}{dt^q} = h(X,t) \tag{1} $$

and a complex fractional order chaotic slave system as:

$$ \frac{d^qY(t)}{dt^q} = f(Y,t) + V \tag{2} $$

where $\mathbf{X} = [\mathbf{X_1'}, \mathbf{X_2'}, ..., \mathbf{X_n'}]^\mathbf{T} \in \mathbf{C^n}$ and

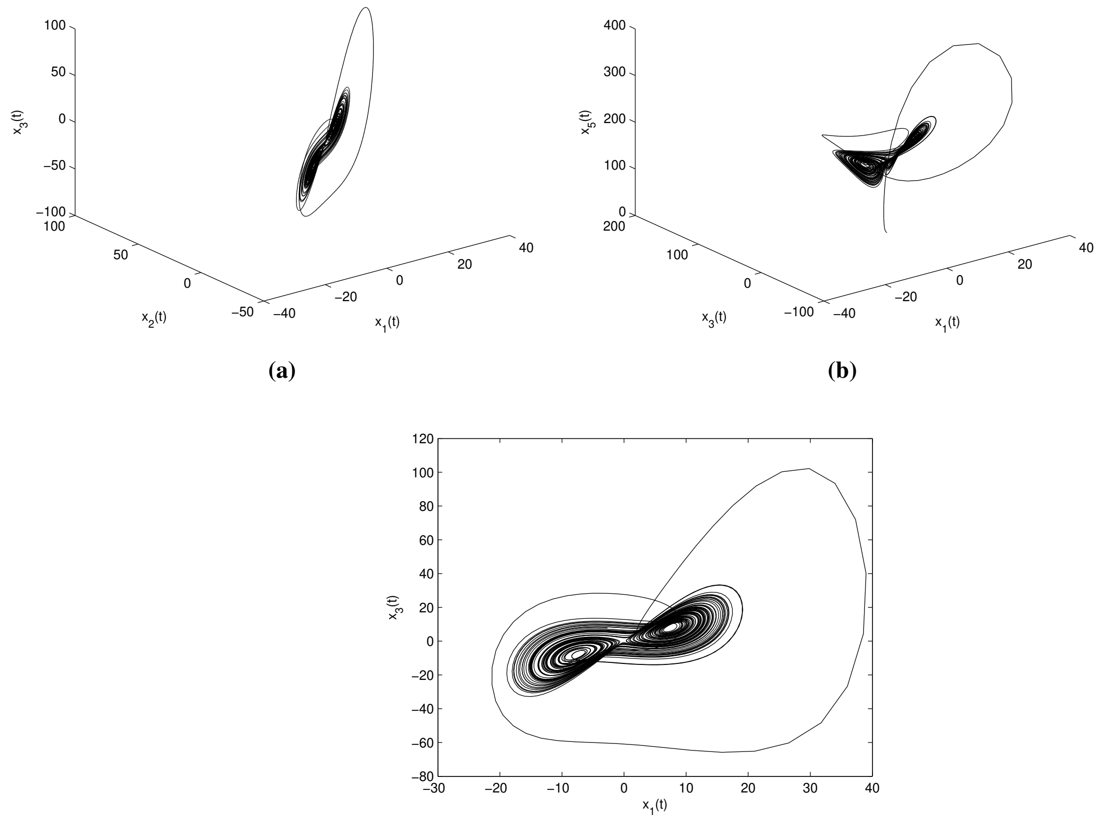
$$ \mathbf{Y} = [\mathbf{Y_1'}, \mathbf{Y_2'}, ..., \mathbf{Y_m'}]^\mathbf{T} \in \mathbf{C^m} $$

are the state vectors and

$$ \mathbf{X_i'} = \mathbf{x_p} + \mathbf{ix_q}, \mathbf{Y_i'} = \mathbf{y_r} + \mathbf{iy_s} $$

where x_p, x_q, y_r, y_s are real variables.

$\mathbf{h} : \mathbf{C^n} \to \mathbf{C^n}$,and
$\mathbf{f} : \mathbf{C^m} \to \mathbf{C^m}$ are non linear function.
$V = (v_1, v_2, ...v_m) \in C^m$ is the controller to be programmed. When $m < n(f \neq g)$, the synchronization is accomplished in the reduced order. In particular, the problem of reduced order is synchronizing a response system with the projection



(a) (b) (c)

Fig 1: The chaotic attractors of complex fractional ordered Lorenz chaotic system

of master system. Consequently, we can split the master system into two parts. The projection part is

$$\dot{x}'_p = h_p(x') \tag{3}$$

where $x'_p = (x'_{p1}, x'_{p2}, ...x'_{pm}) \in C^n$ and $\mathbf{h_p} : \mathbf{C^n} \to \mathbf{C^m}$. Remaining part of the system is

$$\dot{x}'_r = h_r(x') \tag{4}$$

where $x'_r \in C^l, h_r : C^n \to C^l$ and orders m,l satisfy m+l=n.
Between the system (3) and (4), the error states can be defined as $e_{ij} = y_j \pm xpi$, where i,j=1,2,.....,m.
Definition: The master system (1) and the slave system (2) are said to be in reduced order dislocated hybrid synchronization, if there exists suitable controller $V = (v_1, v_2, ..., v_m)$ such that

$$lim_{t\to\infty} e_{ij} = lim_{t\to\infty}[y_j \pm x_{pi}] = 0 \tag{5}$$

$\imath, j = 1, 2, ..., m$ and $i \neq j$ for at least one state variable.

3 System description

3.1 Master system

The fractional order complex Lorenz system is given by

$\frac{d^q x_1'}{dt^q} = a_1(x_2' - x_1')$
$\frac{d^q x_2'}{dt^q} = a_2 x_1' - x_2' - x_1' x_3'$
$\frac{d^q x_3'}{dt^q} = \frac{1}{2}(\bar{x_1'} x_2' + x_1'(\bar{x_2'})) - a_3 x_3'$
where $x' = [x_1', x_2', x_3']^T$ is the state variable vector,$x_1' = x_1 + ix_2$ and $x_2' = x_3 + ix_4$ are the complex variables, while $x_3' = x_5$ is the real variable and a_1, a_2, a_3 are parameters.
Resolving into imaginary and real parts, we get

$$\frac{d^q x_1}{dt^q} = a_1(x_3 - x_1)$$

$$\frac{d^q x_2}{dt^q} = a_1(x_4 - x_2)$$

$$\frac{d^q x_3}{dt^q} = a_2 x_1 - x_3 - x_1 x_5 \tag{6}$$

$$\frac{d^q x_4}{dt^q} = a_2 x_2 - x_4 - x_2 x_5$$

$$\frac{d^q x_5}{dt^q} = x_1 x_3 + x_2 x_4 - a_3 x_5$$

For the values of parameters as $a_1 = 10, a_2 = 180, a_3 = 1$, $x(0) = [2,3,5,6,9]^T$ as initial condition and at $q = 0.95$, the system is chaotic .

3.2 Slave System:

The fractional order complex Duffing system is given by
$\frac{d^q y_1'}{dt^q} = y_2'$
$\frac{d^q y_2'}{dt^q} = y_1' - \alpha y_2' - y_1'^3 + \delta cos(\omega t)$
where $y' = [y_1', y_2']^T$ is the state variable and α, δ, ω are parameters.

Resolving into the imaginary and real parts, we get

$$\begin{aligned} \frac{d^q y_1}{dt^q} &= y_3 \\ \frac{d^q y_2}{dt^q} &= y_4 \\ \frac{d^q y_3}{dt^q} &= y_1 - \alpha y_3 - y_1^3 + 3y_1 y_2^2 + \delta cos(\omega t) \\ \frac{d^q y_4}{dt^q} &= y_2 - \alpha y_4 + y_2^3 - 3y_1^2 y_2 \end{aligned} \tag{7}$$

For the values of parameters as $\alpha = 1, \delta = 8, \omega = 0.5$, initial conditions $y(0) = [-1, 0, -1, 1]^T$ and for $q = 0.95$, the system is chaotic.

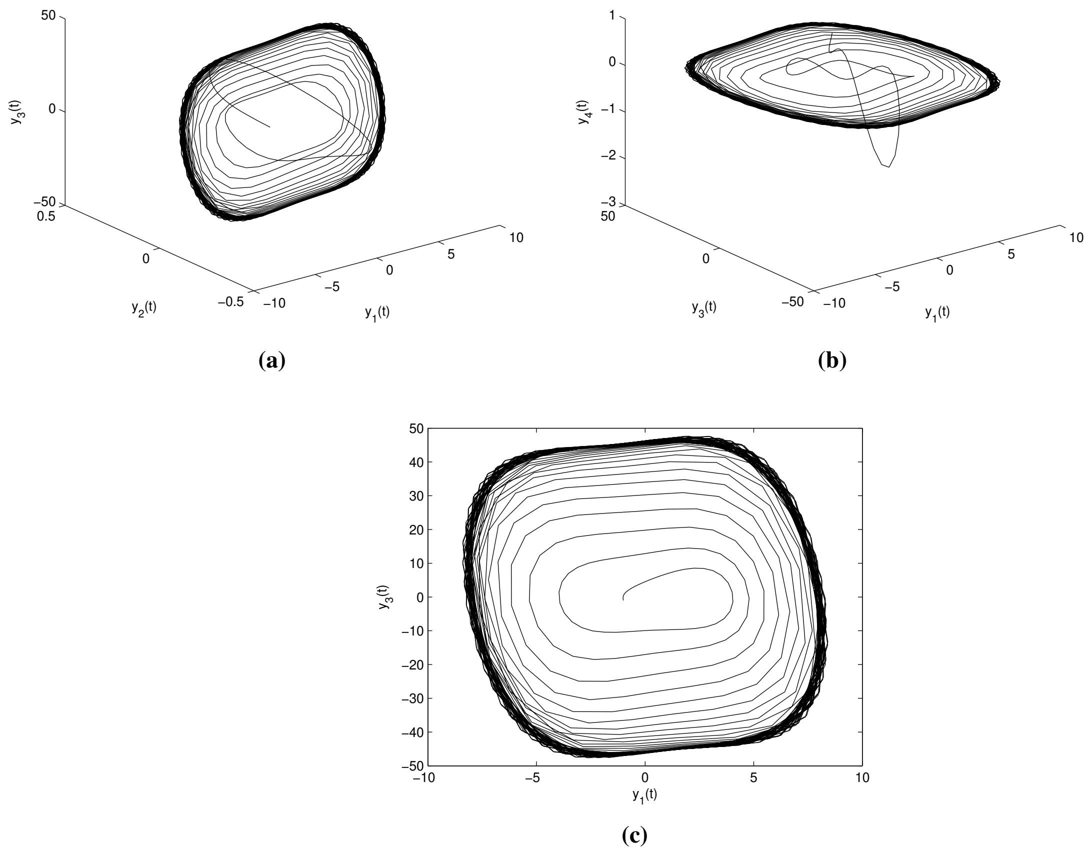


Fig 2: lThe chaotic attractors of complex fractional ordered Duffing chaotic system

4 Reduced order hybrid dislocated synchronization

Here, we study the reduced order hybrid dislocated synchronization between the complex Lorenz system (master) and Complex Duffing system (slave). To attain the intended synchronization, we take the projection of the master system and synchronize it with the slave system. For doing so,there are various possibilities of combinations. Here, we do the simulations for one arbitrarily selected hybrid combination.The outcome of other possibilities of other hybrid possibilities are self explanatory.
The projection of the Lorenz system is

$$\begin{aligned}\frac{d^q x_1}{dt^q} &= a_1(x_3 - x_1)\\ \frac{d^q x_2}{dt^q} &= a_1(x_4 - x_2)\\ \frac{d^q x_3}{dt^q} &= a_2 x_1 - x_3 - x_1 x_5\\ \frac{d^q x_4}{dt^q} &= a_2 x_2 - x_4 - x_2 x_5\end{aligned} \tag{8}$$

Then, the corresponding slave system is given by

$$\begin{aligned} \frac{d^q y_1}{dt^q} &= y_3 + v_1 \\ \frac{d^q y_2}{dt^q} &= y_4 + v_2 \\ \frac{d^q y_3}{dt^q} &= y_1 - \alpha y_3 - y_1^3 + 3y_1 y_2^2 + \delta cos(\omega t) + v_3 \\ \frac{d^q y_4}{dt^q} &= y_2 - \alpha y_4 + y_2^3 - 3y_1^2 y_2 + v_4 \end{aligned} \tag{9}$$

where v_1, v_2, v_3, v_4 are the controllers to be constructed in such a way that system (8) and system (9) are synchronized. As there are various possibilities of combination, we choose the states that are to be synchronised for any random combination.
To examine the hybrid dislocated synchronization between these systems, we define the error states as:

$$\begin{aligned} E_1 &= y_2 - x_1 \\ E_2 &= y_3 + x_4 \\ E_3 &= y_1 - x_3 \\ E_4 &= y_4 + x_2 \end{aligned} \tag{10}$$

The error dynamics is obtained as:

$$\begin{aligned} \frac{d^q E_1}{dt^q} &= y_4 - a_1 x_3 - a_1 E_1 + a_1 y_2 + v_2 \\ \frac{d^q E_2}{dt^q} &= y_1 - \alpha y_3 - y_1^3 + 3y_1 y_2^2 + \delta cos(\omega t) + a_2 x_2 - x_2 x_5 + y_3 - E_2 + v_3 \\ \frac{d^q E_3}{dt^q} &= y_3 - a_2 x_1 + y_1 + x_1 x_5 - E_3 + v_1 \\ \frac{d^q E_4}{dt^q} &= y_2 + y_2^3 - 3y_1^2 y_2 + a_1 x_4 - a_1 x_2 + \alpha x_2 - \alpha E_4 + v_4 \end{aligned} \tag{11}$$

Theorem: If the control functions v_1, v_2, v_3, v_4 are designed as follows:

$$\begin{aligned} v_1 &= -y_3 + a_2 x_1 - x_1 x_5 - y_1 + w_1 \\ v_2 &= -y_4 + a_1 x_3 - a_1 y_2 + w_2 \\ v_3 &= -y_1 + \alpha y_3 + y_1^3 - 3y_1 y_2^2 - \delta cos(\omega t) - a_2 x_2 + x_2 x_5 - y_3 + w_3 \\ v_4 &= -y_2 - y_2^3 + 3y_1^2 y_2 - a_1 x_4 + a_1 x_2 - \alpha x_2 + w_4 \end{aligned} \tag{12}$$

where w_1, w_2, w_3, w_4 are functions of the error states E_1, E_2, E_3, E_4, then the drive system (8) and the slave system (9) will accomplish the reduced order complete hybrid dislocated synchronization.

Proof: Using the controllers defined by (12), (11)can be given by

$$\begin{aligned} \frac{d^q E_1}{dt^q} &= -a_1 E_1 + w_2 \\ \frac{d^q E_2}{dt^q} &= -E_2 + w_3 \\ \frac{d^q E_3}{dt^q} &= -E_3 + w_1 \\ \frac{d^q E_4}{dt^q} &= -\alpha E_4 + w_4 \end{aligned} \tag{13}$$

w_1, w_2, w_3, w_4 are chosen in such a way that (13) becomes stable. Since this feedback stabilizes the system, the errors E_1, E_2, E_3, E_4 will asymptotically converge to zero. Let us choose

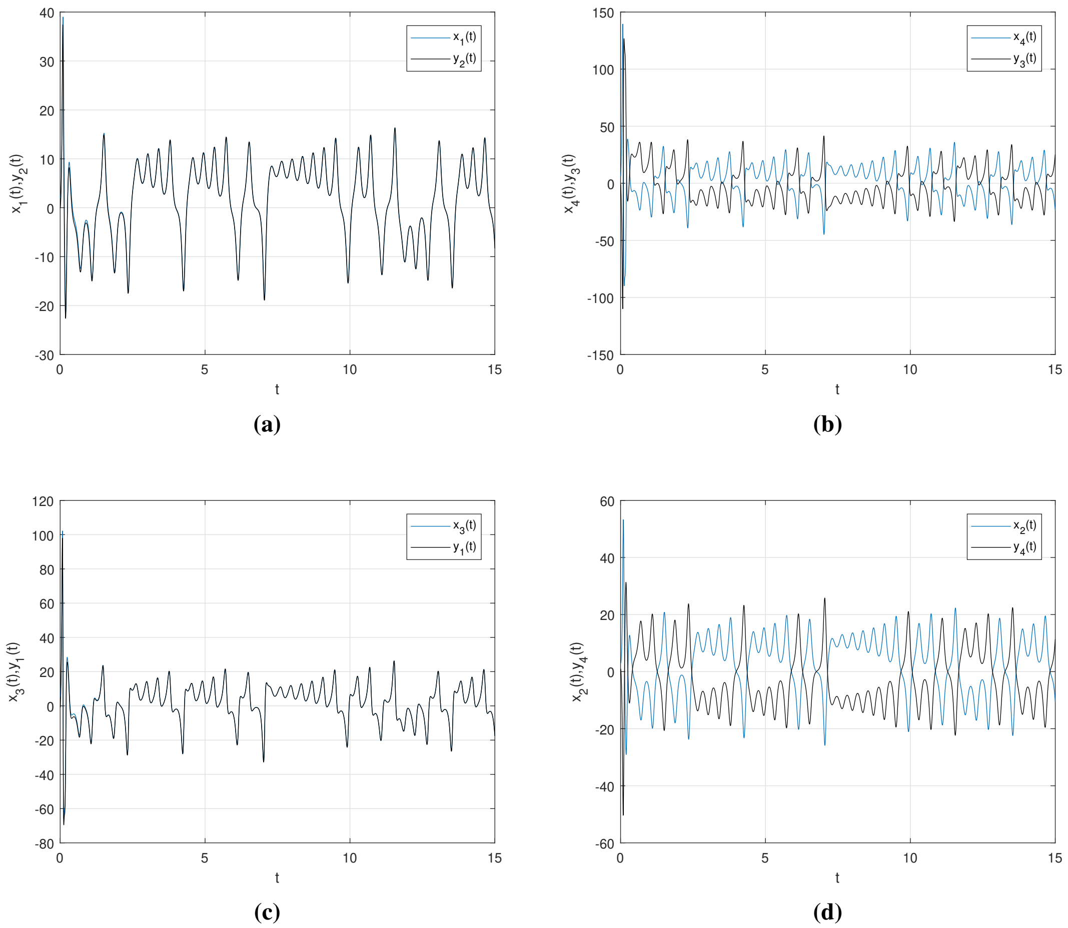


Fig 3: The synchronized trajectories of the master and slave systems

$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = A \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix}$ where A is a 4×4 constant matrix whose elements are selected such that w_1, w_2, w_3, w_4 make(13) stable. There are various possibilities of choosing A . We choose a particular form of given matrix A given by $\begin{bmatrix} 0 & 0 & -1 & 0 \\ (a_1-1) & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. Thus, $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} -E_3 \\ (a_1-1)E_1 \\ -E_2 \\ -E_4 \end{bmatrix}$ the error system(13) with these values of w_1, w_2, w_3, w_4 becomes

$$\begin{aligned} \frac{d^qE_1}{dt^q} &= -E_1 \\ \frac{d^qE_2}{dt^q} &= -2E_2 \\ \frac{d^qE_3}{dt^q} &= -2E_3 \\ \frac{d^qE_4}{dt^q} &= -(\alpha+1)E_4 \end{aligned} \tag{14}$$

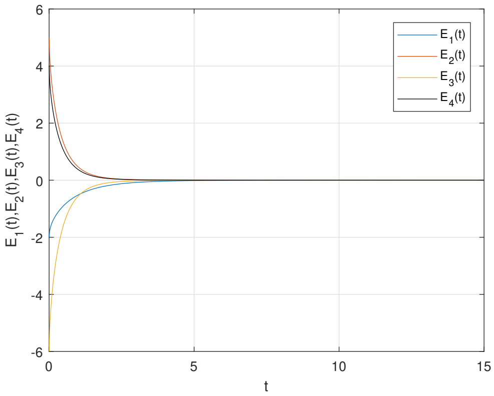


Fig 4: The simultaneous error plot

To display that the systems (13) and (14) are stable under this choice of w_1, w_2, w_3, w_4, we design the Lyapunov function $V(t)$, as follows:

$$V(t) = \frac{1}{2}(k_1E_1^2 + k_2E_2^2 + k_3E_3^2 + k_4E_4^2) \tag{15}$$

where $k_1, k_2, k_3 and k_4$ are positive numbers. The function $V(t)$ is positive definite. The time derivative of V is given by

$$\frac{dV}{dt} = k_1E_1\dot{E}_1 + k_2E_2\dot{E}_2 + k_3E_3\dot{E}_3 + k_4E_4\dot{E}_4$$

$$= k_1E_1(-E_1) + -k_2E_2(-2E_2) + k_3E_3(-2E_3) + k_4E_4(-(\alpha+1)E_4)$$

$$= -k_1E_1^2 - 2k_2E_2^2 - 2k_3E_3^2 - (\alpha+1)k_4E_4^2 < 0$$

Here, we have positive definite and negative definite function V and $\dot{V}$ respectively.By stability theorem of Lyapunov, (13) becomes stable. Similarly, zero solution of (14) is asymptotically stable. Thus systems (8) and (9) accomplish the desired synchronization, respectively.

5 Numerical simulations and discussions

Simulations have been carried out making use of MATLAB. The parameters for Lorenz and Duffing system are taken as ($a_1 = 10, a_2 = 180, a_3 = 1$) and ($\alpha = 1, \delta = 8, \omega = .5$). The initial values for drive and response system are taken as ($x_1(0) = 2, x_2(0) = 3, x_3(0) = 5, x_4(0) = 6, x_5(0) = 9$) and ($y_1(0) = -1, y_2(0) = 0, y_3(0) = -1, y_4(0) = 1$).
lFigures 1 and 2 display the chaotic attractors of complex fractional Lorenz and Duffing system, respectively.Fig. 3 shows the synchronized trajectories of the master and slave systems, i.e. Fig. 3 (a) and Fig. 3 (c) display complete synchronization and Fig. 3 (b) and Fig. 3 (d) display anti-synchronization. Fig. 4 displays the simultaneous error plot and the errors converging to zero which imply the accomplishment of synchronization.

6 Conclusion

In this article, we have introduced the reduced order dislocated synchronization method. Here, the different state variables of the driven system(response) were synchronized with the desired state variables of projection of the driving system(master). This type of synchronization scheme can be used to enhance the security of information transmission. Due to the possibility of several synchronization combinations, it would be challenging to the hackers to interrupt or track the combination in which synchronization would occur. The controllers have been designed successfully to accomplish reduced order dislocated complex fractional order hybrid synchronization between the Lorenz(drive) and Duffing system(slave). Numerical simulations have been performed to validate the scheme. This type of synchronization scheme can be applied to other chaotic systems, as well.

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