Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/070205

Numerical Solution of Fractional Bratu's Initial Value Problem Using Compact Finite Difference Scheme

Raziyeh Gharechahi[1], *Maryam Arabameri*[1,*] *and Morteza Bisheh-Niasar*[2]

[1] Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran
[2] Department of Applied Mathematics, University of Kashan, Kashan, Iran

Received: 7 Dec. 2019, Revised: 1 Feb. 2020, Accepted: 10 Feb. 2020
Published online: 1 Apr. 2021

Abstract: In this paper, we consider initial value problem of Bratu-type equations of fractional order $1 < \alpha \leq 2$. Compact finite difference schemes corresponding to $\alpha = 2$ and $1 < \alpha < 2$ are proposed. Also, convergence analysis of the methods are discussed separately. Some examples are also presented to show the efficiency of the methods.

Keywords: Bratu-type equations, Compact finite difference methods, convergence, fractional calculus.

1 Introduction

In real world, we need fractional differential equations for modeling and analyzing a large number of problems. Fractional calculus has different applications in mathematics, physics, chemistry and engineering fields, such as electromagnetism, control theory, fluid mechanics and viscoelastic [1–10]. In recent years, solving fractional ordinary differential equations (FODEs), fractional integral equations and fractional partial differential equations (FPDEs) have been studied by researchers. Since most of fractional equations do not have analytical solution, numerical methods are developed to find the approximate solutions. One of the FODEs that has a considerable scientific importance is Bratu-type equation.

In this study, we consider the following fractional Bratu-type equation

$$D^{\alpha}u(x) + \lambda exp(u(x)) = 0, \qquad 1 < \alpha \leq 2, \qquad 0 < x < 1, \tag{1}$$

along with the initial conditions

$$u(0) = C_1, \qquad u'(0) = C_2, \tag{2}$$

where C_1, C_2 and λ are given constants and D^{α} is the Caputo derivative operator of fractional order α, that is defined as follows:

$$D^{\alpha}u(x) = \frac{1}{\Gamma(2-\alpha)} \int_0^x \frac{u''(s)}{(x-s)^{\alpha-1}} ds. \tag{3}$$

Several chemical and physical processes in science and engineering can be modeled using Bratu-type equation. The Bratu-type equation is also used in a large variety of applied fields, such as modeling thermal reaction process in combustible non-deformable materials, including the solid fuel ignition model, the electrospinning process for production of ultra-fine polymer fibers, modeling some chemical reaction-diffusion, questions in geometry and relativity about the Chandrasekhar model, radiative heat transfer, and nanotechnology [11–19]. There are various numerical methods for solving Bratu initial value problem. For instance, Laplace transform decomposition algorithm is used in [20] and some numerical methods, based on finite difference technique, are proposed in [21, 22]. Caglar et al. [23] suggested B-spline method. In [24], Jalilian applied non-polynomial spline method and Boyd used Chebyshev polynomial expansions for Bratu equation [25].

* Corresponding author e-mail: arabameri@math.usb.ac.ir

Compared to the considerable works on the Bratu problem, only a little work has been done on the fractional Bratu problem. Babolian et al. [26] proposed reproducing kernel method (RKM) for solving initial value problem (1-2) and Ghomanjani and Shateyi [27] used the Bezier curve method (BCM) for solving fractional Bratu's initial value problem.

The present paper aims to obtain numerical solutions of (1-2) using a high order compact finite difference method. The present study is organized as follows: In Sections 2, 3, 4, the compact finite difference methods are reviewed and applied to solve Eqs. (1-2). Also, their convergence is discussed. In Section 5, the numerical results obtained by the proposed methods are presented. We also compare between the results of our method and those of the proposed methods in [26,27]. Conclusion and the advantages of the proposed technique are presented in Section 6.

2 Compact finite difference scheme

In this work, our main goal is applying the compact finite difference scheme to solve Eq. (1) with initial values (2). For this, we first subdivide the range of $0 \leq x \leq 1$ into N subintervals of width $h = \frac{1}{N}$. Set

$$u_i \approx u(x_i), \qquad u'_i \approx u'(x_i), \qquad u''_i \approx u''(x_i).$$

For the first derivatives, the following compact finite difference scheme was given in [28]:

$$\begin{cases} 4u'_1 + u'_2 = \frac{1}{h}(\frac{-11}{12}u_0 - 4u_1 + 6u_2 - \frac{4}{3}u_3 + \frac{1}{4}u_4), \\ u'_{i-1} + 4u'_i + u'_{i+1} = \frac{3}{h}(-u_{i-1} + u_{i+1}), \qquad i = 1, ..., N-1, \\ u'_{N-2} + 4u'_{N-1} = \frac{1}{h}(-\frac{1}{4}u_{N-4} + \frac{4}{3}u_{N-3} - 6u_{N-2} + 4u_{N-1} + \frac{11}{12}u_N). \end{cases} \tag{4}$$

All above relations have the accuracy of $O(h^4)$. The matrix form of (4) is

$$A_1 U' = \frac{1}{h} B_1 U, \tag{5}$$

where

$$A_1 = \begin{pmatrix} 0 & 4 & 1 & 0 & \dots & 0 \\ 1 & 4 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \dots & 0 & 1 & 4 & 1 \\ 0 & \dots & 0 & 1 & 4 & 0 \end{pmatrix}_{(N+1)\times(N+1)}, B_1 = \begin{pmatrix} -\frac{11}{12} & -4 & 6 & -\frac{4}{3} & \frac{1}{4} & 0 & \dots & 0 \\ -3 & 0 & 3 & 0 & 0 & 0 & \dots & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \dots & 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & \dots & 0 & -\frac{1}{4} & \frac{4}{3} & -6 & 4 & \frac{11}{12} \end{pmatrix}_{(N+1)\times(N+1)},$$

$U = [u_0, u_1, ..., u_N]^T, U' = [u'_0, u'_1, ..., u'_N]^T$.

Lemma 1 *The coefficient matrix A_1 is invertible.*

Proof. Let's expand A_1 along the first column, so

$$det(A_1) = -det \begin{pmatrix} 4 & 1 & 0 & \dots & 0 \\ 1 & 4 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 4 & 1 \\ 0 & \dots & 1 & 4 & 0 \end{pmatrix}_{N\times N}.$$

Now, expanding along the last column, we have

$$det(A_1) = (-1)^N det \begin{pmatrix} 4 & 1 & 0 & \dots & 0 \\ 1 & 4 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 4 & 1 \\ 0 & \dots & 0 & 1 & 4 \end{pmatrix}_{(N-1)\times(N-1)} \neq 0.$$

$\square$

According to lemma 1, from Eq. (5), we have $U' = \frac{1}{h}A_1^{-1}B_1U$, by defining $C = A_1^{-1}B_1$, the following relation holds for U',

$$U' = \frac{1}{h}CU, \tag{6}$$

and in component form

$$u_i' = \frac{1}{h}\sum_{j=0}^{N} c_{i+1,j+1}u_j, \qquad i = 0,\dots,N. \tag{7}$$

Similarly, for the second derivatives, we have the following compact finite difference scheme [28]:

$$\begin{cases} 14u_1'' - 5u_2'' + 4u_3'' - u_4'' = \frac{12}{h^2}(u_0 - 2u_1 + u_2), \\ u_{i-1}'' + 10u_i'' + u_{i+1}'' = \frac{12}{h^2}(u_{i-1} - 2u_i + u_{i+1}), \qquad i = 1,\dots,N-1, \\ -u_{N-4}'' + 4u_{N-3}'' - 5u_{N-2}'' + 14u_{N-1}'' = \frac{12}{h^2}(u_{N-2} - 2u_{N-1} + u_N), \end{cases} \tag{8}$$

where the truncation error for system (8) is $O(h^4)$ and its matrix form is, as follows:

$$A_2U'' = \frac{1}{h^2}B_2U, \tag{9}$$

where

$$A_2 = \begin{pmatrix} 0 & 14 & -5 & 4 & -1 & 0 & \dots & 0 \\ 1 & 10 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 10 & 1 & 0 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 & 10 & 1 \\ 0 & \dots & 0 & -1 & 4 & -5 & 14 & 0 \end{pmatrix}_{(N+1)\times(N+1)}, B_2 = 12\begin{pmatrix} 1 & -2 & 1 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 1 & 0 & 0\dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & 1 & -2 & 1 \\ 0 & \dots & 0 & 0 & 1 & -2 & 1 \end{pmatrix}_{(N+1)\times(N+1)},$$

$U = [u_0, u_1, \dots, u_N]^T$, $U'' = [u_0'', u_1'', \dots, u_N'']^T$.

Similar to lemma 1, it is apparent that the matrix A_2 is invertible. Thus, if we define $D = A_2^{-1}B_2$, we have

$$U'' = \frac{1}{h^2}DU. \tag{10}$$

Therefore, in component form, we have

$$u_i'' = \frac{1}{h^2}\sum_{j=0}^{N} d_{i+1,j+1}u_j, \qquad i = 0,\dots,N. \tag{11}$$

3 Compact finite difference scheme for Bratu problem in $\alpha = 2$ case and its convergence

In this section, we use the compact finite difference scheme for non-fractional Bratu problem and investigate its convergence. Consider the following classical nonlinear Bratu initial value problem

$$u''(x) + \lambda exp(u(x)) = 0, \qquad 0 < x < 1, \tag{12}$$

its initial conditions are

$$u(0) = C_1, \qquad u'(0) = C_2. \tag{13}$$

Using Eq. (7), condition $u'(0) = C_2$ can be written as

$$u_0' = \frac{1}{h}\sum_{j=0}^{N} c_{1,j+1}u_j = C_2. \tag{14}$$

For $x = x_i$, one can write Eq. (12) as

$$u''(x_i) + \lambda e^{u(x_i)} = 0, \qquad i = 1, \ldots, N-1. \tag{15}$$

Thus, by Eq. (11)

$$\frac{1}{h^2}\sum_{j=0}^{N} d_{i+1,j+1} u_j + \lambda e^{u_i} = 0, \qquad i = 1, \ldots, N-1. \tag{16}$$

Eq. (14) and Eq. (16) form a system including N equations and N unknowns $u_1, \ldots, u_N$, that can be solved by Maple software.

Now, the convergence analysis of the proposed method for Eq. (12) along with initial conditions (13) are investigated. Before expressing the main result, we define the following matrices:

$$M = \begin{pmatrix} c_{12} & c_{13} & \ldots & c_{1,N+1} \\ d_{22} & d_{23} & \ldots & d_{2,N+1} \\ \vdots & \vdots & \ddots & \vdots \\ d_{N,2} & d_{N,3} & \ldots & d_{N,N+1} \end{pmatrix}_{N\times N},$$

$$J = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ \frac{\partial e^{u(x_1)}}{\partial x_1} & 0 & 0 & \ldots & 0 \\ 0 & \frac{\partial e^{u(x_2)}}{\partial x_2} & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ldots & 0 \\ 0 & \ldots & 0 & \frac{\partial e^{u(x_{N-1})}}{\partial x_{N-1}} & 0 \end{pmatrix}_{N\times N}, \tag{17}$$

where $\frac{\partial e^{u(x_i)}}{\partial x_i} = \frac{\partial e^{u(x)}}{\partial x}|_{x=x_i}$.

Theorem 1 *Let $U = [u(x_1), \ldots, u(x_N)]^T$ be the vector of exact solution of Eq. (1) along with its initial conditions, and $u = [u_1, \ldots, u_N]^T$ be the numerical solution at same points obtained by Eqs. (14) and (16). Then, we have*

$$\|E\| \leq O(h^3), \tag{18}$$

provided $h^2\lambda\|M^{-1}\|\|J\| \leq 1$ where $E = [e_1, \ldots, e_N]^T$ and $e_i = u(x_i) - u_i, i = 1, \ldots, N$. ($\|.\|$ is the infinity norm)

Proof. According to Eqs. (14) and (16), for numerical solution, we have

$$\begin{cases} \frac{1}{h}\sum_{j=0}^{N} c_{1,j+1} u_j = C_2, \\ \frac{1}{h^2}\sum_{j=0}^{N} d_{i+1,j+1} u_j + \lambda e^{u_i} = 0, \quad i = 1, \ldots, N-1, \end{cases} \tag{19}$$

and for exact solution

$$\begin{cases} \frac{1}{h}\sum_{j=0}^{N} c_{1,j+1} u(x_j) + O(h^4) = C_2, \\ \frac{1}{h^2}\sum_{j=0}^{N} d_{i+1,j+1} u(x_j) + \lambda e^{u(x_i)} + O(h^4) = 0, \quad i = 1, \ldots, N-1. \end{cases} \tag{20}$$

Using the above-mentioned equations, one concludes that

$$\begin{cases} \sum_{j=0}^{N} c_{1,j+1} e_j = O(h^5), \\ \sum_{j=0}^{N} d_{i+1,j+1} e_j + h^2\lambda(e^{u(x_i)} - e^{u_i}) = O(h^6), \quad i = 1, \ldots, N-1, \end{cases} \tag{21}$$

where $e_j = u(x_j) - u_j, j = 0, \ldots, N$. Since $u_i \approx u(x_i)$, we have $e^{U(x)} - e^U \approx JE$ where J is introduced in (17).

Therefore, Eq. (21) can be written as

$$\begin{cases} c_{12}e_1 + c_{13}e_2 + \ldots + c_{1,N+1}e_N = O(h^5), \\ d_{22}e_1 + d_{23}e_2 + \ldots + d_{2,N+1}e_N + h^2\lambda e_1 \frac{\partial e^{u(x_1)}}{\partial x_1} = O(h^6), \\ d_{32}e_1 + d_{33}e_2 + \ldots + d_{3,N+1}e_N + h^2\lambda e_3 \frac{\partial e^{u(x_2)}}{\partial x_2} = O(h^6), \\ \vdots \\ d_{N,2}e_1 + d_{N,3}e_2 + \ldots + d_{N,N+1}e_N + h^2\lambda e_{N-1} \frac{\partial e^{u(x_{N-1})}}{\partial x_{N-1}} = O(h^6). \end{cases} \tag{22}$$

The matrix form of the above equations is, as follows:

$$(M+h^2\lambda J)E = R, \tag{23}$$

where $R = \begin{pmatrix} O(h^5) \\ O(h^6) \\ O(h^6) \\ \vdots \\ O(h^6) \end{pmatrix}_{N\times 1}$.

Now, if $h^2\lambda\|M^{-1}\|\|J\| \leq 1$, then $(I+h^2\lambda M^{-1}J)$ is invertible and $\|(I+h^2\lambda M^{-1}J)^{-1}\| \leq \frac{1}{1-h^2\lambda\|M^{-1}\|\|J\|}$. So, we have

$$E = (I+h^2\lambda M^{-1}J)^{-1}M^{-1}R \Rightarrow \|E\| \leq \|(I+h^2\lambda M^{-1}J)^{-1}\|\|M^{-1}\|\|R\|.$$

We note that $\|R\| \equiv O(h^5)$, thus $\|E\| \equiv \frac{O(h^5)}{O(h^2)} \leq O(h^3)$.

4 Implement of compact finite difference scheme for fractional Bratu problem and its convergence

In this section, we introduce a compact finite difference scheme for fractional Bratu problem of order $1<\alpha<2$. According to Eq. (3), we rewrite the Caputo derivative in $x = x_i, i = 1,\dots,N$ as

$$D^\alpha u(x_i) = \frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{i-1}\int_{x_k}^{x_{k+1}} \frac{u''(s)}{(x_i-s)^{\alpha-1}}ds. \tag{24}$$

Now, we discretize the second order derivative of u, as follows:

$$u''(x) = \frac{u'(x_{k+1})-u'(x_k)}{h} - \frac{h}{2}u'''(x_k) + O(h^2), \qquad x\in[x_k,x_{k+1}]. \tag{25}$$

By neglecting the truncation error, approximation of Caputo derivative can be reduced to:

$$\begin{aligned} D^\alpha u(x_i) &\approx \frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{i-1}\int_{x_k}^{x_{k+1}} \frac{u'_{k+1}-u'_k}{h}(x_i-s)^{1-\alpha}ds, \\ &= \frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{i-1}\frac{u'_{k+1}-u'_k}{h}\int_{x_k}^{x_{k+1}}(x_i-s)^{1-\alpha}ds, \\ &= \frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{i-1}\frac{u'_{k+1}-u'_k}{h}\left[\frac{(x_i-x_k)^{2-\alpha}-(x_i-x_{k+1})^{2-\alpha}}{2-\alpha}\right]. \end{aligned} \tag{26}$$

Substituting $x_i = ih$ in Eq. (26), we have

$$\begin{aligned} D^\alpha u(x_i) &\approx \frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{i-1}\frac{u'_{k+1}-u'_k}{h}\left[\frac{h^{2-\alpha}((i-k)^{2-\alpha}-(i-k-1)^{2-\alpha})}{2-\alpha}\right] \\ &= \frac{1}{h^{\alpha-1}\Gamma(3-\alpha)}\sum_{k=0}^{i-1}a_{i-k}(u'_{k+1}-u'_k), \end{aligned} \tag{27}$$

where $a_{i-k} = (i-k)^{2-\alpha}-(i-k-1)^{2-\alpha}, i = 1,\dots,N, k = 0,\dots,i-1$.

Thus, the solution of Eq. (1) can be approximated using the following equations

$$\frac{1}{h^{\alpha-1}\Gamma(3-\alpha)}\sum_{k=0}^{i-1}a_{i-k}(u'_{k+1}-u'_k) = -\lambda e^{u_i}, \qquad 1<\alpha<2 \qquad i = 1,\dots,N, \tag{28}$$

where $u'_i = \frac{1}{h}\sum_{j=0}^{N} c_{i+1,j+1}u_j, i = 1,...,N$. In matrix form, Eq. (28) is equivalent to

$$F_1U' = \rho e^U + C_2V, \tag{29}$$

where $e^U = [e^{u_1},...,e^{u_N}]^T$, $V = [a_1,...,a_N]^T$, $\rho = -\lambda h^{\alpha-1}\Gamma(3-\alpha)$ and

$$F_1 = \begin{pmatrix} a_1 & 0 & 0 & 0 & \dots & 0 \\ a_2-a_1 & a_1 & 0 & 0 & \dots & 0 \\ a_3-a_2 & a_2-a_1 & a_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{N-1}-a_{N-2} & a_{N-2}-a_{N-3} & \dots & a_2-a_1 & a_1 & 0 \\ a_N-a_{N-1} & a_{N-1}-a_{N-2} & \dots & a_3-a_2 & a_2-a_1 & a_1 \end{pmatrix}. \tag{30}$$

Similar to $\alpha = 2$ case, for $i = 1,...,N$, Eq. (28) can be used to form a system including N equations and N unknowns $u_1,...,u_N$, that can be solved by Maple software.

Now, we discuss the issue of convergence. For convergence analysis of fractional case, we need the following Lemma.

Lemma 2 *[29] Suppose $u \in C^3[0,x_i]$, then*

$$\begin{aligned} &\left|\int_0^{x_i}\frac{u''(s)}{(x_i-s)^{\alpha-1}}ds - \sum_{k=0}^{i-1}\frac{u'(x_{k+1})-u'(x_k)}{h}\int_{x_k}^{x_{k+1}}\frac{1}{(x_i-s)^{\alpha-1}}ds\right| \\ &\le \frac{1}{2-\alpha}\left[\frac{2-\alpha}{12}+\frac{2^{3-\alpha}}{3-\alpha}-(1+2^{1-\alpha})\right]\max_{0\le s\le x_i}|u'''(s)|h^{3-\alpha}. \end{aligned} \tag{31}$$

□

From Eq. (27), we have

$$D^\alpha u(x_i) = \frac{1}{h^{\alpha-1}\Gamma(3-\alpha)}\sum_{k=0}^{i-1}a_{i-k}(u'(x_{k+1})-u'(x_k)) + R_i, \qquad i = 1,...,N, \tag{32}$$

where according to Lemma 2

$$R_i \le \frac{1}{2-\alpha}\left[\frac{2-\alpha}{12}+\frac{2^{3-\alpha}}{3-\alpha}-(1+2^{1-\alpha})\right]\max_{0\le s\le x_i}|u'''(s)|h^{3-\alpha}.$$

For $x = x_i$, by replacing Eq. (32) in Eq. (1), we have

$$\sum_{k=0}^{i-1}a_{i-k}(u'(x_{k+1})-u'(x_k)) = \rho e^{u(x_i)} + \tilde{R}_i, \qquad i = 1,...,N, \tag{33}$$

where $\rho = -\lambda h^{\alpha-1}\Gamma(3-\alpha)$ and $\tilde{R}_i = h^{\alpha-1}\Gamma(3-\alpha)R_i, i = 1,...,N$.

In matrix form, Eq. (33) is equivalent to

$$F_1U'(x) = \rho e^{U(x)} + C_2V + \tilde{R}, \tag{34}$$

where F_1 is the matrix defined in relation (30), $U'(x) = [u'(x_1),...,u'(x_N)]^T$, $e^{U(x)} = [e^{u(x_1)},...,e^{u(x_N)}]^T$ and $\tilde{R} = h^{\alpha-1}\Gamma(3-\alpha)[R_1,...,R_N]^T$.

Theorem 2 *Let $U(x) = [u(x_1),...,u(x_N)]^T$ be the vector of exact solution of Eq. (1) along with its initial conditions at points $x_0,x_1,...,x_N$, and let $U = [u_1,...,u_N]^T$ be the numerical solution obtained by Eq. (28). Then, we have*

$$\|E\| \le O(h^{3-\alpha}), \tag{35}$$

provided $h\rho\|C^{-1}\|\|F_1^{-1}\|\|J\| \leq 1$ *where* $E = U(x) - U$ *and*

$$J = \begin{pmatrix} e^{u_1} & 0 & \dots & 0 \\ 0 & e^{u_2} & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & e^{u_N} \end{pmatrix}.$$

Proof. According to Eq. (34) and Eq. (29), for exact and numerical solution, we have

$$\begin{cases} F_1 U'(x) = \rho e^{U(x)} + C_2 V + \tilde{R}, \\ F_1 U' = \rho e^{U} + C_2 V. \end{cases} \tag{36}$$

Using Eq. (36), one concludes that

$$F_1(U'(x) - U') = \rho(e^{U(x)} - e^{U}) + \tilde{R}. \tag{37}$$

Therefore, by replacing $U' = \frac{1}{h}CU$ from (6) and $U'(x) = \frac{1}{h}CU(x) + T$ in Eq. (37), we have

$$\frac{1}{h}C(U(x) - U) - \rho F_1^{-1}(e^{U(x)} - e^{U}) = F_1^{-1}\tilde{R} + T, \tag{38}$$

where $T \equiv O(h^4)$ is the local trunction error of system (4).

We now $e^{U(x)} - e^{U} \approx JE$, so it can be written

$$(C - h\rho F_1^{-1}J)E = h(F_1^{-1}\tilde{R} + T) \Rightarrow (I - h\rho C^{-1}F_1^{-1}J)E = hC^{-1}(F_1^{-1}\tilde{R} + T).$$

Now, if $h\rho\|C^{-1}\|\|F_1^{-1}\|\|J\| \leq 1$ then $(I - h\rho C^{-1}F_1^{-1}J)$ is invertible and

$$E = h(I - h\rho C^{-1}F_1^{-1}J)^{-1}C^{-1}(F_1^{-1}\tilde{R} + T),$$

$$\|E\| \leq h\|(I - h\rho C^{-1}F_1^{-1}J)^{-1}\|\|C^{-1}\|(\|F_1^{-1}\|\|\tilde{R}\| + \|T\|).$$

It follows that

$$\|E\| \leq \frac{h\|C^{-1}\|(\|F_1^{-1}\|\|\tilde{R}\| + \|T\|)}{1 - h\rho\|C^{-1}\|\|F_1^{-1}\|\|J\|},$$

we note that $\|\tilde{R}\| \equiv O(h^2)$, so

$$\|E\| \leq \frac{O(h^3)}{O(h^\alpha)} + \frac{O(h^5)}{O(h^\alpha)} = O(h^{3-\alpha}) + O(h^{5-\alpha}) \equiv O(h^{3-\alpha}).$$

5 Numerical results

In this section, we apply our compact finite difference schemes to three example, to illustrate its effectiveness. Maple 17 is used for obtaining numerical results.

Example 1 *Consider the following Bratu type differential equation of fractional order*

$$\begin{cases} D^\alpha u(x) - 2exp(u(x)) = 0, \qquad 1 < \alpha \leq 2, \qquad 0 < x < 1. \\ u(0) = 0, \qquad u'(0) = 0. \end{cases} \tag{39}$$

The exact solution for $\alpha = 2$ *is* $u(x) = -2\ln(\cos x)$ *[26].*

In Figure 1, comparison between the exact solution for $\alpha = 2$ *and numerical solution for* $\alpha = 1.5, 1.6, 1.7, 1.8, 1.9, 1.95, 2$ *is shown. Also, Table 1 presents numerical solutions at some points of* $[0,1]$ *and for* *different values of* α.

We have calculated the rate of convergence of our methods (denoted by ROC) with the following formula

$$ROC = \log_2(\frac{Error^{2h}}{Error^{h}}). \tag{40}$$

Table 2 shows the obtained maximum errors and ROC for $\alpha = 2$ *and* $N = 5, 10, 20, 40, 80$. *Also, Figure 2 shows the numerical and exact solutions for* $\alpha = 2$ *and* $N = 20$.

In Table 3, we compare between the error of solutions of the present method with RKM [26] and BCM [27] for $\alpha = 2$. *Also, in Table 4, we compare between the approximate solution of the present method with RKM [26] and BCM [27] at points* $0.1, 0.2, ..., 1$, *for* $\alpha = 1.9$.

Table 1. Numerical solutions of Example 1 for $\alpha = 1.5, 1.6, 1.7, 1.8, 1.9, 1.95, 2$ **and** $N = 10$

α N	1.5 10	1.6 10	1.7 10	1.8 10	1.9 10	1.95 10	2 10
x							
0.1	-7.55×10^{-2}	-1.22×10^{-2}	8.15×10^{-3}	1.11×10^{-2}	1.03×10^{-2}	9.63×10^{-3}	1.00×10^{-2}
0.2	1.01×10^{-2}	5.42×10^{-2}	6.10×10^{-2}	5.45×10^{-2}	4.65×10^{-2}	4.29×10^{-2}	4.02×10^{-2}
0.3	9.72×10^{-2}	1.41×10^{-1}	1.40×10^{-1}	1.23×10^{-1}	1.05×10^{-1}	9.80×10^{-2}	9.14×10^{-2}
0.4	2.47×10^{-1}	2.70×10^{-1}	2.50×10^{-1}	2.19×10^{-1}	1.90×10^{-1}	1.78×10^{-1}	1.64×10^{-1}
0.5	4.02×10^{-1}	4.23×10^{-1}	3.90×10^{-1}	3.43×10^{-1}	3.00×10^{-1}	2.82×10^{-1}	2.61×10^{-1}
0.6	6.31×10^{-1}	6.28×10^{-1}	5.70×10^{-1}	5.01×10^{-1}	4.41×10^{-1}	4.15×10^{-1}	3.84×10^{-1}
0.7	8.90×10^{-1}	8.76×10^{-1}	7.91×10^{-1}	6.98×10^{-1}	6.16×10^{-1}	5.82×10^{-1}	5.36×10^{-1}
0.8	1.26	1.21	1.07	9.45×10^{-1}	8.35×10^{-1}	7.89×10^{-1}	7.23×10^{-1}
0.9	1.77	1.65	1.50	1.26	1.11	1.04	9.51×10^{-1}
1.0	2.67	2.35	1.97	1.68	1.46	1.37	1.23

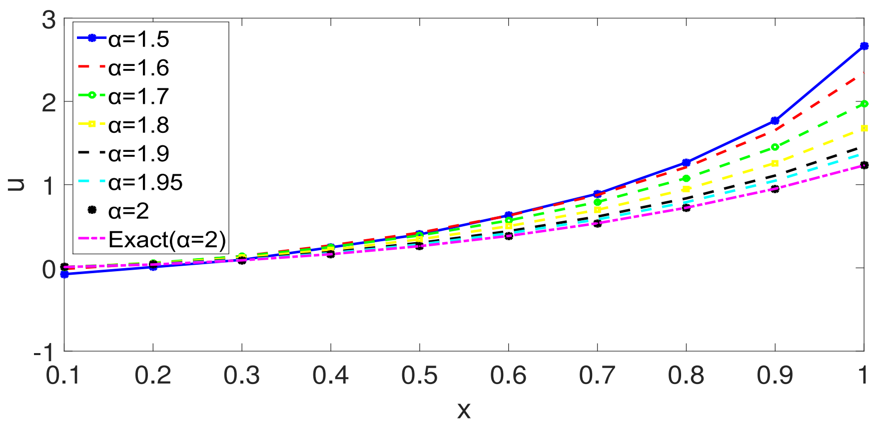

Figure 1. Comparison between exact solution of Example 1 for $\alpha = 2$ **and numerical solutions for** $\alpha = 1.5, 1.6, 1.7, 1.8, 1.9, 1.95, 2$ **and** $N = 10$

Example 2 *Consider the following Bratu-type differential equation*

$$\begin{cases} D^{\alpha}u(x) - exp(2u(x)) = 0, & 1 < \alpha \leq 2, \quad 0 < x < 1. \\ u(0) = 0, \quad u'(0) = 0. \end{cases} \tag{41}$$

Table 2. Maximum absolute errors and ROC of Example 1 for $\alpha = 2$ and $N = 5, 10, 20, 40, 80$

N	$MaxError$	ROC
5	1.67×10^{-3}	–
10	8.32×10^{-5}	4.33
20	4.43×10^{-6}	4.23
40	2.38×10^{-7}	4.22
80	1.36×10^{-8}	4.11

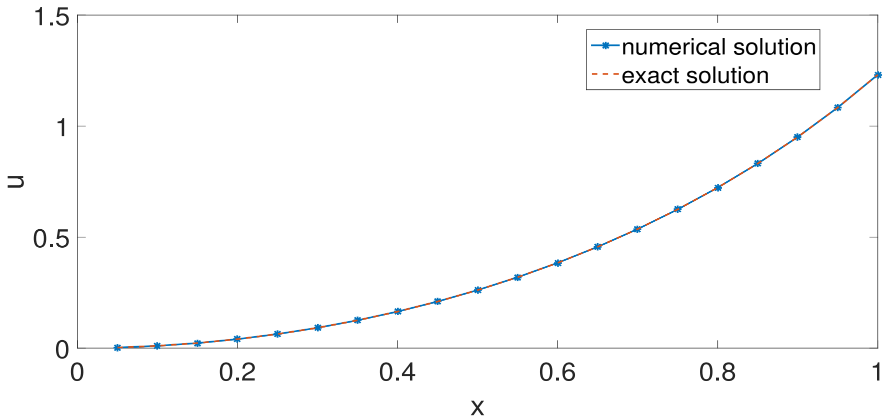

Figure 2. Comparison between exact solution and numerical solution of Example 1 for $\alpha = 2$ and $N = 10$

Table 3. Comparison between the absolute error of solution by our method with RKM [26] and BCM [27] for $\alpha = 2$, for Example 1

x	error of proposed method	error of RKM [26]	error of BCM [27]
0.1	7.89×10^{-6}	1.67×10^{-5}	2.98×10^{-3}
0.2	1.23×10^{-5}	3.10×10^{-7}	0
0.3	1.71×10^{-5}	1.13×10^{-6}	1.69×10^{-4}
0.4	2.26×10^{-5}	2.12×10^{-4}	1.11×10^{-4}
0.5	2.90×10^{-5}	2.90×10^{-6}	0
0.6	3.69×10^{-5}	4.10×10^{-6}	0
0.7	4.72×10^{-5}	6.50×10^{-6}	7.77×10^{-4}
0.8	6.14×10^{-5}	7.50×10^{-6}	0
0.9	8.32×10^{-5}	3.35×10^{-5}	3.47×10^{-3}
1.0	1.29×10^{-5}	4.37×10^{-5}	0

Table 4. Comparison between the approximate solution of the present method with RKM [26] and BCM [27] for $\alpha = 1.9$ and $N = 10$ for Example 1

x	solution of proposed method	solution of RKM [26]	solution of BCM [27]
0.1	1.03×10^{-2}	1.31×10^{-2}	1.03×10^{-2}
0.2	4.65×10^{-2}	5.14×10^{-2}	4.02×10^{-2}
0.3	1.05×10^{-1}	1.13×10^{-1}	9.12×10^{-2}
0.4	1.90×10^{-1}	1.99×10^{-1}	1.64×10^{-1}
0.5	3.00×10^{-1}	3.11×10^{-1}	2.61×10^{-1}
0.6	4.41×10^{-1}	4.51×10^{-1}	3.84×10^{-1}
0.7	6.16×10^{-1}	6.23×10^{-1}	5.36×10^{-1}
0.8	8.35×10^{-1}	8.38×10^{-1}	7.23×10^{-1}
0.9	1.11	1.11	9.51×10^{-1}
1.0	1.46	1.43	1.23

The exact solution for $\alpha = 2$ is $u(x) = \ln(\sec x)$ [26].

In Figure 3, we compare between the exact solution for $\alpha = 2$ and numerical solution for $\alpha = 1.5, 1.6, 1.7, 1.8, 1.9, 1.95, 2$ is shown. Also, Table 5 presents numerical solutions at some points of $[0,1]$ and for different values of α. Table 6 shows the obtained maximum errors and ROC for $\alpha = 2$ and $N = 5, 10, 20, 40, 80$. Figure 4 shows the exact solution and approximate solution for $\alpha = 2$ and $N = 20$.

In Table 7, we compare between the maximum absolute errors of the present method with RKM [26] and BCM [27] for $\alpha = 2$. In Table 8, we compare between the approximate solution of the present method with RKM [26] and BCM [27] for $\alpha = 1.9$ and $N = 10$.

Table 5. Numerical solutions of Example 2 for $\alpha = 1.5, 1.6, 1.7, 1.8, 1.9, 1.95, 2$ and $N = 10$

α N	1.5 10	1.6 10	1.7 10	1.8 10	1.9 10	1.95 10	2 10
x							
0.1	-3.77×10^{-2}	-6.09×10^{-3}	4.08×10^{-3}	5.58×10^{-3}	5.17×10^{-3}	4.82×10^{-3}	5.01×10^{-3}
0.2	5.08×10^{-3}	2.71×10^{-2}	3.05×10^{-2}	2.73×10^{-2}	2.33×10^{-2}	2.14×10^{-2}	2.01×10^{-2}
0.3	4.86×10^{-2}	7.05×10^{-2}	6.97×10^{-2}	6.13×10^{-2}	5.28×10^{-2}	4.90×10^{-2}	4.57×10^{-2}
0.4	1.23×10^{-1}	1.35×10^{-1}	1.25×10^{-1}	1.09×10^{-1}	9.49×10^{-2}	8.86×10^{-2}	8.22×10^{-2}
0.5	2.01×10^{-1}	2.11×10^{-1}	1.95×10^{-1}	1.71×10^{-1}	1.50×10^{-1}	1.40×10^{-1}	1.31×10^{-1}
0.6	3.15×10^{-1}	3.14×10^{-1}	2.85×10^{-1}	2.50×10^{-1}	2.21×10^{-1}	2.07×10^{-1}	1.92×10^{-1}
0.7	4.45×10^{-1}	4.38×10^{-1}	3.96×10^{-1}	3.49×10^{-1}	3.08×10^{-1}	2.91×10^{-1}	2.68×10^{-1}
0.8	6.33×10^{-1}	6.04×10^{-1}	5.39×10^{-1}	4.72×10^{-1}	4.18×10^{-1}	3.95×10^{-1}	3.61×10^{-1}
0.9	8.84×10^{-1}	8.27×10^{-1}	7.25×10^{-1}	6.30×10^{-1}	5.55×10^{-1}	5.23×10^{-1}	4.75×10^{-1}
1.0	1.33	1.17	9.86×10^{-1}	8.38×10^{-1}	7.30×10^{-1}	6.87×10^{-1}	6.16×10^{-1}

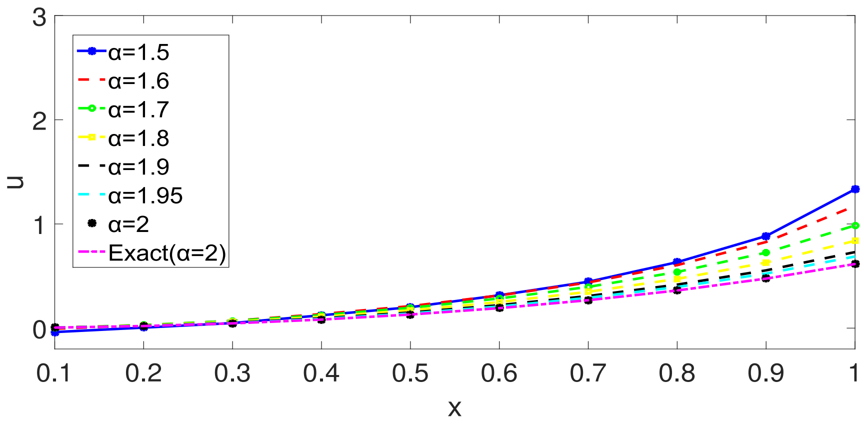

Figure 3. Comparison between exact solution of Example 2 for $\alpha = 2$ and numerical solutions obtained by the present method for $\alpha = 1.5, 1.6, 1.7, 1.8, 1.9, 1.95, 2$ and $N = 10$

Example 3 *We consider Bratu-type equation with the following initial conditions*

$$\begin{cases} D^{\alpha}u(x) + \pi^2 exp(-u(x)) = 0, & 1 < \alpha \leq 2, \qquad 0 < x < 1. \\ u(0) = 0, \qquad u'(0) = \pi. \end{cases} \tag{42}$$

For $\alpha = 2$, the exact solution is $u(x) = \ln(1 + \sin \pi x)$ [26].

In Figure 5, we compare between the exact solution for $\alpha = 2$ and approximate solution for $\alpha = 1.7, 1.8, 1.9, 1.95, 1.99$. In Table 9, these approximate solutions are presented.

Table 6. Maximum absolute errors and ROC of Example 2 for $\alpha = 2$ and $N = 5, 10, 20, 40, 80$

N	$MaxError$	ROC
5	8.34×10^{-4}	–
10	4.16×10^{-5}	4.32
20	2.22×10^{-6}	4.23
40	1.92×10^{-7}	3.53
80	6.80×10^{-9}	4.82

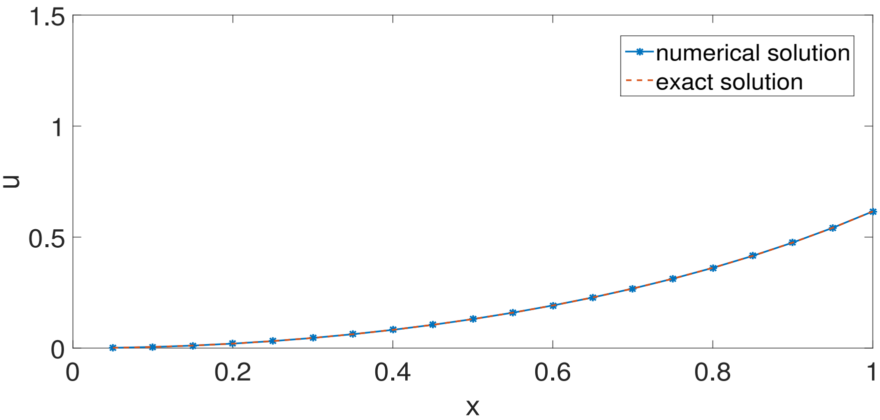

Figure 4. Comparison between exact solution and numerical solution of Example 2 for $\alpha = 2$ and $N = 10$

Table 7. Comparison between the absolute error of solution by present method with RKM [26] and BCM [27] for $\alpha = 2$, for Example 2

x	error of proposed method	error of RKM [26]	error of BCM [27]
0.1	3.95×10^{-6}	8.33×10^{-6}	1.40×10^{-5}
0.2	6.15×10^{-6}	1.51×10^{-7}	2.05×10^{-11}
0.3	8.57×10^{-6}	6.40×10^{-7}	2.63×10^{-5}
0.4	1.13×10^{-5}	1.02×10^{-6}	3.54×10^{-10}
0.5	1.45×10^{-5}	1.35×10^{-6}	1.08×10^{-4}
0.6	1.85×10^{-5}	1.90×10^{-6}	2.40×10^{-4}
0.7	2.36×10^{-5}	3.20×10^{-6}	2.51×10^{-4}
0.8	3.07×10^{-5}	3.50×10^{-6}	1.00×10^{-10}
0.9	4.16×10^{-5}	1.65×10^{-5}	3.97×10^{-4}
1.0	6.48×10^{-6}	2.14×10^{-5}	0

Table 8. Comparison between the approximate solution of the present method with RKM [26] and BCM [27] for $\alpha = 1.9$ and $N = 10$ for Example 2

x	solution of proposed method	solution of RKM [26]	solution of BCM [27]
0.1	5.17×10^{-3}	6.54×10^{-3}	6.54×10^{-3}
0.2	2.33×10^{-2}	2.57×10^{-2}	2.33×10^{-2}
0.3	5.28×10^{-2}	5.66×10^{-2}	4.84×10^{-2}
0.4	9.49×10^{-2}	9.96×10^{-2}	8.22×10^{-2}
0.5	1.50×10^{-1}	1.56×10^{-1}	1.27×10^{-1}
0.6	2.21×10^{-1}	2.25×10^{-1}	1.85×10^{-1}
0.7	3.08×10^{-1}	3.12×10^{-1}	2.61×10^{-1}
0.8	4.18×10^{-1}	4.18×10^{-1}	4.75×10^{-1}
0.9	5.55×10^{-1}	5.52×10^{-1}	4.75×10^{-1}
1.0	7.30×10^{-1}	7.15×10^{-1}	6.15×10^{-1}

In Table 10, we compare between the maximum absolute errors of the present method with RKM [26] and BCM [27] for $\alpha = 2$ is accomplished.

Table 9. Numerical solutions of Example 3 obtained by present method for $\alpha = 1.7, 1.8, 1.9, 1.95, 1.99, 2$ and $N = 10$

α N	1.7 10	1.8 10	1.9 10	1.95 10	1.99 10	2 10
x						
0.1	3.28×10^{-1}	2.71×10^{-1}	2.68×10^{-1}	2.71×10^{-1}	2.74×10^{-1}	2.69×10^{-1}
0.2	4.59×10^{-1}	4.32×10^{-1}	4.56×10^{-1}	4.70×10^{-1}	4.81×10^{-1}	4.62×10^{-1}
0.3	5.43×10^{-1}	5.31×10^{-1}	5.81×10^{-1}	6.09×10^{-1}	6.30×10^{-1}	5.92×10^{-1}
0.4	5.42×10^{-1}	5.65×10^{-1}	6.54×10^{-1}	6.98×10^{-1}	7.29×10^{-1}	6.68×10^{-1}
0.5	5.08×10^{-1}	5.48×10^{-1}	6.76×10^{-1}	7.38×10^{-1}	7.83×10^{-1}	6.93×10^{-1}
0.6	3.86×10^{-1}	4.68×10^{-1}	6.48×10^{-1}	7.32×10^{-1}	7.91×10^{-1}	6.68×10^{-1}
0.7	2.11×10^{-1}	3.26×10^{-1}	5.68×10^{-1}	6.78×10^{-1}	7.55×10^{-1}	5.92×10^{-1}
0.8	-9.15×10^{-2}	9.72×10^{-2}	4.29×10^{-1}	5.72×10^{-1}	6.70×10^{-1}	4.62×10^{-1}
0.9	-5.38×10^{-1}	-2.42×10^{-1}	2.19×10^{-1}	4.06×10^{-1}	5.32×10^{-1}	2.69×10^{-1}
1.0	-1.37	-7.77×10^{-1}	-8.53×10^{-2}	1.67×10^{-1}	3.30×10^{-1}	0

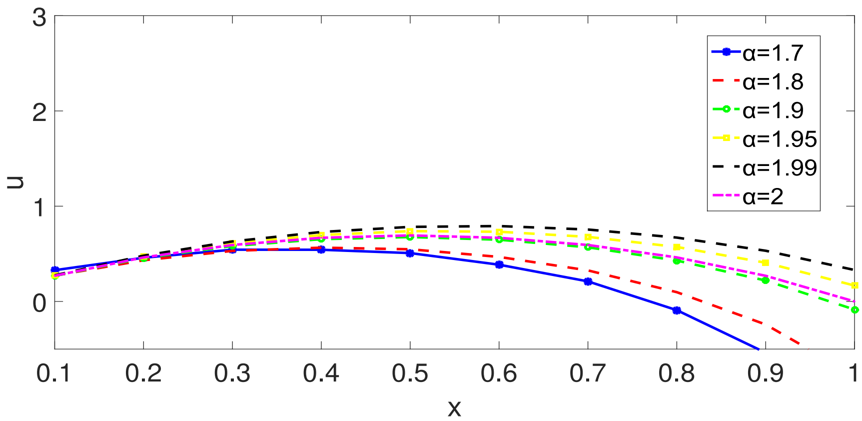

Figure 5. Comparison between exact solution of Example 3 for $\alpha = 2$ and numerical solutions by the proposed method for $\alpha = 1.7, 1.8, 1.9, 1.95, 1.99, 2$ and $N = 10$

Table 10. Comparison between the maximum absolute errors of solution by our method with RKM [26] and BCM [27] for $\alpha = 2$

	proposed method	RKM [26]	BCM [27]
Example 1	8.32×10^{-5}	2.12×10^{-4}	3.47×10^{-3}
Example 2	4.16×10^{-5}	2.14×10^{-5}	3.97×10^{-4}

6 Conclusion

In this paper, the compact finite difference schemes were proposed for the numerical solution of the Bratu-type differential equation of fractional and non-fractional order. Convergence analysis for them was investigated. Numerical

solutions presented in Tables 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10 showed that the proposed methods are effective.
Another advantage of the adopted technique is that the presented method has a simple implementation process, i.e. its computer programming is very easy and fast.

References

[1] K. Diethelm and N. J. Ford, Analysis of fractional differential equations, *J. Math. Anal. Appl.* **265 (2)**, 229–248 (2002).
[2] K. Diethelm and N. J. Ford, Multi-order fractional differential equations and their numerical solution, *Appl. Math. Comput.* **154 (3)**, 621–640 (2004).
[3] D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, *Fractional calculus: models and numerical methods*, World Scientific, 2012.
[4] D. Baleanu, J. A. T. Machado and A. C. Luo, *Fractional dynamics and control*, Springer, 2011.
[5] D. Baleanu, Z. B. Guvenc and J. A. T. Machado, *New trends in nanotechnology and fractional calculus applications*, New York, Springer, 2010.
[6] S. Salahshour, A. Ahmadian, M. Salimi, M. Ferrara and D. Baleanu, Asymptotic solutions of fractional interval differential equations with nonsingular kernel derivative, *Chaos* **29**8, 1–18 (2019).
[7] S. Salahshour, A. Ahmadian, S. Abbasbandy and D. Baleanu, M-fractional derivative under interval uncertainty: Theory, properties and applications, *Chaos Solit. Fract.* **117**, 84–93 (2018).
[8] A. Ahmadian, F. Ismail, S. Salahshour, D. Baleanu and F. Ghaemi, Uncertain viscoelastic models with fractional order: A new spectral tau method to study the numerical simulations of the solution, *Commun. Nonlin. Sci. Numer. Simul.* **53**, 44-64 (2017).
[9] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, (1999).
[10] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives, theory and applications*, Gordon and Breach, Yverdon, 1993.
[11] R. Buckmire, Application of a Mickens finite-difference scheme to the cylindrical Bratu–Gelfand problem, *Numer. Methods Partial Differ. Equ.* **20**, 327–337 (2004).
[12] J. S. McGough, Numerical continuation and the Gelfand problem, *Appl. Math. Comput.* **89**, 225–239 (1998).
[13] A. S. Mounim and B. M. De Dormale, From the fitting techniques to accurate schemes for the Liouville–Bratu–Gelfand problem, *Numer. Meth. Part. Differ. Equ.* **22**, 761–775 (2006).
[14] M. I. Syam and A. Hamdan, An efficient method for solving Bratu equations, *Appl. Math. Comput.* **176**, 704–713 (2006).
[15] S. Li and S. J. Liao, An analytic approach to solve multiple solutions of a strongly nonlinear problem, *Appl. Math. Comput.* **169**, 854–865 (2005).
[16] A. M. Wazwaz, Adomian decomposition method for a reliable treatment of the Bratu–type equations, *Appl. Math. Comput.* **166**, 652–663 (2005).
[17] J. H. He, Some asymptotic methods for strongly nonlinear equations, *Int. J. Mod. Phys. B.* **20**, 1141–1199 (2006).
[18] J. H. He, Variational approach to the Bratu's problem, it J. Phys. Conf. Ser. **96**, 012087 (2008).
[19] S. Liao and Y. Tan, A general approach to obtain series solutions of nonlinear differential equations, *Stud. Appl. Math.* **119**, 131–136 (2007).
[20] S. A. Khuri, A new approach to Bratu's problem, *Appl. Math. Comput.* **147**, 297–354 (2004).
[21] H. Temimi and M. Ben–Romdhane, An iterative finite difference method for solving Bratu's problem, *J. Comput. Appl. Math.* **292**, 76–82 (2016).
[22] R. Gharechahi, M. Arab Ameri and M. Bisheh–Niasar, High order compact finite difference schemes for solving Bratu–type equations, *J. Comput. Appl. Mech.* **5**(1), 91–102 (2019).
[23] H. Caglar, N. Caglar, M. Özer, A. Valaristos and A. N. Anagnostopoulos, B–spline method for solving Bratu's problem, *Int. J. Comput. Math.* **87**, 1885–1891 (2010).
[24] R. Jalilian, Non–polynomial spline method for solving Bratu's problem, *Comput. Phys. Commun.* **181**, 1868–1872 (2010).
[25] J. P. Boyd, Chebyshev polynomial expansions for simultaneous approximation of two branches of a function with application to the one–dimensional Bratu equation, *Appl. Math. Comput.* **143**, 189–200 (2003).
[26] E. Babolian, S. Javadi and E. Moradi, RKM for solving Bratu-type differential equations of fractional order, *Math. Meth. Appl. Sci.* **39**(6), 1548–1557 (2016).
[27] F. Ghomanjani and S. Shateyi, Numerical solution for fractional Bratu's initial value problem, *Open Physics.* **15**(1), 1045–1048 (2017).
[28] P. G. Zhang and J. P. Wang, A predictor–corrector compact finite difference scheme for Burgers' equation, *Appl. Math. Comput.* **219**(3), 892–898 (2012).
[29] Z. Z. Sun and X. Wu, A fully discrete difference scheme for a diffusion–wave system, *Appl. Numer. Math.* **56**(2), 193–209 (2006).